On a Duality of Modules over Valuation Rings (*) (**).

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Sunto. – *Rispondendo ad una questione posta da A. Orsatti in* [8], *si dà esempio di dualità, non soddisfacente alle proprietà di estensione di caratteri, sopra un anello di valutazione discreto completo e si ottiene una caratterizzazione dei domini noetheriani per cui una tale dualità esiste.*

Let R denote a commutative ring with 1. All modules considered in this note are unital R-modules.

In his paper [8], ORSATTI considers, for any R-module E, two classes of R-modules: $\mathfrak{D}(E)$ consists of all R-submodules of products of copies of E, while $\mathfrak{C}(E)$ consists of all closed R-submodules of topological products of copies of E where E is viewed in the discrete topology. A character of $M \in \mathfrak{D}(E)\,[\in \mathfrak{C}(E)]$ is an R-homomorphism [continuous R-homomorphism] of M into the discrete module E.

The dual M^* of M is the R-module of characters of M; if $M \in \mathfrak{D}(E)$, it is given the topology of simple convergence and $M^* \in \mathfrak{C}(E)$, while if $M \in \mathfrak{C}(E)$, then M^* carries no topology and belongs to $\mathfrak{D}(E)$. There is a canonical homomorphism $\omega_M\colon M \to M^{**}$ acting via $x \mapsto \tilde{x}$ $(x \in M)$ where $\tilde{x}\colon \xi \mapsto \xi(x)$ $(\xi \in M^*)$. If ω_M is a (topological) isomorphism for every M in $\mathfrak{D}(E)$ and in $\mathfrak{C}(E)$, we then say that the pair (R, E) defines a *duality*.

This duality has been investigated by Orsatti in order to get a general setting for several dualities studied previously (see the bibliography of [8]). His main concerns were dualities with the socalled extension property of characters, and he raised the question of existence of dualities without this property. Our purpose here is to show that there is such a duality, namely (R, R) where R is a complete discrete valuation ring. We complement this by showing that if R is a noetherian domain and (R, R) defines a duality, then R has to be a complete discrete valuation ring.

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Let R be a complete discrete valuation ring and let Rp be its unique maximal ideal. The field Q of quotients of R as well as $K = Q/R$ are injective R-modules. Every rank 1 torsion-free R-module is isomorphic either to R or to Q.

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It is easy to see that for a complete discrete valuation ring R, the class $\mathfrak{D}(R)$ consists of all torsion-free R-modules which are reduced (i.e. they do not contain any copy of Q). R is a cotorsion R-module (see MATLIS [6]) whence we deduce that every finite rank pure submodule H of an R-module $M \in \mathfrak{D}(R)$ is a free summand of M (H pure means M/H torsion-free).

In order to get acquainted with our class $\mathfrak{C}(R)$, let us note that every $M \in \mathfrak{C}(R)$ is algebraically isomorphic to a product of copies of R. To verify this, we consider R both in its discrete topology and in its p-adic topology; we use the notations R and R_0, respectively, to distinguish between these topologies. Both topologies make R linearly compact. Now any $M \in \mathfrak{C}(R)$ can also be considered as a topological submodule M_0 of a product $\prod R_0$ with the relative topology. Clearly, $1_M\colon M \to M_0$ is a continuous map, and since M is linearly compact, so is M_0. Kaplansky's duality [4] implies that M_0 is topologically isomorphic to $\mathrm{Hom}_R(N, K)$ for some discrete R-module N. Since M is reduced torsion-free, N cannot have any summands isomorphic to R, Q or R/Rp^n for any $n \geqq 1$. Hence N is a direct sum of copies of K whence M algebraically is isomorphic to a product of $\mathrm{Hom}_R(K, K) = R$.

We now exhibit an example for an Orsatti type duality without the extension property.

THEOREM 1. – *For any complete discrete valuation ring R, (R, R) defines a duality.*

Before entering into the proof, let us note right away that ω_M is always an algebraical and topological isomorphism between M and $\omega_M(M)$ (see ORSATTI [8]); this is an easy consequence of the definitions of the classes. Thus it suffices to show that ω_M is always surjective.

