Periodic Solutions of a Third Order Nonlinear Differential Equation.

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Dedicated to Professor WOLFGANG HAHN
on his 60th birthday

Summary. – *The differential equation* $x''' + \psi(x')x'' + \varphi(x)x' + f(x) = p(t)$ *is considered where the forcing term* p *is an* ω*-periodic function of* t. *In the special cases* $\varphi(x) = k^2$ *respectively* $\psi(x') = a$ *the existence of periodic solutions is proved on the basis of the Leray-Schauder fixed point technique. The conditions imposed on the nonlinear terms do not include the ultimate boundedness of all solutions.*

It is obvious that the linear differential equation

$$x''' + k^2 x' = p(t)\,, \qquad p(t+\omega) \equiv p(t) \tag{1}$$

possesses ω-periodic solutions if the conditions

$$0 < k \neq \frac{2n\pi}{\omega} \qquad (n \text{ positive integer})$$

$$\int_0^\omega p(t)\,dt = 0$$

are satisfied.

These conditions play a role, too, for a certain class of nonlinear equations which comprises the special case (1):

$$x''' + \psi(x')x'' + k^2 x' + f(x) = p(t)\,, \qquad p(t+\omega) \equiv p(t)\,, \tag{2}$$

$$[\psi(y),\ f(x),\ p(t) \text{ continuous functions}]\,.$$

The following theorem is valid:

THEOREM 1. – Differential equation (2) admits at least one ω-periodic solution if

(i) $\quad 0 < k \neq \dfrac{2n\pi}{\omega} \qquad (n = 1, 2, 3, \ldots)$

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(ii) $\int_0^\omega p(t)\,dt = 0$ $\left[\text{that is, } P(t) = \int_0^t p(\tau)\,d\tau \ \omega\text{-periodic}\right]$

(iii) $|\Psi(y)| \leqq G\left[\Psi(y) = \int_0^y \psi(\eta)\,d\eta\right]$

(iv$_1$) $\dfrac{|f(x)|}{|x|} \to 0 \qquad (|x| \to \infty)$

(iv$_2$) $f(x)\,\mathrm{sgn}\,x \geqq 0 \qquad (|x| \geqq h)$.

The proof by means of the LERAY-SCHAUDER method is simple. We consider a differential equation containing the parameter μ, $0 \leqq \mu \leqq 1$:

$$x''' + k^2 x' + cx = \mu\{p(t) - f(x) + cx - \psi(x')x''\} \tag{3}$$

where c is an arbitrary positive constant. For $\mu = 0$ we obtain a homogeneous linear equation the only ω-periodic solution of which is the trivial one; for $\mu = 1$ equation (3) is identical with the original one (2). It is a well-known fact (cp. [2], [5], [6]) that equation (3) admits at least one periodic solution for each parameter value $\mu \in [0, 1]$ if for $0 < \mu < 1$ all periodic solutions as well as their derivatives of the first and second order are uniformly bounded. Consequently the stated theorem can be proved with the aid of an a priori estimate.

Let $x(t) \equiv x(t + \omega)$ be a solution of equation (3) and let $0 < \mu < 1$. We denote

$$R = \operatorname*{Max}_{[0,\omega]} |x(t)| , \qquad F = F(R) = \operatorname*{Max}_{|x| \leqslant R} |f(x)| .$$

The derivative $y = x'$ fulfils the equation

$$y'' + k^2 y = \mu\{p(t) - f(x(t)) - \psi(y(t))y'(t)\} - (1 - \mu)\,cx(t)$$

which can be considered as a nonhomogeneous linear one:

$$y'' + k^2 y = q(t) , \qquad q(t + \omega) \equiv q(t) . \tag{4}$$

Introducing the GREEN matrix

$$G(t - \tau) = \begin{pmatrix} g_{11}(t - \tau) & g_{12}(t - \tau) \\ g_{21}(t - \tau) & g_{22}(t - \tau) \end{pmatrix}$$

$$[g_{21} = g'_{11},\ g_{22} = g'_{12};\quad g'_{21} = -k^2 g_{11},\ g'_{22} = -k^2 g_{12}]$$

of the boundary value problem

$$y' = z , \qquad z' = -k^2 y + q(t) \qquad \text{for } 0 \leqq t \leqq \omega; \tag{5}$$

$$y(0) = y(\omega) , \qquad z(0) = z(\omega)$$

we obtain the following representation of the solution $y(t)$, $z(t)$:

