Remarks on an Inequality of Schulenberger and Wilcox (*) (**).

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Summary. – *J. R. Schulenberger and C. H. Wilcox* [1], [2], *have proven a coerciveness inequality for a class of nonelliptic first-order partial differential operators of the form* $\Lambda = E^{-1}A_jD_j$, *where* A_j $(j = 1, \ldots, n)$ *is a constant* $m \times m$ *Hermitian matrix,* $E = E(x)$ *is uniformly positive definite, bounded, and uniformly differentiable Hermitian* $m \times m$ *matrix, and where the symbol* $\Lambda(p, x) = E(x)^{-1}A_jp_j$ *has constant rank for all* $p \in R^n - \{0\}$ *and* $x \in R^n$. *They prove coerciveness on* $N(\Lambda)^{\perp}$, *the orthogonal complement of the null space* $N(\Lambda)$ *relative to the inner product*

$$\langle u, v\rangle = \int_{R^n} u(x)\cdot E(x)v(x)\,dx\,.$$

Their proof is rather long, a simpler and shorter proof is given here. This proof leads naturally to a generalization of their results to the case where the A_j*'s need not be Hermitian.*

Let $A_1, \ldots, A_n$ be constant Hermitian $m \times m$ matrices with the property that the symbol

$$A(p) = \sum A_j p_j$$

has constant rank for all real $p = (p_1, \ldots, p_n) \neq 0$. Let $E(x)$ be a matrix function of $x = (x_1, \ldots, x_n) \in R^n$ which is uniformly positive definite, bounded, and with uniformly bounded first derivatives in R^n. Let

$$\Lambda = E(x)^{-1}\sum A_j D_j\,, \qquad x \in R^n,\; D_j = \frac{1}{i}\,\partial/\partial x_j. \tag{1}$$

SCHULEMBERGER and WILCOX define the space $\mathcal{H}$ as the completion of the space C_0^∞ with respect to the norm

$$\|u\|^2_{\mathcal{H}} = \langle u, u\rangle = \int_{R^n} u(x)\cdot E(x)u(x)\,dx\,, \tag{2}$$

and prove the following theorem:

THEOREM 1. – Let N be the null space of Λ. There exists $c > 0$ such that if u is in the domain of Λ and is orthogonal (with respect to the $\mathcal{H}$ inner product) to N,

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then the first derivatives of u are $\mathcal{H}$ and satisfy

$$\sum \|D^i u\|_{\mathcal{H}} \leqslant c(\|u\|_{\mathcal{H}} + \|\Lambda u\|_{\mathcal{H}}) . \tag{3}$$

We now present our shorter proof.

We first observe that because of the conditions on E, the spaces $\mathcal{H}$ and $L_2(R^n)=H$ have the same elements and equivalent norms, and that the operations of multiplication by $E(x)$ or $E^{-1}(x)$ are bounded in $\mathcal{H}$, H, and in the space H^1 with norm

$$\|u\|_1^2 = \|u\|^2 + \sum_j \|D_j u\|^2 ,$$

where $\|.\|$ denotes the norm in $L_2(R^n)$. Also, one can write

$$\langle u, v\rangle = (u, Ev),$$

where (,) denotes the inner product in H.

Let L be the operator

$$L = \sum A_j D_j = E\Lambda ,$$

and let N be its null space. Because of the above remarks we can restate Theorem 1 in the following form:

THEOREM 1′. – There exists $c>0$ such that if u is in the domain of L and in orthogonal in H to EN, then u is in H^1, and

$$\|u\|_1 \leqslant c(\|u\| + \|Lu\|). \tag{3'}$$

PROOF OF THEOREM 1′. – Denote the orthogonal complements in H of N and of EN by $N^\perp$ and $(EN)^\perp$ respectively.