First, let $M \in \mathfrak{D}(R)$ and $\alpha \in M^{**}$. Thus α is a continuous character $M^* \to R$, so its kernel contains the annihilator $H^\perp$ of some finitely generated submodule H of M. Since R is torsion-free, H can be replaced by its pure closure; in other words, we can assume that H is a pure submodule of finite rank in M. Then H is a free summand of M, $M = H \oplus N$ for some submodule N of M. Thus $M^* = H^* \oplus N^*$ where $N^* = H^\perp$. Let $\pi\colon M^* \to H^*$ be the projection map with kernel $H^\perp$, and β a character of H^* satisfying $\alpha = \beta\pi$. Duality evidently holds for H, so $\beta = \tilde{x}$ for some $x \in H$. Writing $\xi \in M^*$ in the form $\xi = \xi_H + \xi_N$ with $\xi_H \in H^*$, $\xi_N \in N^*$, we have:

$$\alpha(\xi) = \beta\pi(\xi_H + \xi_N) = \beta\xi_H = \tilde{x}(\xi_H) = \xi_H(x) = \xi(x) = \tilde{x}(\xi)\,,$$

i.e. α is induced by some $x \in H$; thus ω_M is surjective.

Secondly, let $M \in \mathfrak{C}(R)$. We prove that $\omega_M(M)$ is dense in M^{**}; since $\omega_M(M)$ as a continuous image of a linearly compact module M is closed in M^{**}, this will suffice to establish that ω_M is surjective.

We show that if $\alpha \in M^{**}$ and F is any finitely generated submodule of M^*, then some $x \in M$ satisfies $\alpha - \tilde{x} \in F^\perp$, i.e. $\alpha|F = \tilde{x}|F$. Without loss of generality, F can be assumed to be pure in, and hence a summand of M^*. Let $j\colon F \to M^*$ be the inclusion map, so that $j^*\colon M^{**} \to F^*$ is the restriction map $\alpha \mapsto \alpha|F$. By the stacked basis theorem on P.I.D., we argue that the free R-module $F^* \cong F$ has a basis $\eta_1^*, \ldots, \eta_n^*$ such that for suitable $r_1, \ldots, r_n \in R$, $r_1\eta_1^*, \ldots, r_n\eta_n^*$ form a basis of $j^*(\omega M)$. Let $\eta_1, \ldots, \eta_n$ be the dual basis of F, i.e. $\eta_i^*(\eta_k) = \delta_{ik}$ for $i, k = 1, \ldots, n$. Then, for every i, we have

$$j^*(\omega(M))(\eta_i) = r_i\eta_i^* R\eta_i = Rr_i\,.$$

Observe that each η_i is surjective, since $R\eta_i$ is pure in M^* (if $\operatorname{Im}\eta_i = p^kR\,(k \geqq 1)$, then $p^{-k}\eta_i$ would again be a character of M). Consequently,

$$j^*(\omega(M))(\eta_i) = \{\tilde{x}(\eta_i)\,|\,x \in M\} = \{\eta_i(x)\,|\,x \in M\} = \eta_i(M) = R\,.$$

We conclude that all the r_i are units of R, whence $j^*(\omega(M)) = F^*$ follows at once. This establishes our claim that $\alpha|F$ is equal to some $\tilde{x}|F$. The proof of Theorem 1 is completed.

Note that CHASE [1] has established an interesting duality between certain modules in $\mathfrak{D}(R)$ and all modules in $\mathfrak{C}(R)$ where R is a P.I.D. His correspondence $M \mapsto M^*$ for $M \in \mathfrak{D}(R)$ is canonical, but this is not the case for $M \in \mathfrak{C}(R)$. If R is a complete discrete valuation ring, then Chase's result gives a duality between all of $\mathfrak{D}(R)$ and $\mathfrak{C}(R)$, and in order to derive our theorem 1 from his result it suffices to show that our ω_M is an isomorphism for every $M \in \mathfrak{C}(R)$. He proved that for each such M there is an $N \in \mathfrak{D}(R)$ along with a topological isomorphism $\varphi\colon N^* \to M$. Thus in the commutative diagram

$$\begin{array}{ccc} N^* & \xrightarrow{\ \varphi\ } & M \\ {\scriptstyle \omega_{N^*}}\big\downarrow & & \big\downarrow{\scriptstyle \omega_M} \\ N^{***} & \xrightarrow[\ \varphi^{**}\ ]{} & M^{**} \end{array}$$

the maps φ, φ^{**} and ω_{N^*} are all isomorphisms. Therefore, ω_M is likewise an isomorphism.

