μ-Homotopic Maps are Loop Homotopic (*) (**).

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Summary. – *In [4] we have defined a notion of μ-homotopy in the category of simplicial groups and we have made the conjecture that μ-homotopy is equivalent to loop homotopy. The purpose of this paper is to prove this cojecture.*

Introduction.

In [4] we have developed a theory of principal cofibre bundles, dual in the sense of ECKMANN-HILTON [1] to the theory principal fibre bundles with group G. We have shown that any such bundle with « cobase » A and « cofibre » FK where FK is MILNOR's F construction [3] over the simplicial set K, is induced via a map from GK into A. Here G is the construction of KAN [2]. The question whether loop-homotopic maps induce equivalent cofibre bundles remained open; it was only shown that if we replace the usual definition of homotopy in the category $\mathfrak{G}$ of simplicial groups, namely that of loop-homotopy by an ad hoc notion of μ-homotopy, then it is true that two maps from GK into A are μ-homotopic if and only if they induce equivalent bundles. We conjectured that μ-homotopy is equivalent to loop-homotopy. We shall prove (Theorem 1.3) that this is indeed the case and we conclude that two loop-homotopic maps from A to B induce equivalent.bundles and that two maps from GK into A induce equivalent bundles if and if they are loop-homotopic.

1. – Let K be a simplicial set which is reduced i.e. it has only one 0-simplex k_0, and let K_n represent the set of all simplexes of K of dimension n. Using the same notations as in [4] we define the simplicial group GK by constructing $(GK)_n$ as the group with generators $\tau(x)$, $x \in K_{n+1}$ and relations $\tau(s_0 x) = e_n$ (the unit element). The face and degeneracy operators in K are defined as follows:

$$\partial_0(\tau(x)) = \tau(\partial_0 x)^{-1}(\partial_1 x)$$

$$\partial_i(\tau(x)) = \tau(\partial_{i+1}(x)) \qquad \text{if } i \neq 0$$

$$s_i(\tau(x)) = \tau(s_{i+1}(x)) \qquad \text{if } i \geqslant 0$$

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Our notations and conventions are different from those of [2]; also we would like to point out that there is an abvious misprint in the definition of $\partial_0(\tau(x))$ given in [4].

Given a simplicial group G and a simplicial set K, we shall construct following [2] the « tensor product » $G \otimes K$ as the simplicial group generated by the simbols $g \in x$, where $g \in G$ and $x \otimes K$, with the relations $(g \otimes x)(h \otimes x) = gh \otimes x$. The face and degeneracy operators are defined by $\partial_i(g \otimes x) = (\partial_i g \otimes \partial_i x)$ and $s_i(g \otimes x) = (s_i g \otimes s_i x)$. The unit interval I is the simplicial set consisting of one non-degenerate 1-simplex 01, its vertices 0 and 1 and their degeneracies. Any simplex of I can be represented as $0^p 1^q$, $p+q \leqslant 1$, $p \geqslant 0$, $q \geqslant 0$. There are two obvious imbeddings of G into $G \otimes I$ namely i_0 defined by $i_0(g) = g \otimes 1^{n+1}$ and i_1 defined by $i_1(g) = g \otimes 0^{n+1}$. It is obvious from the contractibility of I, that as simplicial sets $i_j(G)$ are deformations retracts of $G \otimes I$.

DEFINITION 1.1. [2]. – *Two morphisms $f_0, f_1\colon G \to H$, $G, H \in \mathfrak{G}$ are called loop-homotopic if there is a morphism $F\colon G \otimes I \to H$ such that $f_0 = Fi_0$, $f_1 = Fi_1$.*

It has been shown in [2] that if G is free, the loop-homotopy relation is an equivalence relation.

Let $G = GK$ where K is a reduced simplicial set.