We first prove a coerciveness inequality of the form (3′) for functions $u \in D(L) \cap N^\perp$. Setting $Lu = f$, we take Fourier transforms and find

$$A(p)\,\hat{u}(p) = \hat{f}(p) \tag{4}$$

If $u \in N^\perp$, then for each $p \in R^n$, $\hat{u}(p)$ is orthogonal to the null space $N(p)$ of $A(p)$. Since $A(p)$ is Hermitian and has constant rank for $p \neq 0$, the eigenvalues λ_j of $A(p)$ are bounded above $k|p|$ for some $k>0$ for all $p \neq 0$. Hence the restriction of $A(p)$ to $N(p)^\perp$ has an inverse with bound $(k|p|)^{-1}$, and we conclude immediately that

$$\|D_j u\| = \|p_j u\| \leqslant k^{-1}\|f\| = k^{-1}\|Lu\| ,$$

which implies (3′) with $c = k^{-1}$.

Next, suppose that $u \in D(L) \cap (EN)^{\perp} \cap H^1$.

Write $u = u_1 + u_2$, with $u_1 \in N$ and $u_2 \in N$. Since $u \in (EN)^{\perp}$, we have

$$(u_1 + u_2, Eu_2) = 0$$

and hence

$$(u_1, Eu_2) = -(u_2, Eu_2) .$$

Thus

$$\|E\| \|u_1\| \|u_2\| \geqslant |u_1, Eu_2| = (u_2, Eu_2) \geqslant C\|u_2\|^2 ,$$

where C is a lower bound for E, and we conclude that

(5) $$\|u_2\| \leqslant C^{-1}\|E\| \|u_1\| .$$

We can also estimate the first derivatives of u_2 in terms of u_1 and its first derivatives. As we observed above, $v \in N^{\perp}$ implies that $v(p) \in N(p)^{\perp}$ for all p. Similarly, $v \in N$ implies that $v(p) \in N(p)$ for all p. Hence $D_j u_1 \in N^{\perp}$ and $D_j u_2 \in N$, $j = 1, \ldots, n$.

If $w \in N$, then w_T, defined by

$$\widehat{w}_T(p) = \begin{cases} \widehat{w}(p) , & |p| \leqslant T, \\ 0 , & |p| > T' \end{cases}$$

is in $N \cap H^1$, and $D_j w_T$ is in N. Further, as $T \to \infty$, we have $ED_j w_T \to ED_j w$ in the topology of H^{-1}, the dual of H^1.

Since $u \in H^1 \cap (EN)^{\perp}$,

$$(u, ED_j w) = \lim_{T \to \infty} (u, ED_j w_T) = 0$$

and we have

(6) $$(D_j u, Ew) = (u, (D_j E)w) + (u, ED_j w) = (u, (D_j E)w) .$$

Substituting $D_j u_2$ for w in (6) gives

(7) $$(D_j u, ED_j u_2) = (u, (D_j E)D_j u_2)$$

which implies, because of the positivity of E, that with some positive constants c an c_1,

(8) $$c\|D_j u_2\|^2 \leqslant (u, (D_j E)D_j u_2) + \|D_j u_1\| \|E\| \|D_j u_2\| \leqslant c_1\|u_1\|_1 \|D_j u_2\| + c_1\|u_2\| \|D_j u_2\| .$$

Now (8) together with (5) implies

(9) $$\|D_j u_2\| \leqslant C\|u_1\|_1 ,$$

and (9) together with (5) implies

$$\|u_2\|_1 \leqslant C\|u_1\|_1, \text{ some } C > 0. \tag{10}$$

Since $u_1 \in N^{\perp}$, we have already shown that u_1 obeys (3′):

$$\|u_1\|_1 \leqslant c(\|u_1\| + \|Lu_1\|) = c(u_1\| + \|Lu\|). \tag{11}$$

Combining (10) and (11) and using the inequality

$$\|u\|_1 \leqslant \|u_1\|_1 + \|u_2\|_1,$$

we see that u does satisfy (3′).

There remains to drop the assumption that $u \in H^1$. To do this we use mollifiers, which we now describe. Let $j \in C_0^\infty(R^n)$ be and even, non-negarive function satisfying

$$\int j(x)\,dx = 1\,.$$

Then the mollifiers $J_M : H \to H$, defined by

$$J_M u(x) = M^n \int j(N(x-y))\,u(y)\,dy$$

satisfy

(i) $\|J_M\| = 1$,

(ii) $J_M \to I$ strongly as $M \to \infty$,

(iii) J_M commutes with constant-coefficient differential operators, and if L_1 is a first-order partial differential operator with coefficients uniformly in C_1, then

$$J_M L_1 - L_1 J_M \to 0 \text{ strongly as } M \to \infty,$$

iv J_M is self-adjoint.

