Dirichlet Problem for a Class of Linear Second Order Elliptic Partial Differential Equations with Discontinuous Coefficients.

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Summary. – *I give a sufficient condition in order that a Dirichlet problem is solvable in $H^2(\Omega)$ for a class of linear second order elliptic partial differential equations. Such a class includes some particular cases for which the result is known.*

Sunto. – *Si prova una condizione sufficiente affinchè un problema di Dirichlet sia risolubile in $H^2(\Omega)$ per una classe di equazioni differenziali alle derivate parziali lineari ellittiche del secondo ordine. Tale classe comprende alcuni casi particolari per i quali il risultato è noto.*

1. – Introduction.

Let us consider the uniformly elliptic operator

$$L = -\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c \tag{1}$$

defined in an open set Ω of R^n. Given f in $L_2(\Omega)$, we want to solve the Dirichlet problem

$$\begin{cases} Lu = f & \text{a.e. in } \Omega, \\ u \in H^2(\Omega) \cap H_0^1(\Omega) \end{cases} \tag{2}$$

under suitable hypotheses on the coefficients of L.

While for $n = 2$ such a problem has one and only one solution (at least with proper c) even if the coefficients a_{ij} are only in $L^\infty(\Omega)$, this assumption is not sufficient for $n \geqslant 3$. In such cases additional hypotheses are necessary, for example the following ones: $a_{ij} \in C^0(\bar{\Omega})$ (see [1], [8], [4]), $a_{ij} \in H^{1,n}(\Omega)$ (see [9]), $\operatorname{ess\,inf}_{\Omega} \left(\sum_{i=1}^{n} a_{ii}\right)^2 \cdot \left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{-1} > n-1$ (see [12], [2], [3]).

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The aim of the present work is to find a more general condition for the coefficients a_{ij} which assures the solvability of problem (2), at least for some value of c. All the types of equations mentioned above satisfy this condition and therefore are particular cases of the class here considered.

2. – Notations and hypotheses.

The following hypotheses will be assumed in the sequel without mention. Let Ω be a bounded open set in R^n with $n \geq 3$. We suppose that $\partial\Omega$ (boundary of Ω) is represented locally by a function with continuous second derivatives. Let $H^{1,p}(\Omega)$, $H_0^{1,p}(\Omega)$ be the spaces obtained by completing $C^1(\bar{\Omega})$, $C_0^1(\Omega)$ respectively according to the norm

$$\|u\|_{H^{1,p}(\Omega)} = \|u\|_{L_p(\Omega)} + \sum_{i=1}^{n} \|u_{x_i}\|_{L_p(\Omega)} .$$

For $p = 2$ we shall write simply $H^1(\Omega)$, $H_0^1(\Omega)$ instead of $H^{1,2}(\Omega)$, $H_0^{1,2}(\Omega)$. Let $H^2(\Omega)$ be the space obtained by completing $C^2(\bar{\Omega})$ according to the norm

$$\|u\|_{H^2(\Omega)} = \|u\|_{L_2(\Omega)} + \sum_{i,j=1}^{n} \|u_{x_i x_j}\|_{L_2(\Omega)} . \tag{3}$$

It can be proven (see e.g. [8]) that in $H^2(\Omega) \cap H_0^1(\Omega)$ a norm equivalent to (3) is the following:

$$\|u_{xx}\|_{L_2(\Omega)} = \left\{ \sum_{i,j=1}^{n} \|u_{x_i x_j}\|_{L_2(\Omega)}^2 \right\}^{\frac{1}{2}} .$$

If $u \in H^1(\Omega)$ and h is a real number, we say that $u \leq h$ on $\partial\Omega$ in the sense of $H^1(\Omega)$ if there exists a sequence $\{u_j\}_{j \in N}$ such that $u_j \in C^1(\bar{\Omega})$, $u_j \leq h$ on $\partial\Omega$ $(j = 1, 2, \ldots)$ and $\lim_{j \to +\infty} \|u - u_j\|_{H^1(\Omega)} = 0$. Then we suppose $a_{ij} \in L_\infty(\Omega)$, $a_{ij} = a_{ji}$ $(i, j = 1, 2, \ldots, n)$, $\sum_{i,j=1}^{n} a_{ij} t_i t_j \geq \nu_0 |t|^2$ a.e. in Ω with ν_0 positive constant, $b_i \in L_n(\Omega)$ $(i = 1, 2, \ldots, n)$, $c \in L_q(\Omega)$ with $q = 2$ if $n = 3$, $q > 2$ if $n = 4$, $q = n/2$ if $n \geq 5$.