DEFINITION 1.2. [4]. – *Two morphisms $f_0, f_1\colon GK \to A$, $A \in \mathfrak{G}$ are called μ-homotopic if there is a map $\mu\colon K \to A$, $\mu(K_n) = A_n$ such that:*

$$\partial_0(\mu(x)) = f_0(\tau(x))^{-1}\mu(\partial_0 x) f_1(\tau(x))$$

$$\partial_i(\mu(x)) = \mu(\partial_i x)\,, \qquad i \neq 0$$

$$s_i(\mu(x)) = \mu(s_i x)\,, \qquad i \geqslant 0\,.$$

Our main result is the following

THEOREM 1.3. – *If K is a reduced simplicial set and A is a simplicial group, two morphisms from GK to A are μ-homotopic if and only if they are loop-homotopic.*

In view of the preceding result, Theorem 2.6.5. of [4] can now be rephrased as

COROLLARY 1.4. – *Two principal cofibre bundles with cobase A and cofibre FK are equivalent if and only if they are induced by loop-homotopic morphisms $GK \to A$.*

This is a dual of the classification theorem of principal fibre bundles. (For a definition of principal cofibre bundles see [4]). In general if A is any simplicial group and P is a principal cofibre bundles with cobase A and cofibre FK then any morphism $f\colon A \to A$ induces a principal cofibre bundles with cobase B and cofibre FK. Since P itself is induced by $k\colon GK \to A$ two cofibre bundles induced by loop-homotopic morphisms $f_1, f_2\colon A \to A$ are induced from the universal bundle b the bloop-homotopic morphisms $f_0 k, f_1 k\colon GK \to A$ and are therefore equivalent. Thus we have:

Corollary 1.5. – *If P is a principal cofibre bundle over A and $f_0, f_1\colon A \to B$ are loop-homotopic, then the induced bundles $f_0 P$, $f_1 P$ are equivalent.*

Since the unit interval is not reduced, the cartesian product between a reduced simplicial set K and I is not a reduced simplicial set. This forces us to use the following

Definition 1.6. – *The reduced cartesian product $Kx'I$ where K is reduced, is the simplicial set obtained from KxI by identifying all the simplexes of the form $(s_0^n k_0, x)$ $x \in I_n$ with one completely degenerated simplex $s_0^n y_0$.*

The next sections will be concerned with the proof of Theorem 1.3.

2. – Let j_0 denote the imbedding of K into $Kx'I$ defined by $j_0(x) = (x, 1^{n+1})$ and let $j_1(x) = (x, 0^{n+1})$.

Lemma 2.1. – *Let $f_0, f_1\colon GK \to A$ be two μ-homotopic morphisms. Then there exists a morphism $F\colon G(Kx'I) \to A$ such that $F_0 G j_i = f_i$, $i = 0, 1$.*

Proof. – Let $F\colon G(Kx'I) \to A$ be defined on the free generators of $G(Kx'I)$ as follows:

$$F(\tau(x, 1^N)) \;= f_0(\tau(x))$$

$$F(\tau(x, 01^n)) = \mu(\partial_0 x) f_1(\tau(x))$$

$$F(\tau(x, 0^p 1^q)) = f_1(\tau(x)), \qquad n + 1 = p + q = N, \qquad p \geqslant 2,$$

It can be checked directly, by using the definition of μ-homotopy and that of the face and degeneracy operators in GK, that the mapping F satisfies our requirements.

Lemma 2.2. – *If two morphisns $f_0, f_1\colon GK \to A$ satisfy the conclusions of Lemma 2.1., they are loop-homotopic.*

Proof. – Let $\overline{W}$ be the functor from $\mathfrak{G}$ to the category $\mathfrak{S}$ of reduced simplicial sets defined in [2]. There is a natural map α which establishes a 1-1 correspondence between $\mathfrak{G}$-morphisms: $GK \to A$ and $\mathfrak{S}$-morphisms: $K \to \overline{W}A$. Therefore the map $\alpha(F)$ is a homotopy in $\mathfrak{S}$ between $\alpha(f_0)$ and $\alpha(f_1)$ (see Proposition 10.5 of [2]). By applying Theorem 11.1 [2] we conclude that f_0 and f_1 are loop-homotopic.