Now we turn our attention to the converse result.

THEOREM 2. – *Suppose that R is a commutative noetherian domain such that (R, R) is a duality. Then*

either R *is a field* (*and the duality is the one of Lefschetz's* [5])

or R *is a complete discrete valuation ring* (*and the duality is the one above*).

To prove this, suppose (R, R) yields a duality and R is not a field. We verify Theorem 2 through a number of steps.

1) *R is reflexive in the sense of* MATLIS [7], i.e. for all R-modules M of finite rank in $\mathfrak{D}(R)$, ω_M is an isomorphism (here no topology is involved in forming M^*).

By induction of the rank, it follows readily that M is finitely generated, hence its dual $M^* \in \mathfrak{C}(M)$ carries the discrete topology. We are therefore justified in viewing our M^{**} as the R-module of all R-homomorphisms of M^* into R. Thus if (R, R) is a duality, then R is reflexive.

2) *Let H be a finite rank submodule of $M \in \mathfrak{D}(R)$ such that M/H is torsion-free. Then every character of H extends to a character of M.*

The sequence $0 \to \mathrm{Hom}_R(M/H, R) = H^\perp \to M^* \xrightarrow{j^*} H^*$ is exact where $j: H \to M$ stands for the inclusion map; here j^* is continuous. By duality, there is a finite rank R-module $\boldsymbol{C}$ such that $\mathrm{Im}\, j^* = \boldsymbol{C}^*$, and the surjection $f: M^* \to \boldsymbol{C}^*$ induces an injection $f^*: \boldsymbol{C} = \boldsymbol{C}^{**} \to M$ (where f, if followed by the injection $i: \boldsymbol{C}^* \to H^*$, equals j^*). If $\boldsymbol{C}^*$ had a smaller rank than H^*, then some non-zero submodule of H would be annihilated by all characters of M, which is impossible. Thus $H^*/\boldsymbol{C}^*$ is a torsion module and $i^*: H \to \boldsymbol{C}$ is an injection. Since M cannot contain any submodule of the same rank as H that properly contains H, it follows from $j = f^* i^*$ that i^* is an isomorphism and $j^* = if$ is surjective.

Recall that a torsion-free module M over an integral domain R is called *slender* if every homomorphism

$$\varphi: R^{\boldsymbol{N}} \to M$$

satisfies $\varphi e_n \neq 0$ for at most a finite number of indices n; here $e_n \in R^{\boldsymbol{N}}$ denotes the vector whose coordinates are 0 except for the n-th coordinate which is 1. It also follows that $\varphi P_m = 0$ for some $m \in \boldsymbol{N}$ where P_m consists of all vectors in $R^{\boldsymbol{N}}$ whose first m coordinates are 0. If these $P_m (m \in \boldsymbol{N})$ are taken as a base of neighborhoods for a linear topology τ of $R^{\boldsymbol{N}}$, then $(R^{\boldsymbol{N}}, \tau)$ is a non-discrete, metrizable complete R-module.

3) *R is not slender as an R-module.*

By way of contradiction, suppose R is slender and let $a \in R$ be a non-unit in R, $a \neq 0$. The R-module

$$M = aR^{\boldsymbol{N}} + R^{(\boldsymbol{N})}$$

is a submodule of RI that consists of all vectors almost all of whose coordinates are multiples of a. It is evident that $H = R(a, \ldots, a, \ldots)$ is a rank 1 submodule of M such that M/H is torsion-free. The character $\varphi: H \to R$ which sends $(a, \ldots, a, \ldots)$ onto 1 extends, by 2), to a character $\bar{\varphi}: M \to R$. Clearly $\bar{\varphi}$ induces a homomorphism $a \cdot R^{\boldsymbol{N}} \to R$. Since $a \cdot R^{\boldsymbol{N}} \cong R^{\boldsymbol{N}}$ and R is slender, there is an $m \in \boldsymbol{N}$ such that $\bar{\varphi}\ aP_m = 0$. By torsion-freeness, φ has to annihilate all vectors in M with 0 first m coordinates. Setting $\varphi(e_1 + \ldots + e_m) = b \in R$, we see that $\varphi(a, \ldots, a, \ldots) = \varphi(ae_1 + \ldots + ae_m) = ab$. This has to be 1, in contradiction to a being a non-unit.

