Symmetric Spinors in k-Dimensions (*).

B. DODDS (Edinburgh, Scotland)

Summary. – *In his paper, explicit formulae are given for any irreducible spinor, in any number of dimensions, which is symmetric in its suffixes. The method employed in determining these formulae is immediately applicable to the task of obtaining explicit forms for spinors which are unsymmetric in their suffixes.*

1. – Introduction.

Given a set of k anticommuting 2^ν-rowed matrices $X_1, X_2, \dots, X_k$ with squares $\pm I$, corresponding to any orthogonal matrix $A = [a_s^t]$ in $k = 2\nu$ variables, there is a matrix U which is unique apart from sign such that

$$a_i^j X_j = U^{-1} X_i U$$

where $i = 1, 2, \dots, k$. The matrix U is called the basic spin matrix. For $k = 2\nu + 1$ variables, there is a slight modification for transformations of negative determinant and the equation is

$$a_i^j X_j = |a_s^t| \cdot U^{-1} X_i U \, .$$

The matrices U form the basic spin representation of the orthogonal group. A 2^ν-rowed vector, which, in the orthogonal transformation corresponding to the orthogonal matrix A, is transformed by the basic spin matrix, is called a basic spinor. The spur of the basic spin matrix is called the basic spin character and is denoted by $[(\frac{1}{2})^\nu]$. The basic spinor is said to be of type $[(\frac{1}{2})^\nu]$.

There are other spin representations of the orthogonal group besides the basic spin representation, and correspondigly there are other spinors besides the basic spinors. The spin representations in fact correspond to partitions $(\lambda_1 \lambda_2 \dots \lambda_\nu)$ into ν parts, in which each part is equal to half an odd integer and $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_\nu$. The basic spin representation corresponds therefore to the simplest case, namely the partition $((\frac{1}{2})^\nu)$. A sequence of spin representations is obtained by taking the direct product of a true representation of the orthogonal group with the basic spin representation. This product is found to be reducible as witnessed by the character equation given by LITTLEWOOD [4], p. 304. This character equation together with the

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fact that every representation of the full linear group is necessarily a representation of the orthogonal group, either simple or compound, contain the method which is to be used to obtain explicit forms for any symmetric spinor in k-dimensions.

2. – Consider an orthogonal transformation

$$x_i' = a_i^j x_j, \qquad (i = 1, 2, \dots, k).$$

Let W be a basic spinor. Suppose that ξ_{i_r} is a tensor of rank one, *i.e.* a quantity whose components transform as the coordinates x_i. The direct product of n such tensors, $\xi_{i_1}, \xi_{i_2}, \dots \xi_{i_n}$, is a complete tensor of rank n, *i.e.* a quantity whose components transform as the product of n coordinates. A complete tensor of rank n may be expressed as a sum of simple tensors, LITTLEWOOD, [1], the symmetric one, $\xi_{i_1 i_2 \dots i_n}$ say, being of type $\{n\}$. Considering then the direct product of $\xi_{i_1 i_2 \dots i_n}$ and W, one writes $\xi_{i_1 i_2 \dots i_n} W = W_{i_1 i_2 \dots i_n}$ say, is of type $\{n\}[(\frac{1}{2})^\nu]$.

A representation of the full linear group, of type $\{\lambda\}$ say, is reducible under the orthogonal group (LITTLEWOOD [2], p. 238 et seq.). Thus, since n is a positive integer, one has that

$$\{n\} = [n] + [n-2] + \dots + \begin{cases} [2] + [0] & \text{if } n \text{ is even;} \\ [3] + [1] & \text{if } n \text{ is odd:} \end{cases}$$

Hence the product $\{n\}[(\frac{1}{2})^\nu]$ can be expanded, the character equation being

$$\{n\}[(\tfrac{1}{2})^\nu] = [n + \tfrac{1}{2}, (\tfrac{1}{2})^{\nu-1}] + [n - \tfrac{1}{2}, (\tfrac{1}{2})^{\nu-1}] + \dots + [\tfrac{3}{2}, (\tfrac{1}{2})^{\nu-1}] + [(\tfrac{1}{2})^\nu],$$

both when n is even and when n is odd. Thus, if the direct product of a simple tensor of type $\{n\}$ and a basic spinor is considered, on removal of the various contractions an irreducible symmetric spinor of type $[n + \frac{1}{2}, (\frac{1}{2})^{\nu-1}]$ remains. Note that there are two modes of contraction, LITTLEWOOD [3], namely with the metric tensor and with a matrix of the set X_i, and it is assumed that for an irreducible spinor each mode of contraction gives a zero result.