3. – We now prove that loop-homotopy implies μ-homotopy. Let $p\colon G \otimes I \to G$ be the canonical projection $p(g \otimes x) = g$, $g \in G$, $x \in I$, $G \in \mathfrak{G}$. Then we have the fibration

$$1 \to H \to G \otimes I \xrightarrow{p} G \to 1$$

and since $i_0(G)$ is a deformation retract of $G \otimes I$ and $pi_0 = 1$, the fiber (kernel) H has trivial homotopy groups. (In fact it is contractible.)

LEMMA 3.1. – *Given simplexes $x_0, x_1, \ldots, x_n$ in H_{n-1} such that $\partial_i x_j = \partial_{j-1} x_i$ for all $i < j$, there exists a simplex $y \in H_n$ such that $\partial_i y = x_i$ for all i.*

PROOF. – Since H satisfies the KAN extension condition, we can find $z \in H$ such that $\partial_i z = x_i^{-1}$ for $i \neq 0$. Then $x_0 \partial_0 z$ satisfies $\partial_i(x_0 \partial_0 z) = e_{n-1}$ for all i. By the vanishing of the motopy groups of H, there exists $w \in H_n$ such that $\partial_0 w = x_0 \partial_0 z$ and $\partial_i w = e_n$ for all $i \neq 0$. We can take $y = wz^{-1}$.

We shall now prove by induction on n the following statement: (a_n) Let i_0, i_1 be canonical imbeddings: $GK \to GK \otimes I$, where K is a reduced simplicial set. Then there exists a μ-homotopy $\mu\colon K_k \to GK \otimes I$ defined for $k \leqslant n$, such that

i) $\mu(K_k) \subset H_k$

ii)
$$\partial_0 \mu(x) = i_0(\tau(x))^{-1} \mu(\partial_0 x)\, i_1(\tau(x))$$
$$\partial_i \mu(x) = \mu(\partial_i x)\,, \qquad i > 0$$
$$s_i \mu(x) = \mu(s_i x)\,, \qquad i \geqslant 0,\ x \in K_{k-1},\ k \leqslant n-1.$$

PROPOSITION 3.2. – (a_n) is true for all $n \geqslant 0$.

Assuming for the moment the validity of 3.2. we obtain

COROLLARY 3.3. – *If two morphisms $f_0, f_1\colon GK \to A$ are loop-homotopic, they are μ-homotopic.*

PROOF. – By hypothesis, there is a morphisms $F\colon GK \otimes I \to A$ such that $f_0 = Fi_0$, $f_1 = Fi_1$. Then the composition $\bar{\mu} = F\mu$, where μ has been constructed in Proposition 3.2. yields the required μ-homotopy between f_0 and f_1.

PROOF OF PROPOSITION 3.2. – For $n = 0$, set $\mu(x) = \mu(k_0) = e_0$ and there is nothing to verify. Suppose that we have proved (a_{n-1}). If $x = s_i y$, $y \in K_{n-1}$ define $\mu(x) = s_i \mu(y)$. Then $\mu(x) \in H_n$ satisfies conditions i) and ii) of (a_n).

If $x \in K_n$ is non-degenerate, consider the elements

$$y_0 = i_0(\tau(x))^{-1}(\partial_0 x)\, i_1(\tau(x))$$
$$y_i = \mu(\partial_i x)\,, \qquad i > 0\,.$$
$$y_0 = i_0(\tau(x))^{-1}(\partial_0 x)\, i_1(\tau(x))$$

$y_i \in H_{n-1}$ satisfy the hypotheses of Lemma 3.1 and we can define $\mu(x) \in H_n$ as the element satisfying $\partial_i(x) = y_i$. This proves (a_n) and concludes the proof of Proposition 3.2.

Lemma 2.2. and Corollary 3.3 yield Theorem 1.3.

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