Let L be the operator defined in (1), let ν be a positive real number. Let us denote with $A(\nu)$ the following class of square matrices of order n:

$$A(\nu) = \left\{ [\tilde{a}_{ij}] \colon \tilde{a}_{ij} \in H^{1,n}(\Omega),\ \tilde{a}_{ij} = \tilde{a}_{ji}\ (i, j = 1, 2, \ldots, n),\ \sum_{i,j=1}^{n} \tilde{a}_{ij} t_i t_j \geq \nu |t|^2 \right\} .$$

Finally we set

$$G = \{g \colon g \in L_\infty(\Omega),\ \operatorname*{ess\,inf}_{\Omega} g > 0\} .$$

3. – Main result.

The aim of the present work is to prove

THEOREM 1. – *Besides the above mentioned hypotheses, we assume that*

$$\inf_{\nu>0} \nu^{-2}\left\{\inf_{g\in G} \inf_{[\tilde a_{ij}]\in A(\nu)} \operatorname{ess\,sup}_{\Omega} \sum_{i,j=1}^{n} (\tilde a_{ij} - g a_{ij})^2\right\} < 1 . \tag{4}$$

Then there exists a positive constant c_0, depending on n, Ω and the coefficients a_{ij}, b_i of L, such that if $c \geqslant c_0$ a.e. in Ω problem (2) *has one and only one solution. If moreover is $u \in H^2(\Omega) \cap \mathring{H}^1_0(\Omega)$, $Lu \leqslant 0$ a.e. in Ω, $c \geqslant c_0$ a.e. in Ω, it follows $u \leqslant 0$ a.e. in Ω.*

The proof of this theorem is found in n. 5.

We observe that condition (4) is certainly verified in the following cases:

i) $a_{ij} \in H^{1,n}(\Omega)$: it is immediate with $\nu = \nu_0$, $g = 1$, $\tilde a_{ij} = a_{ij}$.

ii) $a_{ij} \in C^0(\bar\Omega)$: again with $\nu = \nu_0$, $g = 1$, remembering that $H^{1,n}(\Omega)$ is dense in $C^0(\bar\Omega)$ in the uniform metric.

iii) Equations « of Cordes type », i.e. with $\operatorname{ess\,inf}_{\Omega} \left(\sum_{i=1}^{n} a_{ii}\right)^2 \cdot \left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{-1} > n-1$. In this case (see [2], p. 704) inequality (4) is verified with $\nu = 1$, $\tilde a_{ij} = \delta_{ij}$, $g = \left(\sum_{i=1}^{n} a_{ii}\right) \cdot \left(\sum_{i,j=1}^{n} a_{ij}^2\right)^{-1}$.

4. – Some preliminary lemmata.

LEMMA 1. – *Let $[\tilde a_{ij}] \in A(\nu)$ and $0 < \varepsilon < \nu$. Let $\tilde L$ be the operator*

$$\tilde L = -\sum_{i,j=1}^{n} \tilde a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c . \tag{5}$$

Then there exists a non negative constant λ_0, depending on ε, n, Ω and the coefficients of $\tilde L$, such that if $\lambda \geqslant \lambda_0$ we have

$$\|u_{xx}\|_{L_2(\Omega)} \leqslant (\nu - \varepsilon)^{-1} \|\tilde L u + \lambda u\|_{L_2(\Omega)}$$

for any $u \in H^2(\Omega) \cap \mathring{H}^1_0(\Omega)$.

