Periodic Solutions and Homogenization of Non Linear Variational Problems (*) (**).

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Summary. – *We prove an homogenization formula for some non linear variational problems that extends the analogous one known in the linear case. Namely the solution* u_ε *of the problem*

$$\int_\Omega \left\{ f\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) - \varphi(x) u_\varepsilon \right\} dx = \text{minimum},$$

where the boundary data are independent of ε *and* $f = f(x, \xi)$ *is periodic in* x, *converges in some* L^p *space as* ε *goes to zero to the solution of an analogous variational problem whose integrand can be evaluated from* f.

1. – Introduction.

Let us recall what homogenization means in the simplest case.

Let $[a_{ij}(x)]$ be a L^∞ symmetric matrix such that $\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$ for some $\lambda > 0$ and any $\xi \in \boldsymbol{R}^N$. Assume $[a_{ij}(x)]$ to be Y-periodic, where $Y = \{x \equiv (x_i) \in \boldsymbol{R}^N: 0 < x_i < \bar{y}_i\}$ and $\bar{y} \equiv (\bar{y}_i)$ $(\bar{y}_i > 0)$ is fixed in $\boldsymbol{R}^N$, namely for any x

$$a_{ij}(x + \bar{y}) = a_{ij}(x) \qquad \forall i, j\,.$$

If Ω is an open bounded set in $\boldsymbol{R}^N$, $\varphi \in L^2(\Omega)$ and $\varepsilon > 0$, we denote by u_ε the unique variational solution of the Dirichlet problem

$$(1.1) \qquad -\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial x_j}\right) = \varphi \quad \text{in } \Omega\,, \qquad u_\varepsilon = 0 \quad \text{on } \partial\Omega\,.$$

It can be proved that there exists a constant symmetrix positive-definite matrix $[\alpha_{ij}]$ such that u_ε converges in $L^2(\Omega)$ as $\varepsilon \to 0$ to the function u, solution of the problem

$$(1.2) \qquad -\sum_{i,j} \alpha_{ij}\frac{\partial^2 u}{\partial x_i\,\partial x_j} = \varphi \quad \text{in } \Omega\,, \qquad u = 0 \quad \text{on } \partial\Omega\,.$$

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The matrix $[\alpha_{ij}]$ is generally different from the matrix

$$\left[\frac{1}{\text{meas}\,(Y)}\int_Y a_{ij}(x)\,dx\right]$$

and can be evaluated using the formula

$$\sum_{i,j}\alpha_{ij}\xi_i\xi_j = \frac{1}{\text{meas}\,(Y)}\int_Y a_{ij}(x)\left(\frac{\partial\tilde{u}}{\partial x_i}+\xi_i\right)\left(\frac{\partial\tilde{u}}{\partial x_j}+\xi_j\right)dx \tag{1.3}$$

where $\tilde{u}$ is the unique (up to an additive constant) Y-periodic solution of the equation

$$-\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial\tilde{u}}{\partial x_j}\right)=\sum_{i,j}\frac{\partial}{\partial x_i}(a_{ij}(x)\,\xi_j)\quad\text{ in } Y. \tag{1.4}$$

The procedure of going from (1.1) to (1.2) is said *homogenization*. It has been introduced by SANCHEZ PALENCIA [20], [21] to give a scheme of composite physical structures by means of homogeneous structures with «equivalent» characteristics. For a more detailed physical motivation of it see also BABUŠKA [2], [3].

This mathematical problem has been studied, besides BABUŠKA and SANCHEZ PALENCIA, by BENSOUSSAN-LIONS-PAPANICOLAOU [7], DE GIORGI-SPAGNOLO [13], LIONS [14], [15], MARCELLINI-SBORDONE [17], SPAGNOLO [25], TARTAR [26]. Some variants have been considered; for instance: non symmetric matrices [7], [15], [26]; semi-definite matrices [17]; higher order operators [8]; non linear lower order terms [1], [8], [15], [16], [22]; quasilinear operators [2]; systems [4]; obstacle problems [5], [9], [10], [14], [19]; evolution problems [6], [11], [12], [14], [22], [24].

