On Representation of Functions as Convolution Transforms (*).

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Summary. — *Necessary and sufficient conditions on $f(x)$ to be a convolution-Stieltjes transform of $\alpha(t)$ where $\alpha(t)$ is such that the transform converges absolutely, are given. The above result is given for Convolution transforms with variation diminishing kernels of class I.*

Introduction.

For the convolution transform

$$(1.1) \quad f(x) = \int_{-\infty}^{\infty} G(x-t) d\alpha(t)$$

the representation problem is that of establishing necessary and sufficient conditions on f , the generating function, so that $\alpha(t)$, the determining function, belongs to a certain class of functions.

In this paper we shall be interested in the convolution transform with variation diminishing kernel $G(t)$. In fact, we shall restrict ourselves to kernels satisfying

$$(1.2) \quad \int_{-\infty}^{\infty} G(t) dt = \left[\exp [bs] \prod_{k=1}^{\infty} \left(1 - \frac{s}{a_k} \right) \exp (s/a_k) \right]^{-1} = [E_G(s)]^{-1}$$

where b, a_k are real and $\sum a_k^{-2} < \infty$. (For the most general variation diminishing kernel the right side is multiplied by $\exp (cs^2)$ for some $c \geq 0$.)

Representation theorems for the transforms defined by (1.1) and (1.2) were investigated by I. I. HIRSCHMAN and D. V. WIDDER in several papers which are summarized in [3, Ch. VII]. Additional results in this direction were obtained recently by G. V. BADALAJAN [1, p. 485] and by D. LEVIATAN [4]. In these research works the following cases were investigated: (1) $\alpha(t) \in B \cdot V(-\infty, \infty)$ [3, Ch. VII] and [4] (in the later different conditions in case $G(t) \in$ class II are given); (2) $\alpha(t)$ increasing [3, Ch. VII]; (3) $\alpha(t)$ increasing and $\alpha(t) \in B \cdot V(-\infty, \infty)$ [1, p. 485]; (4) $\alpha(t) = \int_t^{\infty} \varphi(u) du$ where $\varphi(t) \in L_p(-\infty, \infty)$ $1 < p \leq \infty$ [3, Ch. VII]; (5) $\alpha(c) = \int_c^{\infty} \varphi(u) du$ and $\varphi \in L_M(-\infty, \infty)$ where $L_M(-\infty, \infty)$ is an Orlicz class and $G(x) \in$ class II [4].

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Except in case (2) no allowance is made for the possibility that $\alpha(t)$ may behave quite badly for large $|t|$ and case (2) is obviously very restricted.

The results of this paper will treat this question for $G(t) \in \text{class I}$, that is, when $E_G(S)$ has both positive and negative zeros of $(E_G(S))$ defined by (1.2). A necessary and sufficient condition for $f(x)$ to be given by an absolutely convergent integral in (1.1), and a result for $\varphi \in L^-$ with $G(t)$ as weight will be described in sections 3 and proved later. An interesting result about the dependence of the convergence of (1.1) for a specific $\alpha(t)$ only on the zeros of $E(S)$ nearest to the origin will be obtained in section 2 and used many times though not in its full generality for the representation theorems.

2. – Convergence Theorem.

For stating the theorems of this section we have to define:

$$(2.1) \quad \alpha_1(G) = \max_{a_k < 0} [a_k, -\infty] \quad \alpha_2(G) = \min_{a_k > 0} [a_k, \infty]$$

where a_k and G are related by (1.2). Define $\alpha_1(G) + 1$ and $\alpha_2(G) + 1$ as the multiplicity of $\mu_1(G)$ and $\mu_2(G)$ in (1.2) respectively. Recall that: $G \in \text{class I}$ if both $\alpha_1(G)$ and $\alpha_2(G)$ are finite.

THEOREM 2.1. – Assume: (1) $G \in \text{class I}$; (2) G_2 satisfies (1.2); (3) $\alpha_1(G_1) \geq \alpha_1(G_2)$ and if the equality holds $\mu_1(G_1) \geq \mu_1(G_2)$; (4) $\alpha_2(G_1) \leq \alpha_2(G_2)$ and if the equality holds $\mu_2(G_1) \geq \mu_2(G_2)$. Then, for each function $\alpha(t)$ which is of bounded variation locally and for which

$$\int_{-\infty}^{\infty} G_1(x-t) d\alpha(t) \text{ converges}$$

also

$$\int_{-\infty}^{\infty} G_2(x-t) d\alpha(t) \text{ converges.}$$

REMARK 2.1 a. – Since $\int_{-\infty}^{\infty} G_1(x-t) d\alpha(t)$ converges for all x so will $\int_{-\infty}^{\infty} G_2(x, t) d\alpha(t)$ in spite of the fact that G_2 is not necessarily of class I.

