An Extension of Ezeilo's Result.

ROLF REISSIG (Bochum) (*)

Dedicated to Professor KARL KLOTTER
on his 70th birthday

Summary. – *In a recent paper* [1] *Ezeilo considered the nonlinear third order differential equation* $x''' + \psi(x')x'' + \varphi(x)x' + \theta(x, x', x'') = p(t)$. *He proved the ultimate boundedness of the solutions on rather general conditions for the nonlinear terms* ψ, φ, θ. *These conditions (in a little weaker form) are also sufficient in order to prove the existence of forced oscillations in the case when the excitation is* ω*-periodic. For this purpose the Leray-Schauder principle in a form suggested by G. Güssefeldt* [2] *is applicable.*

We study the differential equation

$$x''' + \psi(x')x'' + \varphi(x)x' + \theta(t, x, x', x'') = p(t) \tag{1}$$

where the functions ψ, φ, θ, p are continuous and ω-periodic with respect to t. According to [1] we define

$$P(t) = \int_0^t p(\tau)\,d\tau\,, \qquad \Phi(x_1) = \int_0^{x_1} \varphi(\xi)\,d\xi\,, \qquad \Psi(x_2) = \int_0^{x_2} \psi(\xi)\,d\xi\,.$$

THEOREM. – Suppose, that

(i) $\Psi(x_2)\operatorname{sgn} x_2 \to +\infty \quad$ as $|x_2| \to \infty$

(ii) $|\theta(t, x_1, \ldots)| \leqq F$ for all values of the independent variables,
$\theta(t, x_1, \ldots)\operatorname{sgn} x_1 \geqq 0 \quad$ for $|x_1| \geqq h$

(iii) $|\Phi(x_1) - bx_1| \leqq M \quad (b > 0$ constant)

(iv) $|P(t)| \leqq m \quad$ for all t $\left(\text{i.e. } \int_0^\omega p(\tau)\,d\tau = 0\right)$.

Then there exists at least one ω-periodic solution of equation (1).

PROOF. – Choosing an arbitrary constant $c \neq 0$ we introduce the following first order system depending on a parameter $\mu \in [0, 1]$:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 + \{(bx_1 - \Phi(x_1, \mu)) - \Psi(x_2, \mu) + \mu P(t)\} \\ x_3' = -cx_1 - bx_2 + \{cx_1 - g(t, x_1, x_2, x_3, \mu)\}\,. \end{cases} \tag{2}$$

(*) Entrata in Redazione il 21 ottobre 1971.

These equations are regarded as components of a vector differential equation

(3) $$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{f}(t, \boldsymbol{x}, \mu); \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \boldsymbol{f} = \begin{pmatrix} 0 \\ bx_1 - \Phi - \Psi + \mu P \\ cx_1 - g \end{pmatrix}.$$

The homogeneous linear equation

(4) $$\boldsymbol{x}' = \boldsymbol{A}\boldsymbol{x}$$

admits only the trivial ω-periodic solution $\boldsymbol{x}(t) \equiv 0$ since the characteristic equation

$$\operatorname{Det}(\lambda \boldsymbol{E} - \boldsymbol{A}) = \lambda^3 + b\lambda + c = 0$$

has no purely imaginary root.

The following denotations are used:

$$\Phi(x_1, \mu) = \frac{\Phi(x_1)}{2-\mu} - \frac{1-\mu}{2-\mu}\Phi(-x_1) = \int_0^{x_1} \varphi(\xi, \mu)\, d\xi,$$

$$\varphi(x_1, \mu) = \frac{\varphi(x_1)}{2-\mu} + \frac{1-\mu}{2-\mu}\varphi(-x_1),$$

$$[\Phi(x_1, 0) = \tfrac{1}{2}\{\Phi(x_1) - \Phi(-x_1)\} = -\Phi(-x_1, 0),\ \Phi(x_1, 1) = \Phi(x_1)],$$

$\Psi(x_2, \mu)$ and $\psi(x_2, \mu)$ are defined in an analogous way,

$$g(t, x_1, x_2, x_3, \mu) = \mu\theta(t, x_1, x_2, x_2') + (1-\mu)F\frac{x_1}{1+|x_1|}$$

$$\left[g(t, x_1, x_2, x_3, 0) = \frac{Fx_1}{1+|x_1|} \text{ odd}\right].$$

Evidently, the succeeding relations are valid:

(5) $$\begin{cases} \Psi(x_2, \mu)\operatorname{sgn} x_2 \to \infty \quad \text{as } |x_2| \to \infty \text{ (uniformly with respect to } \mu \in [0, 1]); \\ |g(t, x_1, x_2, x_3, \mu)| \leqq \mu|\theta| + (1-\mu)F \leqq F \quad \text{for } \mu \in [0, 1]; \\ g(\ldots)\operatorname{sgn} x_1 \geqq (1-\mu)F\dfrac{|x_1|}{1+|x_1|} > 0\,(|x_1| \geqq h) \quad \text{for } \mu \in [0, 1); \\ |\Phi(x_1, \mu) - bx_1| \leqq M \text{ by virtue of } x_1 = \dfrac{x_1}{2-\mu} - \dfrac{1-\mu}{2-\mu}(-x_1). \end{cases}$$

System (2) respectively (3) is equivalent to the third order differential equation

(6) $$x''' + \psi(x', \mu)x'' + \varphi(x, \mu)x' + g(t, x, \ldots) = \mu p(t), \qquad x = x_1$$

which is identical with the original equation (1) if $\mu = 1$.

For $\mu = 0$ the nonlinear terms Ψ, Φ, g are odd with respect to x_1, x_2, x_3 (the last component does not occur); that is

$$\boldsymbol{f}(t, -\boldsymbol{x}, 0) = -\boldsymbol{f}(t, \boldsymbol{x}, 0)$$

where $\boldsymbol{f}(t, \boldsymbol{x}, \mu)$ is the nonlinear perturbance term in equation (3). This term is a continuous function of

$$(t, \boldsymbol{x}, \mu) \in [0, \omega] \times R^3 \times [0, 1] .$$

For $0 \leqq t \leqq \omega$ the ω-periodic solutions $\boldsymbol{x}(t)$ of equation (3) can be represented as the continuous solutions of an integral equation of Hammerstein type:

$$\boldsymbol{x}(t) = T\{\boldsymbol{x}(t), \mu\} = \int_0^\omega \boldsymbol{G}(t-\tau)\boldsymbol{f}(\tau, \boldsymbol{x}(\tau), \mu)\, d\tau \tag{7}$$

or shortly

$$V_\mu \boldsymbol{x} \equiv \boldsymbol{x}(t) - T\{\boldsymbol{x}(t), \mu\} = 0 .$$

The Green matrix $\boldsymbol{G}(s)$, $s \in [-\omega, +\omega]$ is continuous for $s \neq 0$, and it fulfils the jump condition

$$\boldsymbol{G}(+0) - \boldsymbol{G}(-0) = \boldsymbol{E} .$$

Let B be the Banach space of continuous vectors $\boldsymbol{x}(t)$, $0 \leqq t \leqq \omega$ with the boundary condition $\boldsymbol{x}(0) = \boldsymbol{x}(\omega)$ and normed by virtue of

$$\|\ldots\| = \sup_{[0,\omega]} |\boldsymbol{x}(t)| .$$

The operator T generates a continuous and compact (i.e. completely continuous) mapping of the normed product space $B \times [0, 1]$ into the Banach space B. The periodic solutions of (3) correspond to the fixed points of T or to the null vectors of V_μ.

Let

$$S_R = \{\boldsymbol{x}(t) \in B \colon \|\boldsymbol{x}(t)\| = R\} , \qquad B_R^* = \{\boldsymbol{x}(t) \in B \colon \|\boldsymbol{x}(t)\| < R\} .$$

On the sphere S_R we study the vector field V_μ $\big(V_\mu \boldsymbol{x} = \boldsymbol{x} - T\{\boldsymbol{x}, \mu\}\big)$ for every fixed $\mu \in [0, 1]$. If there is no null vector on S_R the vector fields V_0 and V_1 are called homotopic. The rotations of homotopic vector fields are identical. Since V_0 is an odd vector field,

$$V_0(-\boldsymbol{x}) = -\boldsymbol{x}(t) - \int_0^\omega \boldsymbol{G}(t-\tau)\boldsymbol{f}(\tau, -\boldsymbol{x}(\tau), 0)\, d\tau = -V_0 \boldsymbol{x} ,$$

its rotation is an odd number. The rotation of V_0 on the sphere S_R being different from zero the ball B_R contains at least one null vector of V_0. This result is also true of all vector fields V_μ $(0 \leqq \mu \leqq 1)$. Consequently, for every parameter value system (2) admits at least one ω-periodic solution $\boldsymbol{x}(t)$ the norm of which is less than R. (For further details see [2] and [3].)