In this paper we give an approach to the study of homogenization of non linear variational problems. We describe briefly the situation. Let $f\colon \boldsymbol{R}^N\times\boldsymbol{R}^N\to\boldsymbol{R}$ be a function such that

$$\begin{cases} f(x,\xi) \text{ is measurable in } x, \text{ strictly convex and } C^1 \text{ in } \xi; \\ \lambda|\xi-\xi_0|^p\le f(x,\xi)\le \Lambda_1+\Lambda_2|\xi|^p & \forall x,\xi; \\ |f(x,\xi)|^{1/p}-f(x,\xi')^{1/p}|\le L|\xi-\xi'| & \forall x,\xi,\xi'; \end{cases} \tag{1.5}$$

where $p>1$, $\lambda, \Lambda_2, L>0$, $\Lambda_1\ge 0$ and $\xi_0\in\boldsymbol{R}^N$. We also suppose that f is Y-periodic in x, that is

$$f(x+\bar{y},\xi)=f(x,\xi)\,,\qquad \forall x,\xi\,.$$

For Ω bounded open set in $\boldsymbol{R}^N$, $\varepsilon>0$, $\varphi\in L^{p'}(\Omega)$ $(p^{-1}+p'^{-1}=1)$, let u_ε be the unique solution of the Dirichlet problem

$$\int_\Omega\left\{f\left(\frac{x}{\varepsilon}, Du_\varepsilon\right)-\varphi(x)\,u_\varepsilon\right\}dx = \text{minimum on } H_0^{1,p}(\Omega)$$

(other variational boundary conditions are also considered). We prove that there exists a function $g = g(\xi)$ independent of φ and verifying (1.5) such that u_ε converges in $L^p(\Omega)$, as ε goes to zero, to the unique solution u of the problem

$$\int_\Omega \{g(Du) - \varphi(x)u\}\, dx = \text{minimum on } H_0^{1,p}(\Omega)\,.$$

The function g can be evaluated by solving a variational problem relative to the integral of $f(x, Dv)$ over the period Y whose Euler equation is of the type (1.4). Such a result (Theorem 4.4), already announced in [18], generalizes the known theory relative to the linear case described at the beginning.

To obtain the homogenization Theorem 4.4 we are led to study some properties of periodic solutions of variational problems. Roughly speaking, if $f(\cdot, \xi)$ and φ are Y-periodic, any solution u to

$$\int_Y \{f(x, Du) - \varphi(x)\, u\}\, dx = \text{minimum on } v \in H^{1,p}_{\text{loc}}\,,\ v\ Y\text{-periodic}\,,$$

is also a solution to

$$\int_{kY} \{f(x, Du) - \varphi(x)\, u\}\, dx = \text{minimum on } v \in H^{1,p}_{\text{loc}}\,,\ v\ kY\text{-periodic}\,,$$

for any integer $k > 1$. This matter is thoroughly investigated in section 2.

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2. – Periodic solutions.

In this section we consider real functions $f = f(x, u, \xi)$ with $x \in \boldsymbol{R}^N$, $u \in \boldsymbol{R}$, $\xi \in \boldsymbol{R}^N$ such that

$$\text{(2.1)} \qquad \begin{cases} f(x, u, \xi) \text{ is measurable in } x, \text{ convex and } C^1 \text{ in } (u, \xi); \\ f(x + \bar{y}, u, \xi) = f(x, u, \xi)\,, \qquad \forall x, u, \xi; \\ \lambda|\xi - \xi_0|^p \le f(x, u, \xi) \le \Lambda_1 + \Lambda_2(|u|^p + |\xi|^p)\,, \qquad \forall x, u, \xi; \end{cases}$$

where $\bar{y} \equiv (\bar{y}_i) \in \boldsymbol{R}^N$ $(\bar{y}_i > 0)$, $\xi_0 \in \boldsymbol{R}^N$, $p > 1$, $\lambda, \Lambda_2 > 0$, $\Lambda_1 \ge 0$.

For any bounded open subset Ω of $\boldsymbol{R}^N$ let

$$\text{(2.2)} \qquad F(\Omega, v) = \int_\Omega f(x, v, Dv)\, dx\,, \qquad \forall v \in H^{1,p}(\Omega)\,.$$

F is a real convex lower semicontinuous functional on $H^{1,p}(\Omega)$.