REMARK 2.1 b. – In case $\alpha_i(G_1) = \alpha_i(G_2)$ and $\mu_i(G_1) = \mu_i(G_2)$ for $i = 1, 2$ the convergence of $\int_{-\infty}^{\infty} G_1(x-t) d\alpha(t)$ and $\int_{-\infty}^{\infty} G_2(x-t) d\alpha(t)$ are equivalent therefore, for $G \in \text{class I}$ convergence of $\int_{-\infty}^{\infty} G(x-t) d\alpha(t)$ depends only on the a_k 's nearest to the origin from both sides.

REMARK 2.1 c. – In Theorem 2.1 we deliberately did not make the restriction that $G(t)$ is non-finite.

PROOF OF THEOREM 2.1 – Let us prove our theorem when both the sequence of positive zeros and that of negative zeros of $E_{\alpha_1}(s)$ are different then those of $E_{\alpha_2}(s)$. (We assume the subsequences of positive and negative zeros are ordered by magnitude.) Using Lemma 2.1 b of [3, p. 121] the proof follows if we can show: (1) $|G_2(x-t)/G_1(x-t)| \leq M$. (2) $G_2(x-t)/G_1(x-t)$ is continuous in t . (3) $G_2(x-t)/G_1(x-t)$ has at most finite number of changes of trend in $(-\infty, -A)$ and (B, ∞) for some A and B . Condition (2) is obvious and condition (1) follows a well known result [3, Th. 2.1, p. 108]. We shall prove (3) for $t \in (-\infty, -A)$. Using the asymptotic formula [3, p. 110] for $G_1(t)$ and $G_2(t)$ we have

$$(2.2) \quad G_j^{(n)}(t) = \sum_{i=1}^{m_j} \left(\frac{d}{dt} \right)^n P_{i,j}(t) \exp(A_{i,j}t) + o(\exp(k_j t)) \quad t \rightarrow \infty, \quad j=1, 2$$

where $A_{i,j}$ is the sequences of negative zeros of $E_{\alpha_j}(s)$; $P_{i,j}(t)$ a polynomial of degree $n_{i,j}$ that equals the multiplicity of $A_{i,j}$ in $E_{\alpha_j}(s)$ minus one and $A_{m+1,1} < k_j < A_{m,j}$. If $\{A_{n,j}\}$ both are infinite there exists an m for which the first part of the r.h.s. of (2.2) is different $j=1$ and $j=2$, since the sequences $\{A_{i,k}\}$ and $\{A_{i,2}\}$ are different and we choose $m_1=m_2$ to be such. If one of the sequences $\{A_{m,j}\}$ or both are finite we still can express $G(t)$ like (2.2) in which the m may differ but $k_1=k_2$ and the first part would be different. One now can show that $(d/dt) \{G_2(t)/G_1(t)\}$ has no changes of sign in $t > M$ since in $G_2'(t)G_1(t) - G_1'(t)G_2(t)$ the dominant element that (2.2) yield is of a fixed sign for $t > 0$. Therefore $G_2(-t)/G_1(x-t)$ has no change of trend in $(-\infty, -A)$ for some A . Similarly $G_2(x-t)/G_1(x-t)$ has no change of trend in (B, ∞) .

To prove our theorem when either the positive or the negative subsequence of zeros of $E_{\alpha_1}(s)$ and $E_{\alpha_2}(s)$ are equal we use $G_3(t)$ which has different subsequences of zeros but $\alpha_i(G_1) = \alpha_i(G_2) = \mu_i(G_3)$ for $i=1, 2$. Using the above $\int_{-\infty}^{\infty} G_3(x-t) d\alpha(t)$ converges and again using the above $\int_{-\infty}^{\infty} G_2(x-t) d\alpha(t)$ converges.

3. – Representation Theorems for $G(t) \in$ class I.