If we can show that the solutions of integral equation (7) on the parameter interval $0 \leqq \mu < 1$ are a priori bounded $(\|x(t)\| < R$, R a uniform bound) then only two alternative possibilities exist:

a) for $\mu = 1$ there is at least one solution $\boldsymbol{x}(t)$ with norm R of integral equation (7);

b) such a solution does not occur, the vector field V_μ $(0 \leqq \mu \leqq 1)$ has no null vector on the sphere S_R, but at least one null vector within B_R.

At any case the asserted theorem is proved, the differential equation (1) has a periodic solution bounded by R.

For the purpose of an a priori estimate we consider a periodic solution of system (2) where $0 \leqq \mu < 1$. At first we study the components $x_2(t)$, $x_3(t)$ with the aid of the positive auxiliary function

$$W(x_2, x_3) = \frac{1}{2}(bx_2^2 + x_3^2) - F(|x_2| + |x_3| - ||x_2| - |x_3||)\,\mathrm{sgn}\,(x_2x_3) + \frac{1+b}{b}\,2F^2$$

tending to infinity as $|x_2| + |x_3| \to \infty$, cp. [4].

Let W' be the total time derivative by virtue of (2); evaluating this expression (which is depending on t, x_1, x_2, x_3, μ) we could easily show that

$$W' \leqq -1 \qquad \text{if } \operatorname{Max}(|x_2| - H_2, |x_3| - H_3) \geqq 0$$

this domain being the same for all μ. The calculations are omitted because they are similar to those carried out in [4].

Now, following Liapunov's second method we should obtain the estimates

$$|x_2(t)| < B_2\,, \qquad |x_3(t)| < B_3 \tag{8}$$

where the bounds B_2, B_3 are determined by the system characteristics (b, m, M, F and the properties of Ψ). Of course, they do not depend on μ.

In order to obtain some information about the component $x_1(t)$ we integrate the differential equation (6) from $t = 0$ to $t = \omega$:

$$\int_0^\omega g(t, x_1(t), x_2(t), x_3(t), \mu)\,dt = 0\,.$$

This equation contradicts to relation (5) if $|x_1(t)| \geqq h$ for all t. Consequently,

$$|x_1(\tau)| < h \qquad \text{for a value } \tau \in (0, \omega)\,.$$

On the interval $\tau \leqq t \leqq \tau + \omega$ we have

$$|x_1(t) - x_1(\tau)| = (t-\tau)|x_2(\tau + \vartheta(t-\tau))| < \omega B_2 ,$$

(9) $$|x_1(t)| < B_1 = h + \omega B_2 .$$

By virtue of periodicity this estimate must hold for all t.

Finally, introducing $R = \sqrt{B_1^2 + B_2^2 + B_3^2}$ we realise the alternative statements *a*) and *b*) on which the proof of our theorem is based.

REMARK 1. – The theorem remains true if we assume

$$\theta(t, x_1, \ldots)\,\mathrm{sgn}\,x_1 \leqq 0 \text{ (instead of } \geqq 0) \qquad \text{for } |x_1| \geqq h .$$

In this case we should define

$$g = \mu\theta - (1-\mu)F\frac{x_1}{1+|x_1|}$$

preserving the previous argumentation.

REMARK 2. – The theorem remains true if we assume

$$\Psi(x_2)\,\mathrm{sgn}\,x_2 \to -\infty \text{ (instead of } +\infty) \qquad \text{as } |x_2| \to \infty .$$

Namely, introducing the new independent variable $s = -t$ and denoting $\dot{x} = dx/ds$ we derive from (1)

$$\dddot{x} - \psi(-\dot{x})\ddot{x} + \varphi(x)\dot{x} - \theta(-s, x, -\dot{x}, \ddot{x}) = -p(-s)$$

written as

(10) $$\dddot{x} + \bar{\psi}(\dot{x})\ddot{x} + \bar{\varphi}(x)\dot{x} + \bar{\theta}(s, x, \dot{x}, \ddot{x}) = \bar{p}(s) .$$

We obtain

$$\bar{\Psi}(x_2) = \int_0^{x_2}\bar{\psi}(\xi)\,d\xi = -\int_0^{x_2}\psi(-\xi)\,d\xi = \Psi(-x_2) ,$$

$$\bar{\Psi}(x_2)\,\mathrm{sgn}\,x_2 = \Psi(-x_2)\,\mathrm{sgn}\,x_2 \to +\infty \qquad (|x_2| \to \infty)$$

and

$$\bar{\theta}(s, x_1, \ldots)\,\mathrm{sgn}\,x_1 = -\theta(-s, x_1, \ldots)\,\mathrm{sgn}\,x_1 \leqq 0 \qquad (|x_1| \geqq h) .$$

According to the preceding remark equation (10) admits a periodic solution $x(s)$. Thus, equation (1) possesses the periodic solution

$$x(-t)\big[x(-t+\omega) \equiv x(-t) \text{ or } x(-t) \equiv x(-(t+\omega))\big] .$$

If the function θ in equation (1) only depends on x the theorem can be completed by the following

Appendix 1.