PROOF. – Let us start for example from [8], p. 175-178. Here it is foundamentally proved what follows: for any $\eta > 0$ there exists a positive constant K_1, depending on η, n, Ω and the coefficients of $\tilde{L}$ such that

$$(6)\qquad \nu^2\|u_{xx}\|^2_{L_2(\Omega)} \leq \eta^2\|u_{xx}\|^2_{L_2(\Omega)} + \|\tilde{L}u\|^2_{L_2(\Omega)} + K_1\{\|u\|^2_{L_2(\Omega)} + \|u_x\|^2_{L_2(\Omega)}\}$$

for any $u \in H^2(\Omega) \cap H^1_0(\Omega)$. Using known results (see e.g. [6], p. 122) from (6) we get at once

$$(7)\qquad \nu^2\|u_{xx}\|^2_{L_2(\Omega)} \leq 2\eta^2\|u_{xx}\|^2_{L_2(\Omega)} + \|\tilde{L}u\|^2_{L_2(\Omega)} + K_2\|u\|^2_{L_2(\Omega)}$$

valid for the same functions u, where the constant K_2 depends on η, n, Ω and the coefficients of $\tilde{L}$.

Moreover it is easy to get

$$(8)\qquad \|\tilde{L}u + \lambda u\|^2_{L_2(\Omega)} = \|\tilde{L}u\|^2_{L_2(\Omega)} + \lambda^2\|u\|^2_{L_2(\Omega)} + 2\lambda\int_\Omega (\tilde{L}u)u\,dx =$$

$$= \|\tilde{L}u\|^2_{L_2(\Omega)} + \lambda^2\|u\|^2_{L_2(\Omega)} + 2\lambda\int_\Omega\left\{\sum_{i,j=1}^n \tilde{a}_{ij}u_{x_i}u_{x_j} + \sum_{i=1}^n\left[b_i + \sum_{j=1}^n(\tilde{a}_{ij})_{x_j}\right]u_{x_i}u + cu^2\right\}dx\,,$$

$$(9)\qquad \sum_{i,j=1}^n \tilde{a}_{ij}u_{x_i}u_{x_j} \geq \nu\|u_x\|^2_{L_2(\Omega)}$$

where we have written, for shortness,

$$\|u_x\|_{L_2(\Omega)} = \left\{\sum_{i=1}^n \|u_{x_i}\|^2_{L_2(\Omega)}\right\}^{\frac{1}{2}}.$$

Using known properties of the space $H^1_0(\Omega)$ we find, for any $\delta > 0$:

$$(10)\qquad \left\|\sum_{i=1}^n\left[b_i + \sum_{j=1}^n(\tilde{a}_{ij})_{x_j}\right]u_{x_i}u + cu^2\right\|^2_{L_2(\Omega)} \leq \delta\|u_x\|^2_{L_2(\Omega)} + K_3\|u\|^2_{L_2(\Omega)}$$

satisfied for any $u \in H^1_0(\Omega)$, where the constant K_3 depends on n, δ, Ω and the coefficients of $\tilde{L}$.

Now, choosing $\delta = \nu/2$ in (10), from (8), (9), (10) it follows

$$(11)\qquad \|\tilde{L}u + \lambda u\|^2_{L_2(\Omega)} \geq \|\tilde{L}u\|^2_{L_2(\Omega)} + \lambda^2\|u\|^2_{L_2(\Omega)} + \lambda\nu\|u_x\|^2_{L_2(\Omega)} - 2\lambda K_3\|u\|^2_{L_2(\Omega)}\,.$$

From (7), (11) we get

$$(12)\qquad (\nu^2 - 2\eta^2)\|u_{xx}\|^2_{L_2(\Omega)} \leq \|\tilde{L}u + \lambda u\|^2_{L_2(\Omega)} + (K_2 + 2\lambda K_3 - \lambda^2)\|u\|^2_{L_2(\Omega)} - \lambda\nu\|u_x\|^2_{L_2(\Omega)}\,.$$

Let us choose η such that $0 < 2\eta^2 < 2\varepsilon\nu - \varepsilon^2$; in this way the constant K_2 also is determined. Then put $\lambda_0 = K_3 + (K_3^2 + K_2)^{\frac{1}{2}}$: from (12) we conclude

$$\|u_{xx}\|_{L_2(\Omega)} \leqslant (\nu - \varepsilon)^{-1} \|\tilde{L}u + \lambda u\|_{L_2(\Omega)}$$

valid for any $\lambda \geqslant \lambda_0$ and any $u \in H^2(\Omega) \cap H_0^1(\Omega)$. ■

The following lemmata are very similar to the corresponding ones in [3]; I repeat them for readers' convenience.