Our main result which we shall prove in section 5 is the following:

THEOREM 3.1. – For a given $G(t)$ that belongs to class I, a necessary and sufficient condition for $f(x)$ to be equal to $\int_{-\infty}^{\infty} G(x-t) d\alpha(t)$ where $\alpha(t)$ is locally of bounded variation satisfying $\int_{-\infty}^{\infty} G(x_0-t) |d\alpha(t)| \leq M$ is: is $f(x) \in O^\infty$, $f(x) = o(\exp(\alpha_1 x))$ $x \rightarrow -\infty$,

$f(x) = o(\exp(\alpha_2 x))$ $x \rightarrow \infty$ and for $m \geq m_0$

$$(3.1) \quad \int_{-\infty}^{\infty} G(x_0 - t) |P_m(D) f(t)| dt \leq M + o(1) \quad m \rightarrow \infty.$$

From this theorem one can deduce the following corollaries.

COROLLARY 3.1 a. - If a point x_0 for $m \geq m_0$

$$(3.2) \quad \int_{-\infty}^{\infty} G(x_0 - t) |P_m(D) f(t)| dt \leq K$$

$f \in C^\infty$ $f(x) = o(\exp(\alpha_1 x))$ $x \rightarrow -\infty$ and $f(x) = o(\exp(\alpha_2 x))$ $x \rightarrow \infty$ then there exist $\alpha(t)$ locally of bounded variation such that $f(x)$ is equal to $\int_{-\infty}^{\infty} G(x-t) d\alpha(t)$ which converges absolutely for all x .

COROLLARY 3.1 b. - In Corollary 3.1 a condition (3.2) can be replaced by for $m \geq m_0$

$$(3.3) \quad \begin{cases} \int_0^{\infty} t^{\mu_2} \exp(-\alpha_2 t) |P_m(D) f(t)| dt \leq L \\ \int_{-\infty}^0 (-t)^{\mu_1} \exp(\alpha_1 t) |P_m(D) f(t)| dt \leq L \end{cases}$$

where $\mu_i = \mu_i(G)$ and $\alpha_i = \alpha_i(G)$, $i = 1, 2$.

The following representation theorem concerning determining functions in L_p with weight $G(x_0 - t)$ would be proved in section 6.

THEOREM 3.2. - For a given $G(t)$, $G(t) \in \text{class I}$, a necessary and sufficient condition for $f(x)$ to be equal to $\int_{-\infty}^{\infty} G(x-t) \varphi(t) dt$ where $\int_{-\infty}^{\infty} G(x_0 - t) |\varphi(t)|^p dt \leq M$, $1 < p < \infty$ is that $f(x) \in C^\infty$ $f(x) = o(\exp(\alpha_1 x))$ $x \rightarrow -\infty$ $f(x) = o(\exp(\alpha_2 x))$ $x \rightarrow \infty$ and for $m \geq m_0$

$$(3.4) \quad \int_{-\infty}^{\infty} G(x_0 - t) |P_m(D) f(t)|^p dt \leq M + o(1) \quad m \rightarrow \infty.$$

REMARK 3.2 a. - Obviously the analogue Theorem 3.2 for $p = \infty$ is not valid since if $\sup_t G(x_0 - t) |\varphi(t)| \leq M$ $\int_{-\infty}^{\infty} G(x_0 - t) \varphi(t) dt$ does not necessarily converge.

The classes of determining functions in the new representation theorems are, as is easily seen, wider than the classes represented before. For instance for every

$G(t)$ for which $E_G(s) = \prod_{k=1}^{\infty} (1 - s^2/a_k^2)$ every polynomial $p(x)$ can be represented $p(x) = \int_{-\infty}^{\infty} G(x-t) d\alpha(t)$ by the theorems above, while x^2 already is not representable by any of the representation theorems known before.

4. - Some preliminary results.

We recall, for $f(x)$ defined by (1.1), that

$$(4.1) \quad P_m(D)f(x) = \int_{-\infty}^{\infty} G_m(x-t) d\alpha(t),$$

where $G_m(t)$ satisfies

$$(4.2) \quad \int_{-\infty}^{\infty} \exp(-st) G_m(t) dt = [E_m(s)]^{-1} \equiv \left[\exp(b_m s) \prod_{k=m+1}^{\infty} \left(1 - \frac{s}{a_k}\right) \exp(s/a_k) \right]^{-1}$$

and where $b_m = o(1)$ $m \rightarrow \infty$.