Differential equation (1) possesses a periodic solution, too, if

(i) $\theta(x_1)\,\mathrm{sgn}\, x_1 \geqq 0 \qquad (|x_1| \geqq h)$

(ii) $|\Phi(x_1) - bx_1| \leqq M \qquad (b \leqq 0$ constant)

(iii) $|P(t)| \leqq m$.

PROOF. – In equation (6) the function $g = g(x, \mu)$ is constructed by means of an arbitrary positive number F since $\theta(x)$ must not be bounded. Multiplying this equation by x' and integrating from 0 to ω we obtain the result:

$$\left[x''x' + \int_0^{x'} \psi(\eta, \mu)\eta\, d\eta + x' \int_0^{x} \varphi(\xi, \mu)\, d\xi + \int_0^{x} g(\xi, \mu)\, d\xi - \mu x' P(t)\right]_0^{\omega} - \\ - \int_0^{\omega} x''\big(x'' + \Phi(x, \mu) - \mu P(t)\big)\, dt = 0.$$

Assuming that $x(t)$ is a periodic solution the result will be simplified:

$$\int_0^{\omega} x''\Big(x'' + bx + \big(\Phi(x, \mu) - bx\big) - \mu P(t)\Big)\, dt = 0 .$$

Because of

$$|\Phi(x, \mu) - bx| \leqq M , \qquad |P(t)| \leqq m$$

and

$$\int_0^{\omega} x'' x\, dt = -\int_0^{\omega} x'^2\, dt$$

we have the equation

$$\int_0^{\omega} x''^2\, dt - b\int_0^{\omega} x'^2\, dt + \int_0^{\omega} x''\big\{\big(\Phi(x, \mu) - bx\big) - \mu P(t)\big\}\, dt = 0$$

from which we conclude:

$$\int_0^{\omega} x''^2\, dt \leqq \omega(m + M)^2.$$

$x(t)$ being periodic the derivative $x'(t)$ must have a zero on $[0, \omega]$, say τ. Then

$$x'(t) = \int_{\tau}^{t} x''(s)\, ds , \qquad \tau \leqq t \leqq \tau + \omega$$

and

(11) $$|x'(t)| \leqq \omega(m + M) \qquad \text{for all } t\,.$$

Now, following the same conclusion as above (applying condition (i)) we should estimate

(12) $$|x(t)| < h + \omega^2(m + M) \qquad \text{if } \mu \in [0, 1)\,.$$

The estimates (11) and (12) are already sufficient in order to complete the proof. Namely, the third component of the vector integral equation (7) can be omitted since the nonlinear term f only depends on the components $x_1 = x$ and $x_2 = x'$.

Evidently, condition (i) of the Appendix can be modified into

$$\theta(x_1)\,\mathrm{sgn}\,x_1 \leqq 0 \qquad (|x_1| \geqq h)\,.$$

EXAMPLE. – The equation

(13) $$x''' = p(t)\,, \qquad \int_0^\omega p(t)\,dt = 0\,,$$

fulfils the conditions of the appendic $(\psi(x') \equiv 0,\ \varphi(x) \equiv 0$ and $b = 0,\ \theta(x) \equiv 0)$.

Let (as above)

$$P(t) = \int_0^t p(\tau)\,d\tau \qquad (\omega\text{-periodic});$$

moreover, let

$$\int_0^t P(\tau)\,d\tau = P_1(t) + ct$$

$$\left[P_1(t + \omega) \equiv P_1(t)\,, \qquad \text{i.e.} \quad c = \frac{1}{\omega}\int_0^\omega P(t)\,dt\right]$$

and

$$\int_0^t P_1(\tau)\,d\tau = P_2(t) + c_1 t$$

$$\left[P_2(t + \omega) \equiv P_2(t), \qquad \text{i.e.} \quad c_1 = \frac{1}{\omega}\int_0^\omega P_1(t)\,dt\right].$$

Then

$$\int_0^t\left(\int_0^{\tau_1}\left(\int_0^{\tau_2} p(\tau_3)\,d\tau_3\right)d\tau_2\right)d\tau_1 = P_2(t) + c_1 t + \frac{c}{2}t^2$$

and equation (13) possesses the general solution

$$x(t) = a + a_1 t + a_2 t^2 + P_2(t) + c_1 t + \frac{c}{2} t^2$$

(a, a_1, a_2 arbitrary constant values).