For the next result, see DE MARCO and ORSATTI [2] and HEINLEIN [3].

4) *R is complete (Hausdorff) in a non-discrete metrizable linear topology σ where a base of neighborhoods about 0 is formed by principal ideals.*

By 3), R is not slender, thus there is a homomorphism $\varphi: R^N \to R$ such that $\varphi e_n = r_n \neq 0$ for every $n \in \boldsymbol{N}$. Choose again some $a \neq 0$ in R which is not a unit of R. There is an endomorphism η of R^N acting as

$$\eta(x_1, \dots, x_n, \dots) = (x_1, ar_1x_2, \dots, a^{n-1}r_1 \dots r_{n-1}x_n, \dots) .$$

Then the map $\varphi\eta: R^N \to R$ also satisfies $\varphi\eta e_n \neq 0$ for every n. The ideals $I_n = \varphi\eta P_n = a^{n-1}r_1 \dots r_{n-1}R$ form a descending chain with 0 intersection (since $\varphi\eta P_n \subseteq Ra^n$ and $\bigcap_n Ra^n = 0$ in a noetherian domain). If R is now furnished with the linear topology σ where the principal ideals I_n form a base of neighborhoods about 0, then $\varphi\eta$ becomes a continuous open map of the complete R-module (R^N, τ) into (R, σ); hence (R, σ) is complete.

5) *R is complete in the R-topology.*

Let (R, ϱ) be the topological R-module R, furnished with the R-topology. Then the identity map of R is a continuous map $(R, \varrho) \to (R, \sigma)$, thus 1_R has a unique extension f to the respective completions: $f: (\hat{R}, \hat{\varrho}) \to (R, \sigma)$. We infer that there is an algebraic direct decomposition $\hat{R} = R \oplus X$ for some R-module X. Since R is a torsion-free R-module, in view of Matlis [6], $\hat{\varrho}$ is the R-topology of $\hat{R}$. Thus the last direct decomposition holds in the topological sense, too, and so R is complete in the R-topology.

Let us note that HEINLEIN [3] also derives the completeness of R in the R-topology from the property stated in 4); the simple proof given here is due to A. ORSATTI.

6) *R is a complete discrete valuation ring.*

By 5), R is complete in the R-topology, so by MATLIS [6], R is cotorsion and so are its ideals. By MATLIS [7], R reflexive and complete in the R-topology implies that every torsion-free R-module of finite rank is a submodule of a free R-module of finite rank provided that it does not contain any copy of the field of quotients of R. Hence it follows by induction on the rank that all torsion-free R-modules of finite rank are cotorsion. We refer again to a result by Matlis [6] to conclude that R is a complete discrete valuation ring.

This completes the proof of Theorem 2.

Needless to say, if the restriction to domains in the above discussion is removed, then we can get more dualities. For instance, R can be a product of complete discrete valuation rings.

REFERENCES

[1] S. U. CHASE, *Function topologies on abelian groups*, Illinois J. Math., **7** (1963), pp. 593-608.

[2] G. DE MARCO - A. ORSATTI, *Complete linear topologies on abelian groups*, Symposia Math., **13** (1974), pp. 153-161.

[3] G. HEINLEIN, *Vollreflexive Ringe und schlanke Moduln* (Dissertation), Erlangen, 1971.

[4] I. KAPLANSKY, *Dual modules over a valuation ring, I*, Proc. Amer. Math. Soc., **4** (1953), pp. 213-219.

[5] S. LEFSCHETZ, *Algebraic topology*, Coll. Lect. Amer. Math. Soc., **27** (1942).

[6] E. MATLIS, *Cotorsion modules*, Memoirs Amer. Math. Soc., **49** (1964).

[7] E. MATLIS, *Torsion-free modules*, Chicago Lecture Notes in Math. (1972).

[8] A. ORSATTI, *Dualità per alcune classi di moduli E-compatti*, Annali di Matematica, (4), **113** (1977), pp. 211-235.