Now suppose that $u \in D(L) \cap (EN)^{\perp}$. Define the sequence of approximating functions

$$u_M = E^{-1} J_M E u. \tag{12}$$

Because of (i), $\lim_{M\to\infty} u_M = E^{-1} \lim_{M\to\infty} J_M(Eu) = E^{-1}Eu = u$.

Next we claim that $u_M \in D(L) \cap (EN)^{\perp}$. Since $J_M(Eu) \in H^1$, $u_M \in H^1 \subset D(L)$. To show that $u_M \in (EN)^{\perp}$, we let $w \in N$. Because of (iii), $J_M w$ is also in N. Thus

$$(u_M, Ew) = (E^{-1}J_M Eu, Ew) = (J_M Eu, w) = (u, EJ_M w) = 0.$$

Since $u_M \in D(L) \cap (EN) \cap H^1$, u_M satisfies (3'):

$$\|u_M\|_1 \leqslant c(\|u_M\| + \|Lu_M\|). \tag{13}$$

Consider now

$$Lu_M = LE^{-1} J_M Eu = J_M Lu + [(LE^{-1})J_M - J_M(LE^{-1})]\, u\,.$$

Using (ii) and (iii), we find that $\lim_{M\to\infty} Lu_M = Lu$. Taking the limit of (13) as $M \to \infty$, we conclude that u satisfies (3').

Theorem 1' is easily extend to the following:

THEOREM 2. – *The statement of Theorem* 1' *remains true if we drop the requirement that the* A_j's *be Hermitian.*

The proof of Theorem 1 carries over to Theorem 2 except that we must reprove the existence of $k > 0$ such that

$$|A(p)w| \geqslant k|w||p|, \qquad w \in N(p)^\perp,\ p \neq 0. \tag{14}$$

Because of the homogeneity of $A(p)$ it sufficies to prove the inequality for $|w| = 1$. Because $N(p)$ has constant dimension and because $A(p)$ depends continuously on p for $p \neq 0$, $N(p)$ and $N(p)^\perp$ depend continuously on p for $p \in S^{n-1}$. If there did not exist $k > 0$ as claimed, we could construct two sequences, $p_j \in S^{n-1}$ and $w_j \in S^{m-1}$ such that $w_j \in N(p_j)^\perp$ and such that $A(p_j)s_j \to 0$ as $j \to \infty$.

Because of the compactness of S^{n-1} and of S^{m-1} and of S^{m-1}, and because of the continuous dependence of $N(p)$ on p for $p \in S^{n-1}$, we could then find $p \in S^{n-1}$ and $w \in S^{m-1} \cap N(p)^\perp$ such that $A(p)\ w = 0$, a contradiction. Theorem 2 is proved.

REMARK. – SCHULENBERGER and WILCOX assumed that the matrix $E(x)$ tends to a constant matrix E_0 as $|x| \to \infty$. This assumption was for convenience in applying pseudodifferential operators and plays no part in this paper.

REMARK 2. – Most of the technical work in the proof of Theorem 1' lay in proving the inequality (10). Evidently if we had assumed that $E(x)$ was uniformly in C^s, we could have gone on inductively to prove

$$\|u_2\|_s \leqslant C\|u_1\|_s, \qquad s \text{ a positive integer.}$$

Note that if we introduce the orthogonal projector P into N, then $u \in (EN)^\perp$ is equivalent with $PEu = 0$, or $u = E^{-1}(1-P)Eu$. We thus have the following lemma:

LEMMA 1. – Let $E(x)$ be a matrix function of $x \in R^n$ which is Hermitian, uniformly positive, and uniformly in C^s, s a positive integer. Let P be an orthogonal projection operator which commutes with differentiation:

$$P = P^* = P^2, \text{ and } D_j P = P D_j.$$

Then there exists $c_s > 0$ such that all functions $u \in H^s$ and which satisfy $PEu = 0$, also satisfy

$$\|Pu\|_s \leq c_s \|(1 - P)u\|_s \qquad (15)_s$$

BIBLIOGRAPHY

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