If $Y = \{x \equiv (x_i) \in \boldsymbol{R}^N : 0 < x_i < \bar{y}_i\}$ we denote by $W(Y)$ the completion of the set of $C^1(\overline{Y})$ Y-periodic functions with respect to the norm

$$\|v\|_{W(Y)} = \|v\|_{H^{1,p}(Y)} = \left\{\int_Y (|u|^p + |Du|^p)\, dx\right\}^{1/p}.$$

We may assume a function $v \in W(Y)$ defined all over $\boldsymbol{R}^N$ by periodicity.

Finally if k is a positive real number we put $kY = \{x \equiv (x_i) \in \boldsymbol{R}^N : 0 < x_i < k\bar{y}_i\}$.

THEOREM 2.1. – *Let F be as in* (2.2) *with f verifying* (2.1). *If u minimizes $F(Y, v)$ on $W(Y)$ then u minimizes $F(kY, v)$ on $W(kY)$ for every integer $k \geq 1$.*

REMARK 2.2. – By (2.1) there exists a (not necessarily unique) function u that minimizes $F(Y, v)$ on $W(Y)$. We use the conditions $p > 1$ and $f(x, u, \xi) \geq \lambda|\xi - \xi_0|^p$ only at this point, so that it may be replaced by any other coercivity condition which assures the existence of a minimum on $W(Y)$.

Before proving Theorem 2.1 we state two lemmas.

LEMMA 2.3. – *We suppose that f verifies* (2.1) *and F is defined by* (2.2). *If u minimizes $F(Y, v)$ on $W(Y)$ then for every integer $k > 0$ we have*

$$F(kY, u) \leq F(kY, u + \varphi), \qquad \forall \varphi \in H_0^{1,p}(kY).$$

PROOF. – As $F(Y + y, v) = F(Y, v)$ for any $v \in W(Y)$ and any $y \in \boldsymbol{R}^N$ we have

$$F(Y + y, u) \leq F(Y + y, v), \qquad \forall v \in W(Y) = W(Y + y), \ \forall y \in \boldsymbol{R}^N.$$

The corresponding Euler equation is

$$\int_{Y+y} \left\{\sum_i f_{\xi_i}(x, u, Du)\, \psi_{x_i} + f_u(x, u, Du)\, \psi\right\} dx = 0, \qquad \forall \psi \in W(Y + y). \tag{2.3}$$

In particular (2.3) is true for $\psi \in H_0^{1,p}(Y + y)$. If $\varphi \in H_0^{1,p}(\boldsymbol{R}^N)$ we can find a finite number of vectors $y_r \in \boldsymbol{R}^N$ such that $\operatorname{supp} \varphi \subset \bigcup_r (Y + y_r)$. If $\{h_r\}$ is a partition of unity on $\operatorname{supp} \varphi$ subordinate to the cover $\{Y + y_r\}$, putting $\psi_r = \varphi h_r \in H_0^{1,p}(Y + y_r)$ in (2.3) and adding up with respect to r, we obtain

$$\int_{\boldsymbol{R}^N} \left\{\sum_i f_{\xi_i}(x, u, Du)\, \varphi_{x_i} + f_u(x, u, Du)\, \varphi\right\} dx = 0, \qquad \forall \varphi \in H_0^{1,p}(\boldsymbol{R}^N). \tag{2.4}$$

Fixed $k \geq 1$, by the convexity of $F(kY, v)$ on $W(kY)$, for any $v \in W(kY)$ we have

$$F(kY, v) \geq F(kY, u) + \int_{kY} \left\{\sum_i f_{\xi_i}(x, u, Du)(v - u)_{x_i} + f_u(x, u, Du)(v - u)\right\} dx. \tag{2.5}$$

Now if $\varphi \in H_0^{1,p}(kY)$, from (2.5) with $v = u + \varphi$ and (2.4) the conclusion easily follows.