Choosing

$$a_1 = -c_1 , \qquad a_2 = -\frac{c}{2}$$

we obtain a family of ω-periodic solutions:

$$x(t) = a + P_2(t) .$$

In two further special cases based on the assumption $\theta = \theta(x)$ the conditions of the general theorem can be weakened considerably, too.

Appendix 2.

Equation (1) admits a periodic solution if

(i) $\theta(x_1)\,\mathrm{sgn}\,x_1 \geqq 0 \;\; (\leqq 0) \qquad \text{for } |x_1| \geqq h$

(ii) $|\Phi(x_1) - b x_1| \leqq M \qquad \left(0 < b < \dfrac{4\pi^2}{\omega^2}\right)$

(iii) $|P(t)| \leqq m$.

PROOF. – Again, assuming $0 \leqq \mu < 1$ and $x(t) \equiv x(t+\omega)$ we multiply equation (6) by x' and integrate from 0 to ω:

$$\int_0^\omega x''^2 dt - b\int_0^\omega x'^2 dt + \int_0^\omega x''\{(\Phi(x,\mu) - bx) - \mu P(t)\}\, dt = 0 . \tag{14}$$

Let k be any positive integer and

$$a_k'' = \frac{2}{\omega}\int_0^\omega x''(t)\cos\frac{2\pi k}{\omega} t\, dt = \frac{4\pi k}{\omega^2}\int_0^\omega x'(t)\sin\frac{2\pi k}{\omega} t\, dt = \frac{2\pi k}{\omega} b_k' ,$$

$$b_k'' = \frac{2}{\omega}\int_0^\omega x''(t)\sin\frac{2\pi k}{\omega} t\, dt = -\frac{4\pi k}{\omega^2}\int_0^\omega x'(t)\cos\frac{2\pi k}{\omega} t\, dt = -\frac{2\pi k}{\omega} a_k' .$$

Applying the PARSEVAL equations for x'' and x' we obtain the following estimate:

$$\int_0^\omega x''^2 dt = \frac{\omega}{2}\sum_{k=1}^{\infty}(a_k''^2 + b_k''^2) = \frac{4\pi^2}{\omega^2}\,\frac{\omega}{2}\sum_{k=1}^{\infty} k^2(a_k'^2 + b_k'^2) \geqq \frac{4\pi^2}{\omega^2}\int_0^\omega x'^2 dt . \tag{15}$$

Now, we derive from (14):

$$\left(1-\frac{b\omega^2}{4\pi^2}\right)\int_0^\omega x''^2\,dt \leqq \int_0^\omega x''^2\,dt - b\int_0^\omega x'^2\,dt \leqq \sqrt{\omega}(M+m)\sqrt{\int_0^\omega x''^2\,dt}\,,$$

$$\sqrt{\int_0^\omega x''^2\,dt} \leqq \frac{4\pi^2(M+m)}{4\pi^2-b\omega^2}\sqrt{\omega}\,.$$

On the basis of this result the proof can be completed by means of the same consideration as above (Appendix 1).

The last Appendix refers to the special case

$$\theta=\theta(x)\,, \qquad \psi(x')=a\neq 0\,.$$

Appendix 3.

The equation

$$x'''+ax''+\varphi(x)x'+\theta(x)=p(t) \tag{16}$$

has a periodic solution if

(i) $\dfrac{\theta(x_1)}{x_1}\to 0 \quad (|x_1|\to\infty), \qquad \theta(x_1)\operatorname{sgn} x_1 \geqq 0 \;\; (\leqq 0) \qquad \text{for } |x_1|\geqq h$

(ii) $|P(t)|\leqq m\,.$

PROOF. – This time we define the auxiliary system (3) in the following way:

$$A=\begin{pmatrix} 0 & 1 & 0\\ 0 & -a & 1\\ -c & 0 & 0\end{pmatrix}, \qquad f=\begin{pmatrix} 0\\ -\Phi+\mu P\\ cx_1-g\end{pmatrix}.$$

It is evident that the characteristic equation of the linear part $x'=Ax$,

$$\operatorname{Det}(\lambda E-A)=\lambda^3+a\lambda^2+c=0\,,$$

has no purely imaginary root.