The 3-dimensional case is considered first. Suppose that $V_{i_1 i_2 \dots i_n}$ is an irreducible spinor of type $[n + \frac{1}{2}]$. In order to obtain $V_{i_1 i_2 \dots i_n}$ explicity, a number of particular examples are considered whence the form taken by $V_{i_1 i_2 \dots i_n}$ becomes evident. Thus suppose that V_i is an irreducible spinor of type $[\frac{3}{2}]$ and that W_i is of type $\{1\}[\frac{1}{2}]$. One has the character equation

$$\{1\}[\tfrac{1}{2}] = [\tfrac{3}{2}] + [\tfrac{1}{2}],$$

and removing the basic spinor contraction gives

$$V_i = W_i - \tfrac{1}{3} X_i X^q W_q$$

where $X^i V_i = 0$ as required. Since a basic spinor in three dimensions is taken to have two complex components, a spinor of type $[\frac{3}{2}]$ has four components which may be taken as being any two of the three linearly dependent 2-vectors V_i, the dependency relation being $X^i V_i = 0$ of course.

Consider next an irreducible spinor of type $[\frac{5}{2}]$, V_{ij} say. Suppose W_{ij} is of type $\{2\}[\frac{1}{2}]$. The required character equation is

$$\{2\}[\tfrac{1}{2}] = [\tfrac{5}{2}] + [\tfrac{3}{2}] + [\tfrac{1}{2}],$$

and removal of the two contractions gives

$$V_{ij} = W_{ij} - \tfrac{1}{5}(X_i X^q W_{qj} + X_j X^q W_{qi}) - \tfrac{1}{5} g_{ij} g^{qr} W_{qr}$$

where $X^i V_{ij} = 0 = g^{ij} V_{ij}$ as required. Note that in the removal of the two contractions, the two formulae

$$X^i X_j X^q W_{qi} = 2X^q W_{qj} - X_j X^i X^q W_{qi} \qquad \text{and} \qquad X^i X^q W_{qi} = g^{iq} W_{qi}$$

have been used. The formulae follow from the properties of the X_i and the symmetry of W_{ij}. Similar formulae are used in higher order cases. The spinor V_{ij} consists of six 2-vectors of which only three, as required, are independent because of the two zero contractions.

It is convenient here to introduce the set of permutation operators $P_{i_1 i_2 \ldots i_n}$, the operators being defined as follows: A permutation operator $P_{i_1 i_2 \ldots i_n}$ operating on a tensor $Q_{i_1 i_2 \ldots i_n}$ denotes the operation of taking the sum of the $n!$ tensors obtained by permuting the suffixes of $Q_{i_1 i_2 \ldots i_n}$ in all ways.

In the light of this definition, the formula for the irreducible spinor V_{ij} is rewritten as

$$V_{ij} = W_{ij} - P_{ij}(Q_{ij}),$$

where

$$Q_{ij} = (X_i X^q W_{qj})/(5 \cdot 1!) + (g_{ij} g^{qr} W_{qr})/(5 \cdot 2 \cdot 1!\, 0!).$$

Higher order spinors are obtained in a similar manner. Thus if V_{ijp}, V_{ijpq}, V_{ijpqr}, V_{ijpqrs} and $V_{ijpqrst}$ are respectively irreducible spinors of types [7/2], [9/2], [11/2], [13/2] and [15/2], and if W_{ijp}, W_{ijpq}, W_{ijpqr}, W_{ijpqrs} and $W_{ijpqrst}$ are of types $\{3\}[\frac{1}{2}]$, $\{4\}[\frac{1}{2}]$, $\{5\}[\frac{1}{2}]$, $\{6\}[\frac{1}{2}]$ and $\{7\}[\frac{1}{2}]$ respectively, then explicit forms for the spinors are as follows:

$$V_{ijp} = W_{ijp} - P_{ijp}(Q_{ijp}),$$

$$V_{ijpq} = W_{ijpq} - P_{ijpq}(Q_{ijpq}),$$

$$V_{ijpqr} = W_{ijpqr} - P_{ijpqr}(Q_{ijpqr}),$$

$$V_{ijpqrs} = W_{ijpqrs} - P_{ijpqrs}(Q_{ijpqrs}) ,$$

$$V_{ijpqrst} = W_{ijpqrst} - P_{ijpqrst}(Q_{ijpqrst}) ,$$

where

$$Q_{ijp} = X_i X^a W_{ajp})/(7 \cdot 2!) + (g_{ij} g^{ab} W_{abp})/(7 \cdot 2 \cdot 1!1!) - (X_i g_{jp} X^a g^{bc} W_{abc})/(7 \cdot 5 \cdot 2 \cdot 1!0!) ,$$