LEMMA 2.4. – *Assume that f verifies* (2.1) *and F is as in* (2.2). *If u and v belong to $W(Y)$, for any positive ε there exist a positive integer k and a function $\varphi \in H_0^{1,p}(kY)$ such that*

$$|F(kY, v) - F(kY, u + \varphi)| < \varepsilon \operatorname{meas}(kY).$$

PROOF. – For each integer $k \geq 3$ we consider the set $\Omega_k = \{x \equiv (x_i) \in \mathbf{R}^N: y_i \leq x_i \leq (k-1)y_i\}$. $kK \setminus \Omega_k$ is contained in the union of $k^N - (k-2)^N$ sets obtained from $\overline{Y}$ somewhat translating it. We denote by Y' a generic of such sets. Let $\varphi_k = v - u$ on Ω_k. As $v - u$ is Y-periodic, it is possible to extend φ_k to kY so that $\varphi_k = 0$ on the boundary of kY and $\|\varphi_k\|_{H^{1,p}(Y')} \leq \|v - u\|_{H^{1,p}(Y)}$. With this choise we have

$$\int_{kY \setminus \Omega_k} \left(|\varphi_k|^p + |D\varphi_k|^p\right) dx \leq \left(k^N - (k-2)^N\right) \|v - u\|^p_{H^{1,p}(Y)} . \tag{2.6}$$

Moreover, as $v = u + \varphi_k$ on Ω_k,

$$F(kY, v) - F(kY, u + \varphi_k) = \int_{kY \setminus \Omega_k} \{f(x, v, Dv) - f(x, u + \varphi_k, D(u + \varphi_k))\}\, dx . \tag{2.7}$$

We estimate separately the addenda in the right-hand side:

$$\int_{kY \setminus \Omega_k} f(x, v, Dv)\, dx = \left(k^N - (k-2)^N\right) \int_Y f(x, v, Dv)\, dx ; \tag{2.8}$$

$$\begin{aligned} \int_{kY \setminus \Omega_k} f(x, u + \varphi_k, D(u + \varphi_k))\, dx &\leq \int_{kY \setminus \Omega_k} \{\Lambda_1 + \Lambda_2(|u + \varphi_k|^p + |D(u + \varphi_k)|^p)\}\, dx \leq \\ &\leq \left(k^N - (k-2)^N\right)\left\{\Lambda_1 \operatorname{meas}(Y) + 2^{p-1}\Lambda_2\left(\|u\|^p_{H^{1,p}(Y)} + \|v - u\|^p_{H^{1,p}(Y)} + \|v - u\|^p_{H^{1,p}(Y)}\right)\right\}. \end{aligned} \tag{2.9}$$

In (2.9) we have used the inequality $(a + b)^p \leq 2^{p-1}(a^p + b^p)$ and (2.6).
From (2.7), (2.8), (2.9) we get

$$|F(kY, v) - F(kY, u + \varphi_k)| \leq C(k^N - (k-2)^N) = C\left\{1 - \left(\frac{k-2}{k}\right)^N\right\} \frac{\operatorname{meas}(kY)}{\operatorname{meas}(Y)}, \tag{2.10}$$

where C is a constant independent of k. Since

$$\lim_{k \to +\infty} \left\{1 - \left(\frac{k-2}{k}\right)^N\right\} = 0 ,$$

(2.10) gives the assertion.

PROOF OF THEOREM 2.1. – Let $k \geqslant 1$ and $v \in W(kY)$ be fixed. We apply Lemma 2.4 with kY instead of Y. For each $\varepsilon > 0$ there exist a multiple integer k' of k and $\varphi \in H_0^{1,p}(k'Y)$ such that

$$\frac{1}{\text{meas}\,(k'Y)} F(k'Y, v) > \frac{1}{\text{meas}\,(k'Y)} F(k'Y, u + \varphi) - \varepsilon\,,$$

and, by Lemma 2.3,

$$\frac{1}{\text{meas}\,(k'Y)} F(k'Y, v) > \frac{1}{\text{meas}\,(k'Y)} F(k'Y, u) - \varepsilon\,. \tag{2.11}$$

Now, since $v \in W(kY)$ and k' is a multiple of k, it is easy to see that

$$\frac{1}{\text{meas}\,(k'Y)} F(k'Y, v) = \frac{1}{\text{meas}\,(kY)} F(kY, v)\,. \tag{2.12}$$

Formula (2.12) holds also with u instead of v. So from (2.11) we get

$$\frac{1}{\text{meas}\,(kY)} F(kY, v) > \frac{1}{\text{meas}\,(kY)} F(hY, u) - \varepsilon\,.$$

We have the assertion since ε is arbitrarily small.

3. – The functions μ_ε, γ_ε^0, γ_ε^1.