The functions $\Phi(x_1,\mu)$ and $g(x_1,\mu)$ are introduced as above. The constant F (in the definition of g) is arbitrarily chosen (as in the proofs of the foregoing appendices).

We consider the third order differential equation which is equivalent to (3):

$$x'''+ax''+\varphi(x,\mu)x'+g(x,\mu)=\mu p(t)\,, \qquad x=x_1\,. \tag{17}$$

Let $x=x(t)$ be a periodic solution and let $0\leqq\mu<1$.

Multiplying this equation by x and integrating from 0 to ω we obtain

$$\int_0^\omega (\mu P - a x')x'\,dt + \int_0^\omega g(x,\mu)\,x\,dt = 0 \tag{18}$$

whence we conclude

$$|a|\int_0^\omega x'^2\,dt \leqq m\sqrt{\omega}\sqrt{\int_0^\omega x'^2\,dt} + G\sqrt{\omega}\sqrt{\int_0^\omega x^2\,dt}$$

if we set

$$G = G(R) = F + \sup_{|x|\leq R}|\theta(x)|\,, \qquad R = \sup_{[0,\omega]}|x(t)|\,.$$

A further calculation yields:

$$\sqrt{\int_0^\omega x'^2\,dt} \leqq \frac{m\sqrt{\omega}}{2|a|}\left(1 + \sqrt{1 + \frac{4G|a|}{\sqrt{\omega}m^2}\sqrt{\int_0^\omega x^2\,dt}}\right) \leqq \frac{m\sqrt{\omega}}{|a|} + \sqrt{\frac{\omega}{|a|}}\sqrt{GR}\,. \tag{19}$$

Taking into account that the nonlinear perturbance term f in equation (3) depends on t, x_1 and μ we need only an a priori estimate of $x = x_1$ in order to complete the proof.

Let τ be such that $|x(\tau)| < h$. Then we have for $\tau \leqq t \leqq \tau + \omega$

$$|x(t) - x(\tau)| \leqq \int_\tau^t |x'(s)|\,ds \leqq \sqrt{\omega}\sqrt{\int_\tau^{\tau+\omega} x'^2(s)\,ds}\,,$$

hence

$$R < h + \sqrt{\omega}\sqrt{\int_0^\omega x'^2\,ds}\,.$$

Now, we obtain with the aid of the estimate (19):

$$1 < \left(h + \frac{m\omega}{|a|}\right)\frac{1}{R} + \frac{\omega}{\sqrt{|a|}}\sqrt{\frac{G(R)}{R}}\,. \tag{20}$$

We could easily show that condition (i) implies:

$$\frac{G(R)}{R} = \frac{F}{R} + \frac{1}{R}\sup_{|x|\leq R}|\theta(x)| \to 0 \qquad \text{as } R \to \infty\,.$$

Thus, we derive from (20):

$$\sup_{[0,\omega]} |x(t)| = R \leqq R_0 \qquad \text{(only dependent on } a,\ m,\ \omega,\ h)$$

as required.

BIBLIOGRAPHY

[1] J. O. C. EZEILO, *A generalization of a theorem of Reissig for a certain third order differential equation*, Ann. Mat. Pura Appl., **87** (1970), pp. 349-356.

[2] G. GÜSSEFELDT, *Zwei Methoden zum Nachweis von periodischen Lösungen bei gewöhnlichen Differentialgleichungen*, Math. Nachr., **48** (1971), pp. 141-151.

[3] M. A. KRASNOSELSKII, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, Oxford, 1964.

[4] R. REISSIG, *Ein Beschränktheitssatz für gewisse nichtlineare Differentialgleichungen dritter Ordnung*, Monatsber. Deutsch. Akad. Wiss. Berlin, **6** (1964), pp. 481-484.

[5] R. REISSIG - G. SANSONE - R. CONTI, *Nichtlineare Differentialgleichungen höherer Ordnung*, Edizioni Cremonese, Roma, 1969.

[6] G. VILLARI, *Soluzioni periodiche di una classe di equazioni differenziali del terz'ordine quasi lineari*, Ann. Mat. Pura Appl., **73** (1966), pp. 103-110.

[7] G. VILLARI, *Einige Bemerkungen über periodische Lösungen*. Vortrag auf der Tagung « Gewöhnliche Differentialgleichungen », Oberwolfach 19.-25.3.1972.