LEMMA 2. – *Let us suppose that the coefficients a_{ij} of L satisfy condition* (4). *Then there exist a function $g \in G$ and two positive constants λ^* and K_4, depending on n, Ω, g and the coefficients of L, such that*

$$\|u_{xx}\|_{L_2(\Omega)} \leqslant K_4 \|gLu + \lambda u\|_{L_2(\Omega)} \tag{13}$$

for any $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and uniformly for any $\lambda \geqslant \lambda^*$.

PROOF. – Since by hypothesis inequality (4) is satisfied, there exist a positive constant ν, a function $g \in G$ and an operator $\tilde{L}$ like (5) such that

$$0 \leqslant k < \nu \tag{14}$$

where

$$k = \left\{ \operatorname*{ess\,sup}_{\Omega} \sum_{i,j=1}^{n} (\tilde{a}_{ij} - g a_{ij})^2 \right\}^{\frac{1}{2}}. \tag{15}$$

From (14), (15), by proceeding as in [2], it follows

$$\|(gL + \lambda I)u - (\tilde{L} + \lambda I)u\|_{L_2(\Omega)}^2 = \left\| \sum_{i,j=1}^{n} (g a_{ij} - \tilde{a}_{ij}) u_{x_i x_j} \right\|_{L_2(\Omega)}^2 \leqslant \tag{16}$$
$$\leqslant \operatorname*{ess\,sup}_{\Omega} \sum_{i,j=1}^{n} (g a_{ij} - \tilde{a}_{ij})^2 \|u_{xx}\|_{L_2(\Omega)}^2 \leqslant k^2 \|u_{xx}\|_{L_2(\Omega)}^2$$

for any $u \in H^2(\Omega) \cap H_0^1(\Omega)$. From lemma 1 there exists a positive constant λ_0 depending on ε, n, Ω and the coefficients of $\tilde{L}$, such that

$$\|u_{xx}\|_{L_2(\Omega)} \leqslant (\nu - \varepsilon)^{-1} \|\tilde{L}u + \lambda u\|_{L_2(\Omega)} \tag{17}$$

for any $\lambda \geqslant \lambda_0$ and any $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Let us choose now ε such that $0 < \varepsilon < \nu - k$. From (14), (15), (16), (17) and known theorems (see e.g. [5], p. 584) if $\lambda \geqslant \lambda_0$ there exists the inverse operator $(gL + \lambda I)^{-1}$ and inequality (13) is satisfied with $K_4 = (\nu - k - \varepsilon)^{-1}$. ■

LEMMA 3. – *Hypotheses: the coefficents a_{ij} of L satisfy condition* (4), *g and λ^* are defined as in lemma* 2, *$\lambda \geqslant \lambda^*$, $f \in L^2(\Omega)$, $z \in H^2(\Omega)$. Conclusion: there exists one and* *only one solution w of the Dirichlet problem*

$$\begin{cases} gLw + \lambda w = f & \text{a.e. in } \Omega, \\ w - z \in H^2(\Omega) \cap H_0^1(\Omega). \end{cases}$$

Moreover if it is $f \leqslant 0$ a.e. in Ω, $c \geqslant 0$ a.e. in Ω, $z \leqslant M$ on $\partial\Omega$ in the sense of $H^1(\Omega)$, then it follows $w \leqslant M$ a.e. in Ω.