Let $f\colon \boldsymbol{R}^N \times \boldsymbol{R}^N \to \boldsymbol{R}$ be a function such that

$$\begin{cases} f(x,\xi) \text{ is measurable in } x, \text{ convex and } C^1 \text{ in } \xi; \\ f(x+\bar{y},\xi) = f(x,\xi)\,, \qquad \forall x, \xi; \\ \lambda|\xi - \xi_0|^p \leqslant f(x,\xi) \leqslant \Lambda_1 + \Lambda_2|\xi|^p\,, \qquad \forall x, \xi; \\ |f(x,\xi)^{1/p} - f(x,\xi')^{1/p}| \leqslant L|\xi - \xi'|\,, \qquad \forall x, \xi, \xi'; \end{cases} \tag{3.1}$$

with $\bar{y} \equiv (\bar{y}_i)$, $\bar{y}_i > 0$ for $i = 1, 2, \ldots, N$ and ξ_0 fixed in $\boldsymbol{R}^N$; $p > 1$; $\lambda, \Lambda_2, L > 0$ and $\Lambda_1 \geqslant 0$.

For each $\varepsilon > 0$ and $\xi \in \boldsymbol{R}^N$ we introduce the notations (with Y, εY and $W(\varepsilon Y)$ defined as in the previous section):

$$\mu_\varepsilon(\xi) = \min\left\{\frac{1}{\text{meas}\,(\varepsilon Y)} \int_{\varepsilon Y} f\left(\frac{x}{\varepsilon}, Dv + \xi\right) dx \colon v \in W(\varepsilon Y)\right\}; \tag{3.2}$$

$$(3.3)\qquad \gamma_\varepsilon^0(\xi)=\min\left\{\frac{1}{\operatorname{meas}(Y)}\int\limits_Y f\left(\frac{x}{\varepsilon}, Dv+\xi\right)dx\colon v\in H_0^{1,p}(Y)\right\};$$

$$(3.4)\qquad \gamma_\varepsilon^1(\xi)=\min\left\{\frac{1}{\operatorname{meas}(Y)}\int\limits_Y f\left(\frac{x}{\varepsilon}, Dv+\xi\right)dx\colon v\in H^{1,p}(Y),\ \int\limits_Y Dv=0\right\}.$$

It is easy to see that—for any ε—μ_ε, γ_ε^0 and γ_ε^1 are positive convex functions on $\boldsymbol{R}^N$, $\gamma_\varepsilon^0\geqslant\gamma_\varepsilon^1$, and $\gamma_1^0\geqslant\mu_1\geqslant\gamma_1^1$. Our aim in this section is to show some other relationships between these functions.

PROPOSITION 3.1. – *For any $\varepsilon>0$ and $\xi\in\boldsymbol{R}^N$ we have $\mu_\varepsilon(\xi)=\mu_1(\xi)$. Moreover, if u realizes the minimum in* (3.2) *for $\varepsilon=1$, then $u_\varepsilon(x)=\varepsilon u(x/\varepsilon)$ is a minimizing function for any other fixed ε.*

PROOF. – To each $v\in W(Y)$ we can associate $v_\varepsilon(x)=\varepsilon v(x/\varepsilon)\in W(\varepsilon Y)$. This is a one-to-one correspondence between $W(Y)$ and $W(\varepsilon Y)$ which does not modify the functionals to minimize. In other words, with the homotety $x\to x/\varepsilon$ we have

$$\frac{1}{\operatorname{meas}(Y)}\int\limits_Y f(x, Dv+\xi)\,dx=\frac{1}{\operatorname{meas}(Y)}\int\limits_{\varepsilon Y} f\left(\frac{x}{\varepsilon}, Dv\left(\frac{x}{\varepsilon}\right)+\xi\right)\frac{1}{\varepsilon^N}\,dx=$$
$$=\frac{1}{\operatorname{meas}(\varepsilon Y)}\int\limits_{\varepsilon Y} f\left(\frac{x}{\varepsilon}, Dv_\varepsilon+\xi\right)dx\,.$$

LEMMA 3.2. – $\limsup\limits_{\varepsilon\to 0}\gamma_\varepsilon^1(\xi)\leqslant\mu_1(\xi)$, $\forall\xi\in\boldsymbol{R}^N$.