We can obtain now the following result:

LEMMA 4.1. - Let $G(t) \in$ class I and $\int_{-\infty}^{\infty} G(x-t) d\alpha(t)$ converges absolutely then

$$(4.3) \quad \int_{-\infty}^{\infty} G(x-t) \int_{-\infty}^{\infty} G_m(t-v) d\alpha(v)$$

converges absolutely for $m \geq m_0$.

PROOF. - We choose m_0 so that for $k > m_0$, $a_k \neq \alpha_1(G)$ and $a_k \neq \alpha_2(G)$. The integral

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} G(x-t) G_m(t-v) dt \right\} |d\alpha(v)| \equiv \int_{-\infty}^{\infty} \mathfrak{G}(x-v) |d\alpha(v)|$$

converges using Theorem 2.1, whenever $\int_{-\infty}^{\infty} G(x-t) |d\alpha(v)|$ converges. Therefore, using FUBINI's Theorem (4.3) converges absolutely.

LEMMA 4.2. - Suppose: (1) $\psi(t) \in C^\infty$. (2) $G(t) \in$ class II. (3) $\mathfrak{G}(t)$ is finite or infinite satisfying $\alpha_i(\mathfrak{G}) = \alpha_i(G) \equiv \alpha_i$ and $\mu_i(\mathfrak{G}) = \mu_i(G)$ for $i=1, 2$. (4) $f(x) = \int_{-\infty}^{\infty} \mathfrak{G}(x-t) \psi(t) dt$.

Then

$$(4.5) \quad \left| \int_{-\infty}^{\infty} G(x_0 - t) \psi^{(k)}(t) dt \right| \leq M(k),$$

implies

$$(4.6) \quad f^{(k)}(x) = o(\exp(\alpha_2 x)) \quad x \rightarrow \infty \quad f^{(k)}(x) = o(\exp(\alpha_1 x)) \quad x \rightarrow -\infty.$$

REMARQUE 4.2. – We shall need the above Lemma only for the case $\mathfrak{G}$ is finite.

PROOF. – Using Lemma 2.1, $I(k, x) \equiv \int_{-\infty}^{\infty} \mathfrak{G}(x-t) \varphi^{(k)} dt$ converges. Integrating by parts and using (4.5) $I(k, x) = f^{(k)}(x)$. We complete our proof using a theorem by I. I. HIRSCHMAN and D. V. WIDDER [3, Th. 2.1 A p. 147].

LEMMA 4.3. – If $G(t) \in \text{class I}$ (finite or infinite) and

$$(4.7) \quad \int_{-\infty}^{\infty} G(x_0 - t) |d\alpha(t)| \leq M$$

then

$$(4.8) \quad |f(x)| \leq M \max(\exp(\alpha_2(x - x_0)), \exp(\alpha_1(x - x_0)))$$

where $f(x) = \int_{-\infty}^{\infty} G(x-t) d\alpha(t)$.

REMARK 4.3 a. – Though this lemma seems weaker than Th. 2.1 of [3, p. 147], replacing $0(\exp(\alpha_2 x))$ by $M \exp(-\alpha_2 x) \exp(\alpha_2 x)$ as $x \rightarrow \infty$ for instant, it has the advantage that the estimate (4.8) is given for all x and that it depends only on the constant M in (4.7) and not on the particular $\alpha(t)$.

PROOF. – We write

$$\begin{aligned} |f(x)| &\leq \int_{-\infty}^{\infty} G(x-t) |d\alpha(t)| = \int_{-\infty}^{\infty} \left\{ \frac{G(x-t)}{G(x_0-t)} \right\} G(x_0-t) |d\alpha(t)| \leq \\ &\leq \max_t \left(\frac{G(x-t)}{G(x_0-t)} \right) \int_{-\infty}^{\infty} G(x_0-t) |d\alpha(t)| \leq M \cdot \sup_t \left(\frac{G(x-t)}{G(x_0-t)} \right). \end{aligned}$$

Recalling

$$\sup_t \frac{G(x-t)}{G(x_0-t)} = \begin{cases} \exp(\alpha_2(x - x_0)) & x \geq x_0 \\ \exp(\alpha_1(x - x_0)) & x < x_0 \end{cases}$$

we complete the proof.

We shall also need the following theorem proved in [4].

THEOREM 4.4. — Let $b_m = 0$ then

$$(4.9) \quad G_m(t) \leq A S_m^{-\frac{1}{2}} \exp(-\tfrac{1}{4} S_m^{-1}/|t|)$$

where $S_m = \sum_{k=m+1}^{\infty} \alpha_k^{-2}$ and $G_m(t)$ is defined by (4.2).