(6) $$y(t)=\int_0^\omega g_{12}(t-\tau)q(\tau)\,d\tau\,, \qquad z(t)=\int_0^\omega g_{22}(t-\tau)q(\tau)\,d\tau$$

where

$$g_{12}(t-\tau)=\frac{\sin k|t-\tau|+\sin k(\omega-|t-\tau|)}{2k(1-\cos k\omega)}\,, \qquad 0\leqq t,\ \tau\leqq\omega$$

and

$$g_{22}(t-\tau)=\begin{cases}\dfrac{\cos k(t-\tau)-\cos k(\omega-t+\tau)}{2(1-\cos k\omega)}\,, & 0\leqq\tau<t\\[2mm] \dfrac{\cos k(\omega-\tau+t)-\cos k(\tau-t)}{2(1-\cos k\omega)}\,, & t<\tau\leqq\omega\end{cases}$$

$$[g_{22}(+0)-g_{22}(-0)=1]\,.$$

Replacing $q(t)$ by the term $\psi(y)y'=(d/dt)\Psi(y)$ which occurs in the expression for $q(t)$ we obtain instead of (6)

$$\int_0^\omega g_{12}(t-\tau)\frac{d}{d\tau}\Psi(y(\tau))\,d\tau=[g_{12}(t-\tau)\Psi(y(\tau))]_0^\omega+ \\ +\int_0^\omega g_{22}(t-\tau)\Psi(y(\tau))\,d\tau=\int_0^\omega g_{22}(t-\tau)\Psi(y(\tau))\,d\tau\,,$$

$$\int_0^\omega g_{22}(t-\tau)\frac{d}{d\tau}\Psi(y(\tau))\,d\tau=[g_{22}(t-\tau)\Psi(y(\tau))]_0^{t-0}+[g_{22}(t-\tau)\Psi(y(\tau))]_{t+0}^\omega- \\ -k^2\int_0^\omega g_{12}(t-\tau)\Psi(y(\tau))\,d\tau=\Psi(y(t))-k^2\int_0^\omega g_{12}(t-\tau)\Psi(y(\tau))\,d\tau\,.$$

Inserting the explicit expression for $q(t)$ in equations (6) and having regard to the preceding transformations we derive estimates of the type

$$|y(t)|\leqq\varrho(m+F+G+cR)\,,$$
$$|z(t)|=|y'(t)|\leqq\sigma(m+F+G+cR)$$

where ϱ and σ are determined by the Green matrix and therefore only depend on k and ω.

Now a term by term integration of differential equation (3) (for the periodic solution) yields:

$$[x''(t)+\mu\Psi(x'(t))+k^2x(t)-\mu P(t)]_0^\omega+\int_0^\omega\{(1-\mu)cx(t)+\mu f(x(t))\}\,dt=0\,,$$

i.e.

$$\int_0^\omega f_\mu(x(t))\,dt = 0\,, \qquad f_\mu(x) = (1-\mu)cx + \mu f(x)\,.$$

Because of $1-\mu > 0$ we have

$$f_\mu(x)\operatorname{sgn} x = (1-\mu)c|x| + \mu f(x)\operatorname{sgn} x > 0 \qquad \text{for } |x| \geqq h\,.$$

Consequently, $|x(t)| \geqq h$ for $0 \leqq t \leqq \omega$ is excluded, we have $|x(\tau)| < h$ for some $\tau \in (0, \omega)$.

Applying the mean value theorem to an arbitrary interval $[\tau, t] \subset [\tau, \tau+\omega]$ we find

$$\begin{aligned} |x(t) - x(\tau)| &= (t-\tau)\big|y\big(\tau + \vartheta(t-\tau)\big)\big| \\ &\leqq \omega\varrho(m + F + G + cR)\,, \end{aligned}$$

$$|x(t)| < h + \omega\varrho(m + F + G + cR)\,.$$

It is obvious that this estimate must be valid for all t; hence

$$\operatorname*{Max}_{[0,\omega]}|x(t)| = R < h + \omega\varrho(m + F + G + cR)\,.$$

Choosing $0 < c < 1/\omega\varrho$ we obtain

$$1 < \frac{h + \omega\varrho(m+G)}{1-\omega\varrho c}\,\frac{1}{R} + \frac{\omega\varrho}{1-\omega\varrho c}\,\frac{F(R)}{R}\,. \tag{7}$$