$$Q_{ijpq} = (X_i X^a W_{ajpq})/(9 \cdot 3!) + (g_{ij} g^{ab} W_{abpq})/(9 \cdot 2 \cdot 1!2!) -$$
$$- (X_i g_{jp} X^a g^{bc} W_{abcq})/(9 \cdot 7 \cdot 2 \cdot 1!1!) - (g_{ij} g_{pq} g^{ab} g^{cd} W_{abcd})/(9 \cdot 7 \cdot 2^2 \cdot 2!0!) ,$$

$$Q_{ijpqr} = (X_i X^a W_{ajpqr})/(11 \cdot 4!) + (g_{ij} g^{ab} W_{abpqr})/(11 \cdot 2 \cdot 1!3!) -$$
$$- (X_i g_{jp} X^a g^{bc} W_{abcqr})/(11 \cdot 9 \cdot 2 \cdot 1!2!) - (g_{ij} g_{pq} g^{ab} g^{cd} W_{abcdr})/(11 \cdot 9 \cdot 2^2 \cdot 2!1!) +$$
$$+ (X_i g_{jp} g_{qr} X^a g^{bc} g^{de} W_{abcde})/(11 \cdot 9 \cdot 7 \cdot 2^2 \cdot 2!0!) ,$$

$$Q_{ijpqrs} = (X_i X^a W_{ajpqrs})/(13 \cdot 5!) + (g_{ij} g^{ab} W_{abpqrs})/(13 \cdot 2 \cdot 1!4!) -$$
$$- (X_i g_{jp} X^a g^{bc} W_{abcqrs})/(13 \cdot 11 \cdot 2 \cdot 1!3!) - (g_{ij} g_{pq} g^{ab} g^{cd} W_{abcdrs})/(13 \cdot 11 \cdot 2^2 \cdot 2!2!) +$$
$$+ (g_{ij} g_{pq} g_{rs} g^{ab} g^{cd} g^{ef} W_{abcdef})/(13 \cdot 11 \cdot 9 \cdot 2^3 \cdot 3!0!) ,$$

$$Q_{ijpqrst} = (X_i X^a W_{ajpqrst})/(15 \cdot 6!) + (g_{ij} g^{ab} W_{abpqrst})/(15 \cdot 2 \cdot 1!5!) -$$
$$- (X_i g_{jp} X^a g^{bc} W_{abcqrst})/(15 \cdot 13 \cdot 2 \cdot 1!4!) - (g_{ij} g_{pq} g^{ab} g^{cd} W_{abcdrst})/(15 \cdot 13 \cdot 2^2 \cdot 2!3!) +$$
$$+ (X_i g_{jp} g_{qr} X^a g^{bc} g^{de} W_{abcdest})/(15 \cdot 13 \cdot 11 \cdot 2^2 \cdot 2!2!) +$$
$$+ (g_{ij} g_{pq} g_{rs} g^{ab} g^{cd} g^{ef} W_{abcdeft})/(15 \cdot 13 \cdot 11 \cdot 2^3 \cdot 3!1!) -$$
$$- (X_i g_{jp} g_{qr} g_{st} X^a g^{bc} g^{de} g^{fh} W_{abcdefh})/(15 \cdot 13 \cdot 11 \cdot 9 \cdot 2^3 \cdot 3!0!) .$$

Both modes of contraction give zero results for each of V_{ijp}, V_{ijpq}, V_{ijpqr}, V_{ijpqrs} and $V_{ijpqrst}$ as required, which consist of 4, 5, 6, 7 and 8 independent 2-vectors respectively.