5. — Proof of the main representation Theorem.

PROOF OF THEOREM 3.1. — First we shall prove (3.1) is necessary. We write using Lemma 4.1 (twice)

$$\begin{aligned} \int_{-\infty}^{\infty} G(x_0 - t) dt \int_{-\infty}^{\infty} G_m(t - v) |d\alpha(v)| &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x_0 - t) G_m(t - v) dt |d\alpha(v)| = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_m(x_0 - t) G(t - v) dt |d\alpha(v)| = \int_{-\infty}^{\infty} G_m(x_0 - t) dt \int_{-\infty}^{\infty} G(t - v) |d\alpha(v)|. \end{aligned}$$

Since $\int_{-\infty}^{\infty} G(t - v) |d\alpha(v)|$ is continuous and $\int_{-\infty}^{\infty} G(x_0 - v) |d\alpha(v)| \leq M$ there exist $\delta_n > 0$ such that for $|x_0 - t| < \delta_n$ $\int_{-\infty}^{\infty} G(t - v) |d\alpha(v)| \leq M + 1/n$. For those δ_n and any m we have for all m

$$(5.1) \quad \int_{x_0 - \delta_n}^{x_0 + \delta_n} G_m(x_0 - t) \int_{-\infty}^{\infty} G(t - v) |d\alpha(v)| \leq M + \frac{1}{n}.$$

For all t the estimate

$$(5.2) \quad \int_{-\infty}^{\infty} G(t - v) |d\alpha(v)| \leq K \max(\exp(\alpha_2 t), \exp(\alpha_1 t))$$

follows [3, p. 147]. To avoid awkwardness of expressions we take $b_m = 0$ the general case is essentially not different. We estimate now

$$\begin{aligned} (5.3) \quad & \left\{ \int_{-\infty}^{x_0 - \delta_n} + \int_{x_0 + \delta_n}^{\infty} \right\} G_m(x_0 - t) \int_{-\infty}^{\infty} G(t - v) |d\alpha(v)| \leq \\ & \leq \left\{ \int_{-\infty}^{\delta_n} + \int_{\delta_n}^{\infty} \right\} A K \cdot S_m^{-\frac{1}{2}} \exp\{-\tfrac{1}{4} S_m^{-1}|u|\} (\exp(\alpha_1(x_0 - u)) + \exp(x_0 - u)) du = o(1) \quad m \rightarrow \infty \end{aligned}$$

for every fixed n . For $\varepsilon > 0$ we can choose δ_n such that $(1/n) < \varepsilon/2$ and choose m_0 such that for $m \geq m_0$ and for the δ_n chosen above (5.3) is estimated by $\varepsilon/2$. This completes the proof of necessity.

To prove that the condition stated in Theorem 3.1 are sufficient we first show for all k

$$(5.4) \quad f^{(k)}(x) = o(\exp(\alpha_1 x)) \quad x \rightarrow -\infty \quad f^{(k)}(x) = o(\exp(\alpha_2 x)) \quad x \rightarrow \infty.$$

We set in Lemma 4.1 $\psi(t) = P_{m_0}(D)f(t)$, obviously $\psi(t) \in C^\infty$. The absolute convergence of $\int_{-\infty}^{\infty} G(x-t)\{P_m(D)f(t)\}dt$ for $m \geq m_0$ at $x = x_0$ implies its absolute convergence for all x and $m \geq m_0$. Since $P_m(D)f(t) = \prod_{k=m_0+1}^m (1 - a_k^{-1}D \exp(a_k^{-1}D)) \psi(t)$ we have

$$\int_{-\infty}^{\infty} G(x-t)P_m(D)f(t)dt = \int_{-\infty}^{\infty} G\left(x - \sum_{k=m_0+1}^m a_k^{-1} - t\right) \prod_{k=m_0+1}^m (1 - a_k^{-1}D) \psi(t)dt$$

and therefore $\int_{-\infty}^{\infty} G(x_0-t)\psi^{(k)}(t)dt$ converges absolutely. Define $f_*(x) = \int_{-\infty}^{\infty} \mathfrak{G}(x-t)\psi(t)dt$ where $\mathfrak{G}(t)$ is defined by $\int_{-\infty}^{\infty} \mathfrak{G}(t) \exp(-st)dt = [P_{m_0}(s)]^{-1}$. We shall complete the proof of (5.4) if we show that $f_*(x) = f(x)$ since f_* satisfies (5.4) by Lemma 4.2. We know that $P_{m_0}(D)[f_*(x) - f(x)] = 0$ but

$$f_*(x) - f(x) = o(\exp(\alpha_1 x)) \quad x \rightarrow -\infty \quad f_*(x) - f(x) = o(\exp(\alpha_2 x)) \quad x \rightarrow \infty$$

and therefore $f_*(x) - f(x) = 0$.