PROOF. – Let us prolong the definition of the functions $\tilde{a}_{ij}$ $(i, j = 1, 2, \ldots, n)$ satisfying (14) to all of R^n in such a way that, denoting by the same letters the prolonged functions, it turns out

$$\tilde{a}_{ij} \in H^{1,n}(R^n) \quad (i, j = 1, 2, \ldots, n), \qquad \sum_{i,j=1}^{n} \tilde{a}_{ij} t_i t_j \geqslant \nu |t|^2 \qquad \text{a.e. in } R^n. \tag{18}$$

This is possible because $\partial\Omega$ is sufficiently regular: see for example [6]. Then we put

$$\alpha_{ij} = \begin{cases} g a_{ij} & \text{in } \Omega \\ \tilde{a}_{ij} & \text{in } R^n - \Omega \end{cases} \qquad (i, j = 1, 2, \ldots, n). \tag{19}$$

Let ϑ be a function such that $\vartheta \in C_0^\infty(R^n)$, $\vartheta(x) = 0$ if $|x| > 1$, $\int_{R^n} \vartheta(x)\,dx = 1$.
For any positive integer m and for $x \in R^n$ put

$$\tilde{a}_{ij}^{(m)}(x) = m^n \int_{R^n} \vartheta(mx - my)\, \tilde{a}_{ij}(y)\, dy \qquad (i, j = 1, 2, \ldots, n) \tag{20}$$

and similarly define $\alpha_{ij}^{(m)}$. Then we have, for $m = 1, 2, \ldots$ and $i, j = 1, 2, \ldots, n$:

$$\alpha_{ij}^{(m)}, \tilde{a}_{ij}^{(m)} \in C^\infty(R^n), \qquad \sum_{i,j=1}^{n} \tilde{a}_{ij}^{(m)} t_i t_j \geqslant \nu |t|^2 \qquad \text{in } R^n,$$

$$\max_{\overline{\Omega}} \sum_{i,j=1}^{n} (\tilde{a}_{ij}^{(m)} - \alpha_{ij}^{(m)})^2 \leqslant \operatorname*{ess\,sup}_{\Omega} \sum_{i,j=1}^{n} (\tilde{a}_{ij} - g a_{ij})^2. \tag{21}$$

Besides the sequence $\{\alpha_{ij}^{(m)}\}_{m \in N}$ converges to $g a_{ij}$ in every $L_p(\Omega)$, $1 \leqslant p < +\infty$, and the sequence $\{\tilde{a}_{ij}^{(m)}\}_{m \in N}$ converges to $\tilde{a}_{ij}$ in $H^{1,n}(\Omega)$.
If we suppose $0 < \varepsilon < \nu$ and denote by $\tilde{L}^{(m)}$ the operators

$$\tilde{L}^{(m)} = -\sum_{i,j=1}^{n} \tilde{a}_{ij}^{(m)} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c \qquad (m = 1, 2, \ldots) \tag{22}$$

it is easy to prove, through the proof of lemma 1, that the inequality

$$\|u_{xx}\|_{L_2(\Omega)} \leqslant (\nu - \varepsilon)^{-1} \|\tilde{L}^{(m)} u + \lambda u\|_{L_2(\Omega)} \qquad (m = 1, 2, \ldots) \tag{23}$$

is verified for any $u\in H^2(\Omega)\cap H_0^1(\Omega)$ and for any $\lambda\geqslant\lambda_0$ uniformly with respect to m (that is, there exists a constant λ_0 not depending on m such that (23) is satisfied for any $\lambda\geqslant\lambda_0$). Let us define the operators

$$L^{(m)}=-\sum_{i,j=1}^{n}\alpha_{ij}^{(m)}\frac{\partial^2}{\partial x_i\,\partial x_j}+\sum_{i=1}^{n}b_i\frac{\partial}{\partial x_i}+c \qquad (m=1,2,\dots). \tag{24}$$