A distinct pattern in the coefficients appearing in the $Q_{(i)}$ is discernible, from which it is possible to deduce the explicit form taken by an irreducible spinor of type $[n+\frac{1}{2}]$. The two cases n an even positive integer and n an odd positive integer must be treated separately: Suppose that $V_{i_1 i_2 \ldots i_n}$ is an irreducible spinor of type $[n+\frac{1}{2}]$ and that $W_{i_1 i_2 \ldots i_n}$ is of type $\{n\}[\frac{1}{2}]$. Then,

(i) if n is an even positive integer, $2m$ say where m is a positive integer,

$$V_{i_1 i_2 \ldots i_{2m}} = W_{i_1 i_2 \ldots i_{2m}} - P_{i_1 i_2 \ldots i_{2m}}(Q_{i_1 i_2 \ldots i_{2m}})$$

where

$$Q_{i_1 i_2 \ldots i_{2m}} = \sum_{r=0}^{m-1} \frac{(-1)^r X_{i_1} g_{i_2 i_3} \ldots g_{i_{2r} i_{2r+1}} X^{j_1} g^{j_2 j_3} \ldots g^{j_{2r} j_{2r+1}} W_{j_1 \ldots j_{2r+1} i_{2r+2} \ldots i_{2m}}}{(4m+1)(4m-1) \ldots (4m-2r+1)(2m-2r-1)!\,2^r r!} +$$
$$+ \sum_{r=0}^{m-1} \frac{(-1)^r g_{i_1 i_2} \ldots g_{i_{2r+1} i_{2r+2}} g^{j_1 j_2} \ldots g^{j_{2r+1} j_{2r+2}} W_{j_1 \ldots j_{2r+2} i_{2r+3} \ldots i_{2m}}}{(4m+1)(4m-1) \ldots (4m-2r+1)(2m-2r-2)!\,2^{r+1}(r+1)!};$$

(ii) if n is an odd positive integer, $2m+1$ say where m is a positive integer,

$$V_{i_1 i_2 \ldots i_{2m+1}} = W_{i_1 i_2 \ldots i_{2m+1}} - P_{i_1 i_2 \ldots i_{2m+1}}(Q_{i_1 i_2 \ldots i_{2m+1}}),$$

where

$$Q_{i_1 i_2 \ldots i_{2m+1}} = \sum_{r=0}^{m} \frac{(-1)^r X_{i_1} g_{i_2 i_3} \ldots g_{i_{2r} i_{2r+1}} X^{j_1} g^{j_2 j_3} \ldots g^{j_{2r} j_{2r+1}} W_{j_1 \ldots j_{2r+1} i_{2r+2} \ldots i_{2m+1}}}{(4m+3)(4m+1) \ldots (4m-2r+3)(2m-2r)!\,2^r r!} +$$
$$+ \sum_{r=0}^{m-1} \frac{(-1)^r g_{i_1 i_2} \ldots g_{i_{2r+1} i_{2r+2}} g^{j_1 j_2} \ldots g^{j_{2r+1} j_{2r+2}} W_{j_1 \ldots j_{2r+2} i_{2r+3} \ldots i_{2m+1}}}{(4m+3)(4m+1) \ldots (4m-2r+3)(2m-2r-1)!\,2^{r+1}(r+1)!}.$$

Considering next the 4-dimensional case, it is found that the irreducible symmetric spinors obtained are the same as their counterparts in the 3-dimensional case (as just determined), except that whereas the numbers 3, 5, 7, 9, ... appeared in the coefficients in the 3-dimensional case, the numbers 4, 6, 8, 10, ... appear in the 4-dimensional case. For example, if V_i and V_{ij} are respectively irreducible spinors of types $[\frac{3}{2}, \frac{1}{2}]$ and $[\frac{5}{2}, \frac{1}{2}]$, and if W_i and W_{ij} are of types $\{1\}[\frac{1}{2}, \frac{1}{2}]$ and $\{2\}[\frac{1}{2}, \frac{1}{2}]$ respectively, then explicit forms for the spinors are as follows:

$$V_i = W_i - (X_i X^q W_q)/4,$$
$$V_{ij} = W_{ij} - P_{ij}(Q_{ij}),$$

where

$$Q_{ij} = (X_i X^q W_{qj})/6 + (g_{ij} g^{qr} W_{qr})/(6 \cdot 2 \cdot 1!\,0!).$$

Similarly in five dimensions, the numbers 5, 7, 9, 11, ... appear in the coefficients instead of 3, 5, 7, 9 ... . Higher dimensional cases follow the same pattern and one can thus generalise the formulae to k-dimensions as follows: Suppose that the number of variables is k, where $k = 2\nu$ or $2\nu+1$ depending upon whether the number of variables is even or odd respectively, ν being a positive integer. Suppose that $V_{i_1 i_2 \ldots i_n}$ is an irreducible symmetric spinor of type $[n+\frac{1}{2}, (\frac{1}{2})^{\nu-1}]$ and that $W_{i_1 i_2 \ldots i_n}$ is of type $\{n\}[(\frac{1}{2})^\nu]$, where n is a positive integer. Then,