PROOF. – For fixed $\xi\in\boldsymbol{R}^N$ let u be a function which realizes the minimum $\mu_1(\xi)$ on $W(Y)$. In view of the previous proposition $u_\varepsilon(x)=\varepsilon u(x/\varepsilon)$ realizes the minimum $\mu_\varepsilon(\xi)$.

Let $k=k(\varepsilon)$ be the smallest integer such that $Y\subseteq k\varepsilon Y$. Using Theorem 2.1 with integrand $f(x/\varepsilon, Dv+\xi)$ and domain of the integral εY we have

$$(3.5)\qquad \mu_1(\xi)=\mu_\varepsilon(\xi)=\min\left\{\frac{1}{\operatorname{meas}(k\varepsilon Y)}\int\limits_{k\varepsilon Y} f\left(\frac{x}{\varepsilon}, Dv+\xi\right)dx\colon v\in W(k\varepsilon Y)\right\}$$

and u_ε realizes such a minimum.

We put

$$(3.6)\qquad \xi_\varepsilon=\int\limits_Y Du_\varepsilon\,dx\,,\qquad v_\varepsilon(x)=u_\varepsilon(x)-(\xi_\varepsilon, x)\,.$$

We have $v_\varepsilon\in H^{1/p}(Y)$ and $\int\limits_Y Dv_\varepsilon\,dx=0$. So, by definition of $\gamma_\varepsilon^1(\xi)$, we get

$$\gamma_\varepsilon^1(\xi)^{1/p}-\mu_\varepsilon(\xi)^{1/p}\leqslant$$
$$\leqslant\frac{1}{\operatorname{meas}(Y)^{1/p}}\left\{\int\limits_Y f\left(\frac{x}{\varepsilon}, Dv_\varepsilon+\xi\right)dx\right\}^{1/p}-\frac{1}{\operatorname{meas}(k\varepsilon Y)^{1/p}}\left\{\int\limits_{k\varepsilon Y} f\left(\frac{x}{\varepsilon}, Du_\varepsilon+\xi\right)dx\right\}^{1/p}\leqslant$$

$$\leqslant \frac{1}{\operatorname{meas}(Y)^{1/p}}\left[\left\{\int_Y f\left(\frac{x}{\varepsilon}, Dv_\varepsilon+\xi\right)dx\right\}^{1/p}-\left\{\int_Y f\left(\frac{x}{\varepsilon}, Du_\varepsilon+\xi\right)dx\right\}^{1/p}\right]+$$

$$+\left[\frac{1}{\operatorname{meas}(Y)^{1/p}}-\frac{1}{\operatorname{meas}(k\varepsilon Y)^{1/p}}\right]\left\{\int_{k\varepsilon Y} f\left(\frac{x}{\varepsilon}, Du_\varepsilon+\xi\right)dx\right\}^{1/p},$$

and, by Minkowski's inequality,

$$\gamma_\varepsilon^1(\xi)^{1/p}-\mu_\varepsilon(\xi)^{1/p}\leqslant \frac{1}{\operatorname{meas}(Y)^{1/p}}\left\{\int_Y\left|f\left(\frac{x}{\varepsilon}, Dv_\varepsilon+\xi\right)^{1/p}-f\left(\frac{x}{\varepsilon}, Du_\varepsilon+\xi\right)^{1/p}\right|^p dx\right\}^{1/p}+$$

$$+[(k\varepsilon)^{N/p}-1]\mu_1(\xi)^{1/p}\leqslant L|\xi_\varepsilon|+[(k\varepsilon)^{N/p}-1]\mu_1(\xi)^{1/p}.$$

Because of (3.5) and $\lim_{\varepsilon\to 0} k\varepsilon=1$, to get the result it is sufficient to show that $\lim_{\varepsilon\to 0}\xi_\varepsilon=0$ in $\boldsymbol{R}^N$. To this aim, as u_ε is $k\varepsilon Y$-periodic, we have

$$\xi_\varepsilon=\int_Y Du_\varepsilon\,dx=\int_{k\varepsilon Y} Du_\varepsilon\,dx-\int_{k\varepsilon Y\setminus Y} Du_\varepsilon\,dx=-\int_{k\varepsilon Y\setminus Y} Du_\varepsilon\,dx,$$

and the conclusion follows from

$$|\xi_\varepsilon|\leqslant\int_{k\varepsilon Y\setminus Y}|Du_\varepsilon|\,dx\leqslant\int_{k\varepsilon Y\setminus(k-1)\varepsilon Y}|Du_\varepsilon|\,dx=$$

$$=[k^N-(k-1)^N]\int_{\varepsilon Y}\left|Du\left(\frac{x}{\varepsilon}\right)\right|dx=[(k\varepsilon)^N-(k\varepsilon-\varepsilon)^N]\int_Y|Du|\,dx.$$