Remembering lemma 2 with its proof and (14), (21), (23) we get

$$\|u_{xx}\|_{L_2(\Omega)}<K_4\|L^{(m)}u+\lambda u\|_{L_2(\Omega)} \qquad (m=1,2,\dots) \tag{25}$$

valid for any $u\in H^2(\Omega)\cap H_0^1(\Omega)$ and any $\lambda\geqslant\lambda^*$, with λ^* and K_4 independent on m. As $L^{(m)}$ has regular coefficients, from known theorems and (25) the Dirichlet problem

$$\begin{cases} L^{(m)}u^{(m)}+\lambda u^{(m)}=f & \text{a.e. in } \Omega,\\ u^{(m)}-z\in H^2(\Omega)\cap H_0^1(\Omega) & (m=1,2,\dots)\end{cases} \tag{26}$$

has one and only one solution $u^{(m)}$ as soon as $\lambda\geqslant\lambda^*$. From (25) we get the existence of a sequence extracted from $\{u^{(m)}\}_{m\in N}$ which converges weakly in $H^2(\Omega)$ to a function w such that

$$\begin{cases} gLw+\lambda w=f & \text{a.e. in } \Omega,\\ w-z\in H^2(\Omega)\cap H_0^1(\Omega)\,. \end{cases} \tag{27}$$

This can be easily verified by passing to the limit for $m\to+\infty$ in (26). The uniqueness of the solution w is a direct consequence of lemma 2.

Finally let us consider the case $f\leqslant 0$, $c\geqslant 0$ a.e. in Ω, $z\leqslant M$ on $\partial\Omega$. For known results (see e.g. [11]) we have

$$u^{(m)}\leqslant M \quad \text{in } \Omega \qquad (m=1,2,\dots) \tag{28}$$

and since $u^{(m)}$ converges weakly to w we get $w\leqslant M$ a.e. in Ω. ∎

LEMMA 4. – *Let us suppose that the coefficients a_{ij} of L satisfy condition* (4) *and that $c\geqslant 0$ a.e. in Ω. Then there exists $g\in G$ such that among all the eigenvalues of the operator $-gL$ there is one, say λ_1, with maximum real part. Besides, λ_1 is real and is the infimum of the real numbers λ such that: $(gL+\lambda I)u\leqslant 0$ a.e. in Ω, $u\in H^2(\Omega)\cap H_0^1(\Omega)$ implies $u\leqslant 0$ a.e. in Ω.*

PROOF. – Let g be chosen as in lemma 2. From lemma 3 if λ is sufficiently large there exists the inverse operator $(gL+\lambda I)^{-1}$ from $L_2(\Omega)$ to $H^2(\Omega)\cap H_0^1(\Omega)$. So the resolvent set of $-gL$ is not empty and since $H^2(\Omega)\cap H_0^1(\Omega)$ is compact in $L_2(\Omega)$ the spectrum of $-gL$ is discrete and countable. From lemma 3 this spectrum has empty intersection with the set $\{\lambda: \lambda \in R,\ \lambda \geqslant \lambda^*\}$. Moreover, if we set $G_\mu = (gL + \mu I)^{-1}$ when $\mu \geqslant \lambda^*$, lemma 3 proves that it turns out $G_\mu f \leqslant 0$ a.e. in Ω if $f \in L_2(\Omega)$, $f \leqslant 0$ a.e. in Ω.

From known results ([7], Theorem 6.1, p. 262) there exists a real eigenvalue t_1 of G_μ having maximum modulus among all the eigenvalues of G_μ:

$$|t| \leqslant t_1 \qquad \forall t \text{ eigenvalue of } G_\mu. \tag{29}$$

Moreover it is easy to see that if λ is an eigenvalue of the operator $-gL$, the number $t = (\mu - \lambda)^{-1}$ is an eigenvalue of the operator G_μ and conversely. Therefore, if we put $t_1 = (\mu - \lambda_1)^{-1}$, (29) yields

$$|\mu - \lambda| \geqslant \mu - \lambda_1 \qquad \forall \lambda \text{ eigenvalue of } -gL. \tag{30}$$

Since (30) is valid for any sufficiently large μ, we can let μ tend to $+\infty$ in it and get

$$\operatorname{Re} \lambda \leqslant \lambda_1 \qquad \forall \lambda \text{ eigenvalue of } -gL. \tag{31}$$

It remains to show that λ_1 is characterized as the present lemma claims.