Define

$$(5.5) \quad \alpha_n(t) = \int_0^t P_n(D)f(u)f(u)du.$$

Using the above and Theorem 3.1 of [3, p. 150] that

$$(5.6) \quad f(x) = \int_{-\infty}^{\infty} H_n(x-u)d\alpha_n(u)$$

where $\int_{-\infty}^{\infty} H_n(x) \exp(-sx)dx = [P_n(s)]^{-1}$ and $\alpha_n(u) = \int_0^u P_n(D)f(t)dt$. We write now

$$\begin{aligned} I_n(x) - f(x) &\equiv \int_{-\infty}^{\infty} G(x-u)d\alpha_n(u) - f(x) = \int_{-\infty}^{\infty} [G(x-u) - H_n(x-u)]d\alpha_n(u) = \\ &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} [G^*(x-v) - H_n^*(x-v)]\mathfrak{G}(v-u)dv \right\} d\alpha_n(u) \end{aligned}$$

where $G^*(t)$, $G_n^*(t)$ and $G(t)$ are given by

$$\int_{-\infty}^{\infty} G(t) \exp(-st) dt = [(1 - \alpha_1^{-1}s)^{-1} (1 - \alpha_2^{-1}s)^{\mu_2+1}]^{-1} \equiv P(s)^{-1}$$

$$\int_{-\infty}^{\infty} G^*(t) \exp(-st) dt = E_G(s)^{-1} P(s) \equiv E^*(s)^{-1}$$

and

$$\int_{-\infty}^{\infty} H_n^*(t) \exp(-st) dt = P_n(s)^{-1} P(s) \equiv P_n^*(s)^{-1}.$$

Using Lemma 4.3 have

$$(5.7) \quad \left| \int_{-\infty}^{\infty} \mathcal{G}(v-u) d\alpha_n(u) \right| \leq M_1 \max \left(\exp(\alpha_2(v-x_0)), \exp(\alpha_1(v-x_0)) \right).$$

Now, we shall prove

$$(5.8) \quad |I_n(x) - f(x)| \leq o(1) (\exp(\alpha_1 x) + \exp(\alpha_2 x)) \quad n \rightarrow \infty$$

or $|I_n(x) - f(x)| = o(1) \quad n \rightarrow \infty$ uniformly on every compact, and differ by $o(1) (\exp(\alpha_2 x) + \exp(\alpha_1 x)) \quad n \rightarrow \infty$ from 0 as $|x| \rightarrow \infty$.

To prove (5.8) we write

$$\begin{aligned} |I_n(x) - f(x)| &\leq M_2 \int_{-\infty}^{\infty} |G^*(x-v) - H_n^*(x-v)| [\exp(\alpha_2 v) + \exp(\alpha_1 v)] dv = \\ &= \sum_{i=1}^2 M_2 \exp(\alpha_i x) \int_{-\infty}^{\infty} |G^*(u) - H_n^*(u)| \exp(-\alpha_i u) du \end{aligned}$$

If we can show now that

$$a) \quad \int_{-\infty}^{\infty} |G^*(u) - H_n^*(u)| \exp(-\alpha_i u) du = o(1) \quad n \rightarrow \infty \quad i = 1, 2;$$

we obtain $|I_n(x) - f(x)| = o(1) (\exp(\alpha_1 x) + \exp(\alpha_2 x)) \quad n \rightarrow \infty$. It is enough to prove (a) for $i = 1$ since the proof for $i = 2$ is similar. We choose R independent of n for $n \geq n_0$ such that

$$b) \quad \left\{ \int_{-\infty}^{-R} + \int_R^{\infty} \right\} |G^*(u) - H_n^*(u)| \exp(-\alpha_1 u) du < \frac{\varepsilon}{2}.$$