An immediate consequence of assumption (iv_1) is

$$\frac{F(R)}{R} \to 0 \qquad (R \to \infty)\,.$$

Therefore we conclude from (7):

$$R = \operatorname*{Max}_{[0,\omega]}|x(t)| \leqq R_0 \qquad \text{(independent on } \mu)\,,$$

$$F(R) = \operatorname*{Max}_{|x|\leqq R}|f(x)| \leqq F_0 = \operatorname*{Max}_{|x|\leqq R_0}|f(x)|\,.$$

The resulting a priori estimates

$$|x(t)| \leqq R_0\,, \quad |y(t)| \leqq \varrho(m + F_0 + G + cR_0)\,, \quad |z(t)| \leqq \sigma(m + F_0 + G + cR_0)$$

ensure the existence of a periodic solution of equation (2) as we stated in our theorem.

REMARK. – In the case

(iv_3) $\quad f(x)\operatorname{sgn} x \leqq 0 \qquad (|x| \geqq h)$

we introduce a new independent variable

$$\tau = -t$$

and obtain a differential equation of the previous type. Thus, Theorem 1. remains valid if assumption (iv_2) is replaced by (iv_3).

By a similar procedure the following theorem can be proved:

THEOREM 2. – The differential equation

$$x''' + ax'' + \varphi(x)x' + f(x) = p(t) \equiv p(t+\omega) \tag{8}$$

$$[\varphi(x),\ f(x),\ p(t) \text{ continuous functions}]$$

admits at least one ω-periodic solution if

(i) $a \neq 0$

(ii) $\int_0^\omega p(t)\,dt = 0$

(iii) $\lim_{|x|\to\infty} \frac{|\Phi(x)|}{|x|} = 0 \qquad \left[\Phi(x) = \int_0^x \varphi(\xi)\,d\xi\right]$

(iv) $\lim_{|x|\to\infty} \frac{|f(x)|}{|x|} = 0$

$f(x)\operatorname{sgn} x \geqq 0 \qquad (|x| \geqq h)$.

Again the proof is based on an a priori estimate of the ω-periodic solutions of the system

$$x' = y\,, \qquad y' = z\,, \qquad z' + az + cx = \mu\{p(t) - \varphi(x)y - f(x) + cx\} \tag{9}$$

$$(0 < \mu < 1,\ c > 0 \text{ properly chosen})\,.$$

Considering a periodic solution and writing the last equation (9) in the form

$$z' + az = q(t)$$

the component $z(t)$ can be represented as

$$z(t) = (1 - e^{-a\omega})^{-1}\int_0^t \exp\big(-a(t-\tau)\big)q(\tau)\,d\tau + (e^{a\omega} - 1)^{-1}\int_t^\omega \exp\big(-a(t-\tau)\big)q(\tau)\,d\tau\,. \tag{10}$$

Replacing $q(t)$ by $\varphi(x)x'$ we obtain by virtue of partial integration the term

$$\Phi(x(t)) - \frac{a}{1-\exp(-a\omega)}\int_0^t \exp\big(-a(t-\tau)\big)\Phi(x(\tau))\,d\tau$$

$$- \frac{a\exp(-a\omega)}{1-\exp(-a\omega)}\int_t^\omega \exp\big(-a(t-\tau)\big)\Phi(x(\tau))\,d\tau\,.$$

Inserting the expression for $q(t)$ in formula (10) and applying the preceding result we can derive an estimate of the type

$$|z(t)| \leqq \sigma(m + G + F + cR)\,, \qquad \sigma = \sigma(a, \omega) \tag{11}$$

where

$$R = \underset{[0,\omega]}{\mathrm{Max}} |x(t)|\,,$$

$$G = \underset{|x| \leqslant R}{\mathrm{Max}} |\Phi(x)|\,,$$

$$F = \underset{|x| \leqslant R}{\mathrm{Max}} |f(x)|\,.$$

Since the trivial case $p(t) \equiv 0$ is excluded we must have

$$\mathrm{Max}\, y(t) > 0\,, \qquad \mathrm{Min}\, y(t) < 0\,.$$

The application of the mean value theorem yields the result

$$\mathrm{Max}\, y(t) - \mathrm{Min}\, y(t) < \omega\sigma(m + G + F + cR)$$

from which we conclude:

$$|y(t)| \leqq \varrho(m + G + F + cR)\,, \qquad \varrho = \omega\sigma\,. \tag{12}$$

Following the same argumentation as above we finally attain to the required boundedness result.

The previous remark concerning the function $f(x)$ is true again.

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