Let us consider the following set of real numbers:

$$B = \{\lambda: gLu + \lambda u \leqslant 0 \text{ a.e. in } \Omega,\ u \in H^2(\Omega) \cap H_0^1(\Omega) \Rightarrow u \leqslant 0 \text{ a.e. in } \Omega\}\,.$$

This set B has the properties:

i) B contains the half line $\{\lambda: \lambda \geqslant \lambda^*\}$ (see lemma 3).

ii) B is open on the left (for this argument see [10]). In fact let μ be in B and $0 < \mu - \lambda < \|G_\mu\|^{-1}$, then there exists G_λ and

$$G_\lambda = \sum_{j=0}^{\infty} (\mu - \lambda)^j G_\mu^{j+1}$$

whence $\lambda \in B$.

iii) If $\mu \in \bar{B}$ and μ is not an eigenvalue of $-gL$, then $\mu \in B$. In fact in this case it is easy to verify that $\lim\limits_{B \ni \lambda \to \mu} \|G_\lambda - G_\mu\| = 0$ (see again [10]).

This is sufficient to conclude that B is an open half line whose right extreme is an eigenvalue, therefore $B = \{\lambda: \lambda > \lambda_1\}$. ∎

5. – Proof of Theorem 1.

It is sufficient to show that there exists a positive constant c_0 depending on n, Ω and the coefficients a_{ij}, b_i of L such that if $c \geqslant c_0$ a.e. in Ω the operator gL is invertible for a suitable $g \in G$. In fact it is clear that problem (2) is equivalent to the following

$$(32) \qquad \begin{cases} gLu = gf & \text{a.e. in } \Omega, \\ u \in H^2(\Omega) \cap H_0^1(\Omega) . \end{cases}$$

Let us choose g as is lemma 2. Consider the operator

$$L_0 = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}$$

i.e. the operator L where we set $c \equiv 0$. From lemma 4 among all the eigenvalues of $-gL_0$ there exists one, denoted by $\hat{\lambda}$, with maximum real part. Now let us suppose $\operatorname{ess\,inf}_{\Omega} c > \operatorname{ess\,sup}_{\Omega} \hat{\lambda}/g$ and let us show that in this case λ_1 (i.e. the eigenvalue of $-gL$ having maximum real part) is negative. From Theorem 6.1 of [7] there exists a non negative eigenfunction w_1 corresponding to λ_1: $w_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, $w_1 \geqslant 0$ a.e. in Ω, w_1 not identically zero in Ω,

$$(33) \qquad gLw_1 + \lambda_1 w_1 \equiv gL_0 w_1 + gcw_1 + \lambda_1 w_1 = 0 \qquad \text{a.e. in } \Omega .$$

Now choose λ such that $\hat{\lambda} < \lambda \leqslant gc$. If it were $\lambda_1 \geqslant 0$ we should get from (33)

$$gL_0 w_1 + \lambda w_1 = -\lambda_1 w_1 + (\lambda - gc) w_1 \leqslant 0 \qquad \text{a.e. in } \Omega$$

and from Lemma 4, applied to the operator L_0, this would imply $w_1 \leqslant 0$ a.e. in Ω, a contradiction. Therefore $\lambda_1 < 0$ and from lemma 4 we get that $u \in H^2(\Omega) \cap H_0^1(\Omega)$, $gLu \leqslant 0$ a.e. in Ω implies $u \leqslant 0$ a.e. in Ω.

So Theorem 1 is proven taking for c_0 any number greater than $\operatorname{ess\,sup} \hat{\lambda}/g$. ∎

REMARK. – If $n = 3$ we can take any positive number as c_0 in Theorem 1. This can be proven exactly as in [3]. As far as I know, the problem of extending this result to $n \geqslant 4$ is open. It would be sufficient to know whether for other values of n the eigenfunctions of the operator L are sufficiently regular.

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Note added in proofs (December 27, 1971).

Other boundary value problems (NEUMANN, oblique derivative) for the same kind of equations will be considered in a subsequent paper. On that occasion condition (4) will be expressed differently and its local character will be proven. In this connection the following work must be added to the references:

M. GIAQUINTA, *Equazioni ellittiche di ordine 2m di tipo Cordes*, Boll. Un Mat. Ital., (4) **4** (1971), pp. 251-257.