(That this choice is possible we shall prove later). Recalling now that in any finite interval and in particular in $[-R, R]$ and for any $\varepsilon_1, \varepsilon_1 > 0$ there exists $n_1 = n_1(\varepsilon_1, R)$ such that for $n \geq n_1 \geq n_0$ $|G^*(u) - H_n^*(u)| < \varepsilon_1$. Choosing $\varepsilon/2 = 2R\varepsilon_1 \exp(-\alpha_1 R)$ we observe that (b) would imply (a). To prove (b) we recall that the zeros of $E^*(s)$ (and $P_n^*(s)$) are not in $[\alpha_1, \alpha_2]$ and therefore by the well known Theorem 2.1 of [3, p. 108] on asymptotic behaviour of $G(t)$ there exists a constant $R_0 > 0$ such that for $R > R_0$

$$c) \quad \left\{ \int_{-\infty}^{-R} + \int_R^{\infty} \right\} G^*(u) \exp(-\alpha_1 u) du < \frac{\varepsilon}{4}.$$

Following [3, p. 108] and choosing $\beta_1 < \alpha_1 < 0$ such that $\beta_1 > a_k^*$ for all negative a_k^* for which $E^*(a_k^*) = 0$ we have for $n \geq 2$

$$d) \quad H_n^*(u) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\exp(su)}{P_n^*(s)} ds = \frac{1}{2\pi i} \int_{\beta_1 - i\infty}^{\beta_1 + i\infty} \frac{\exp(su)}{P_n^*(s)} ds.$$

Therefore

$$H_n^*(u) \leq \frac{1}{2\pi} \exp(\beta_1 u) \int_{-\infty}^{\infty} |P_n(\beta_1 + iy)|^{-1} dy \quad u > 0.$$

but

$$|P_n(\beta_1 + iy)| \geq \left| \prod_{i=1}^n \left(1 - \frac{\beta_1 + iy}{a_i} \right) \exp(\beta_1/a_i^*) \right| \left| \prod_{i=3}^n \left(1 - \frac{\beta_1}{a_i^*} \right) \exp(\beta_1/a_i^*) \right|$$

and since $\prod_{i=3}^{\infty} (1 - \beta_1/a_i^*) \exp(\beta_1/a_i^*)$ converges, $\left| \prod_{i=3}^n (1 - \beta_1/a_i^*) \exp(\beta_1/a_i^*) \right| > K$ for all n which implies $H_n^*(u) \leq K_1 \exp(\beta_1 u)$ for all $n \geq 2$. Therefore, there exists R such that for

$$n \geq 2 \int_R^{\infty} H_n^*(u) \exp(-\alpha_1 u) du < \frac{\varepsilon}{8}.$$

Similarly (even simpler)

$$\int_{-\infty}^{-R} H_n^*(u) \exp(-\alpha_1 u) du < \frac{\varepsilon}{8}.$$

Combining the above $|I_n(x) - f(x)| = o(1) (\exp(\alpha_1 x) + \exp(\alpha_2 x)) \quad n \rightarrow \infty$.

Define now $\beta_n(t) = \int_0^t G(x_0 - u) d\alpha_n(u)$ it is obvious that $\beta_n(t) \in B \cdot V(-\infty, \infty)$ in fact

$$\int_{-\infty}^{\infty} |d\beta_n(t)| = \int_{-\infty}^{\infty} \left| d \left\{ \int_0^t G(x_0 - u) d\alpha_n(u) \right\} \right| = \int_{-\infty}^{\infty} |G(x - t)| da_n(t) \leq M + \varepsilon.$$

Therefore for all $\psi \in C(\infty, \infty)$ such that both $\lim_{t \rightarrow -\infty} \psi(t)$ and $\lim_{t \rightarrow +\infty} \psi(t)$ exist, we have using Theorem 5.1 of [3, p. 156]

$$(5.9) \quad \lim_{i \rightarrow \infty} \int_{-\infty}^{\infty} \psi(t) d\beta_{n_i}(t) = B_1 \psi(-\infty) + B_2 \psi(+\infty) + \int_{-\infty}^{\infty} \psi(t) d\beta(t),$$

where $\int_{-\infty}^{\infty} |d\beta(t)| \leq M + \varepsilon$ (for all ε). Write for $\psi(t) = G(x-t)/G(x_0-t)$ for which $\psi(-\infty) = \exp(\alpha_1(x-x_0))$ and $\psi(\infty) = \exp(\alpha_2(x-x_0))$. For that $\psi(t)$ the left side of (5.9) is $\lim I_{n_i}(x)$ where $I_{n_i}(x)$ is defined in (5.8) which is $o(\exp(\alpha_1 x) + \exp(\alpha_2 x))$ as $|x| \rightarrow \infty$, $n_i \rightarrow \infty$.