LEMMA 3.3. – $\liminf_{\varepsilon\to 0}\gamma_\varepsilon^0(\xi)\geqslant\mu_1(\xi)$, $\forall\xi\in\boldsymbol{R}^N$.

PROOF. – For ξ fixed in $\boldsymbol{R}^N$ let u_ε be a function which realizes the minimum $\gamma_\varepsilon^0(\xi)$. We define $u_\varepsilon=0$ outside Y. If $k=k(\varepsilon)$ is the smallest integer such that $k\varepsilon\geqslant 1$ we have $u_\varepsilon\in H_0^{1,p}(k\varepsilon Y)\subset W(k\varepsilon Y)$. Then, by Theorem 2.1, we can apply once more (3.5) to obtain

$$\mu_1(\xi)\leqslant\frac{1}{\operatorname{meas}(k\varepsilon Y)}\int_{k\varepsilon Y} f\left(\frac{x}{\varepsilon}, Du_\varepsilon+\xi\right)dx$$

and therefore

$$\mu_1(\xi)-\gamma_\varepsilon^0(\xi)\leqslant\frac{1}{\operatorname{meas}(k\varepsilon Y)}\int_{k\varepsilon Y} f\left(\frac{x}{\varepsilon}, Du_\varepsilon+\xi\right)dx-\frac{1}{\operatorname{meas}(Y)}\int_Y f\left(\frac{x}{\varepsilon}, Du_\varepsilon+\xi\right)dx\leqslant$$

$$\leqslant\frac{1}{\operatorname{meas}(Y)}\int_{k\varepsilon Y\setminus Y} f\left(\frac{x}{\varepsilon},\xi\right)dx\leqslant(A_1+A_2|\xi|^p)[(k\varepsilon)^N-1].$$

Since $\lim_{\varepsilon\to 0}k\varepsilon=1$ the result follows.

Using Lemmas 3.2 and 3.3 we obtain in the next section (see Remark 4.6) more precise relations between μ_1, γ_ε^0 and γ_ε^1.

4. – Homogenization.

Let $f = f(x, \xi)$ a function verifying (3.1) and ε_h a sequence of positive real numbers going to zero. If Ω is a bounded open set in $\mathbf{R}^N$ let

$$F_h(\Omega, v) = \int_\Omega f\left(\frac{x}{\varepsilon_h}, Dv\right) dx\,, \qquad \forall v \in H^{1,p}(\Omega)\,. \tag{4.1}$$

We shall use the following result that can be deduced from Theorem 1 of [23] (cf. [18], Theorem 1):

THEOREM 4.1. – *There exists a function* $g = g(x, \xi)$, *verifying*

$$\begin{cases} g(x, \xi) \text{ is measurable in } x \text{ and convex in } \xi; \\ \lambda|\xi - \xi_0|^p \leqslant g(x, \xi) \leqslant \Lambda_1 + \Lambda_2|\xi|^p\,, & \forall x, \xi; \\ |g(x, \xi)^{1/p} - g(x, \xi')^{1/p}| \leqslant L|\xi - \xi'|\,, & \forall x, \xi, \xi'; \end{cases} \tag{4.2}$$

with the same constants as in (3.1), *and there exists a subsequence* ε_{h_r} *of* ε_h *such that, for any bounded open set* Ω *in* $\mathbf{R}^N$, *we have*:

(i) *For each* $v \in H^{1/p}(\Omega)$ *there exists a sequence* $\{v_r\} \subset H_0^{1,p}(\Omega) + v$ *converging to* v *in* $L^p(\Omega)$ *and such that* $\lim_{r\to+\infty} F_{h_r}(\Omega, v_r) = \int_\Omega g(x, Dv)\,dx$.