(i) if n is an even positive integer, $2m$ say where m is a positive integer,

$$V_{i_1 i_2 \ldots i_{2m}} = W_{i_1 i_2 \ldots i_{2m}} - P_{i_1 i_2 \ldots i_{2m}}(Q_{i_1 i_2 \ldots i_{2m}})$$

where

$$Q_{i_1 i_2 \dots i_{2m}} = \sum_{r=0}^{m-1} \frac{(-1)^r X_{i_1} g_{i_2 i_3} \dots g_{i_{2r} i_{2r+1}} X^{j_1} g^{j_2 j_3} \dots g^{j_{2r} j_{2r+1}} W_{j_1 \dots j_{2r+1} i_{2r+2} \dots i_{2m}}}{(4m+k-2)(4m+k-4) \dots (4m+k-2r-2)(2m-2r-1)!2^r r!} +$$
$$+ \sum_{r=0}^{m-1} \frac{(-1)^r g_{i_1 i_2} \dots g_{i_{2r+1} i_{2r+2}} g^{j_1 j_2} \dots g^{j_{2r+1} j_{2r+2}} W_{j_1 \dots j_{2r+2} i_{2r+3} \dots i_{2m}}}{(4m+k-2)(4m+k-4) \dots (4m+k-2r-2)(2m-2r-2)!2^{r+1}(r+1)!};$$

(ii) if n is an odd positive integer, $2m+1$ say where m is a positive integer,

$$V_{i_1 i_2 \dots i_{2m+1}} = W_{i_1 i_2 \dots i_{2m+1}} - P_{i_1 i_2 \dots i_{2m+1}}(Q_{i_1 i_2 \dots i_{2m+1}})$$

where

$$Q_{i_1 i_2 \dots i_{2m+1}} = \sum_{r=0}^{m} \frac{(-1)^r X_{i_1} g_{i_2 i_3} \dots g_{i_{2r} i_{2r+1}} X^{j_1} g^{j_2 j_3} \dots g^{j_{2r} j_{2r+1}} W_{j_1 \dots j_{2r+1} i_{2r+2} \dots i_{2m+1}}}{(4m+k)(4m+k-2) \dots (4m+k-2r)(2m-2r)!2^r r!} +$$
$$+ \sum_{r=0}^{m-1} \frac{(-1)^r g_{i_1 i_2} \dots g_{i_{2r+1} i_{2r+2}} g^{j_1 j_2} \dots g^{j_{2r+1} j_{2r+2}} W_{j_1 \dots j_{2r+2} i_{2r+3} \dots i_{2m+1}}}{(4m+k)(4m+k-2) \dots (4m+k-2r)(2m-2r-1)!2^{r+1}(r+1)!}.$$

The irreducible spinor $V_{i_1 i_2 \dots i_n}$ consists of $(k+n-1)!/n!(k-1)!$ 2^ν-vectors of which just $(k+n-2)!/n!(k-2)!$ are independent because of the zero contractions.

The above formulae remain valid irrespective of the signature of the metric. Thus, for example, in four dimensions, the metric could be the sum of squares, the signature then being equal to 4, or one could take the metric of space-time, in which case the signature is equal to 2.

All irreducible spinors which are symmetric in their suffixes are thus explicitly determined. Spinors which are unsymmetric in their suffixes can be determined explicitly by employing the same method. For example, in four dimensions, if V_{ij} is an irreducible spinor of type $[\frac{3}{2}, \frac{3}{2}]$ and W_{ij} is of type $\{1^2\}[\frac{1}{2}, \frac{1}{2}]$, so that $W_{ij} + W_{ji} = 0$, one has the character equation

$$\{1^2\}[\tfrac{1}{2}, \tfrac{1}{2}] = [\tfrac{3}{2}, \tfrac{3}{2}] + [\tfrac{3}{2}, \tfrac{1}{2}] + [\tfrac{1}{2}, \tfrac{1}{2}],$$

and correspondingly one has that

$$V_{ij} = W_{ij} - (X_i X^q W_{qj} - X_j X^q W_{qi})/2 - (X_i X_j - X_j X_i) X^q X^r W_{qr}/16,$$

where both modes of contraction give zero results and thus V_{ij} consists of just two independent 4-vectors as required.

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