Defining $\alpha(t) = \int_0^t d\beta(t)/G(x-t)$ we have

$$\int_{-\infty}^{\infty} G(x_0-t) |d\alpha(t)| \leq M \quad \text{and} \quad f_*(x) = \int_{-\infty}^{\infty} \frac{G(x-t)}{G_0(x-t)} d\beta(t) = \int_{-\infty}^{\infty} G(x-t) d\alpha(t).$$

Therefore $f_*(x) = o(\exp(\alpha_1 x))$ $x \rightarrow \infty$, $f^*(x) = o(\exp(\alpha_2 x))$ $x \rightarrow \infty$ and $B_1 - B_2 = 0$. This implies $f_*(x) = f(x)$. Q.E.D.

6. - Proof of the $L_p(G)$ representation Theorem.

PROOF OF THEOREM 3.2. - First we shall prove that the conditions mentioned are necessary. We write

$$\begin{aligned} I(m) &= \int_{-\infty}^{\infty} G(x_0-t) |P_m(D)f(t)|^p dt = \int_{-\infty}^{\infty} G(x_0-t) dt \left| \int_{-\infty}^{\infty} G_m(t-u) \varphi(u) du \right|^p \leq \\ &\leq \int_{-\infty}^{\infty} G(x_0-t) dt \int_{-\infty}^{\infty} G_m(t-u) |\varphi(u)|^p du. \end{aligned}$$

Using Lemma 4.1 and the method of proof of Th. 3.1, we have

$$I(m) \leq \int_{-\infty}^{\infty} G_m(x_0-t) dt \int_{-\infty}^{\infty} G(t-u) |\varphi(u)|^p du \leq M + \varepsilon.$$

We shall prove now sufficiency of the conditions in our theorem. The condition

$$\int_{-\infty}^{\infty} G(x_0-t) |P_m(D)f(t)|^p dt \leq M + \varepsilon \quad \text{implies} \quad \int_{-\infty}^{\infty} G(x_0-t) |P_m(D)f(t)| dt \leq M + 1 + \varepsilon$$

since $|P_m(D)f(t)| \leq |P_m(D)f(t)|^p + 1$. This enables us to use the same technique we used in proving Theorem 3.1 to show that

$$f(x) = \int_{-\infty}^{\infty} H_m(x-t) P_m(D) f(t) dt$$

and

$$\left| f(x) - \int G(x-t) P_m(D) f(t) dt \right| = o(1) (\exp(\alpha_1 x) + \exp(\alpha_2 x)) \quad m \rightarrow \infty.$$

Condition (3.4) of our Theorem implies that $P_m(D)f(t)$ are in a ball of radius $M + \varepsilon$ in L_p with the weight function $G(x_0 - t)$ provided $m \geq m_0$. This implies there exist $\varphi(t) \in L_p(G(x_0 - t))$ and a subsequence $P_{m_i}(D)f(t)$ of $P_m(D)f(t)$ such that for all $\psi(t) \in L_q(G(x_0 - t))$, $p^{-1} + q^{-1} = 1$, we have

$$\lim_{m_i \rightarrow \infty} \int_{-\infty}^{\infty} \psi(t) [P_{m_i}(D)f(t)] G(x_0 - t) dt = \int_{-\infty}^{\infty} \psi(t) \varphi(t) G(x_0 - t) dt$$

and $\int_{-\infty}^{\infty} |\varphi(t)|^p G(x_0 - t) dt \leq M + \varepsilon$. In fact since φ is the weak limit also of subsequences of the above sequence ε is arbitrary and $\int_{-\infty}^{\infty} |\varphi(t)|^p G(x_0 - t) dt \leq M$. We substitute $\psi(t) = G(x - t)/G(x_0 - t)$ this $\psi(t)$ belong to $L_q(G(x_0 - t))$ since it is bounded (for every fixed x) and we have

$$f(x) = \lim_{m_i \rightarrow \infty} \int_{-\infty}^{\infty} P_{m_i}(D)f(t) \cdot G(x - t) dt = \int_{-\infty}^{\infty} \left(\frac{G(x - t)}{G(x_0 - t)} \right) \varphi(t) G(x_0 - t) dt = \int_{-\infty}^{\infty} \varphi(t) G(x - t) dt.$$

Q.E.D.

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