(ii) *For* v *and* v_r *in* $H^{1/p}(\Omega)$ *with* v_r *converging to* v *in* $L^p(\Omega)$ *we have* $\liminf_{r\to+\infty} F_{h_r}(\Omega, v_r) \geqslant \int_\Omega g(x, Dv)\,dx$.

(iii) *If* V *is a closed subspace of* $H^{1,p}(\Omega)$ *containing* $H_0^{1,p}(\Omega)$, $w \in H^{1,p}(\Omega)$ *and* $\varphi \in L^{p'}\,(p^{-1} + p'^{-1} = 1)$ *we have*

$$\lim_{r\to+\infty} \min\left\{F_{h_r}(\Omega, v) - \int_\Omega \varphi v\,dx : v \in V + w\right\} = \min\left\{\int_\Omega \{g(x, Dv) - \varphi v\}\,dx : v \in V + w\right\}.$$

(iv) *Let* V, w, φ, *be as above*; *let* g *be strictly convex in* ξ *and* $V \cap \{v \in H^{1,p}(\Omega): Dv = 0\} = \{0\}$. *If* u_r *minimizes* $F_{h_r}(\Omega, v) - \int_\Omega \varphi v\,dx$ *on* $V + w$, *as* $r \to +\infty$ *the sequence* u_r *converges in* $L^p(\Omega)$ *to the unique* u *that minimizes* $\int_\Omega \{g(x, Dv) - \varphi v\}\,dx$ *on* $V + w$.

Since the period of the integrand of F_h goes to zero as $h \to +\infty$, it is possible to prove that the function g in the above theorem is independent of x:

LEMMA 4.2. – *For each fixed* ξ *the function* $g = g(x, \xi)$ *of Theorem* 4.1 *is constant for almost every* x.

PROOF. – For each $y \in \boldsymbol{R}^N$ we can construct a sequence $\{z_r\} \subset \boldsymbol{R}^N$ such that $y_r = \varepsilon_{h_r} z_r$ converges to y and

$$f(x + z_r, \xi) = f(x, \xi)\,, \qquad \forall x, \xi, r\,. \tag{4.3}$$

Let Ω be a fixed open ball in $\boldsymbol{R}^N$ with radius R and Ω_k the open ball with the same centre of Ω an radius $R_k = R(1 - 1/k)$. If $v(x) = (\xi, x)$ ($\xi \in \boldsymbol{R}^N$), let v_r be a sequence in $H^{1,p}(\Omega)$ for which (i) of Theorem 4.1 holds. Since y_r converges to y in $\boldsymbol{R}^N$, for each k we have $\Omega_k + y \subset \Omega + y_r$ for large r. Therefore by (4.3) we have

$$\int_\Omega g(x, \xi)\, dx = \lim_{r\to+\infty} \int_\Omega f\left(\frac{x}{\varepsilon_{h_r}}, Dv_r\right) dx = \lim_{r\to+\infty} \int_\Omega f\left(\frac{x + y_r}{\varepsilon_{h_r}}, Dv_r\right) dx = \tag{4.4}$$
$$= \lim_{r\to+\infty} \int_{\Omega+y_r} f\left(\frac{x}{\varepsilon_{h_r}}, Dv_r(x - y_r)\right) dx \geqslant \liminf_{r\to+\infty} \int_{\Omega_k+y} f\left(\frac{x}{\varepsilon_{h_r}}, Dv_r(x - y_r)\right) dx \geqslant \int_{\Omega_k+y} g(x, \xi)\, dx\,.$$

We have used (ii) of Theorem 4.1 since $v_r(x - y_r)$ converges in $L^p(\Omega_k + y)$ to $(\xi, x - y)$ as $r \to +\infty$.

We can pass to the limit as $k \to +\infty$ in (4.4). Using Beppo Levi's monotone convergence theorem we get

$$\int_\Omega g(x, \xi)\, dx \geqslant \int_{\Omega+y} g(x, \xi)\, dx\,,$$

and, by symmetry, the equality; that is

$$\int_\Omega g(x, \xi)\, dx = \int_{\Omega+y} g(x, \xi)\, dx = \int_\Omega g(x + y, \xi)\, dx\,. \tag{4.5}$$

Now we suppose x and $x + y$ are Lebesgue points for all the integrands $g(x, \xi_k)$, where ξ_k is a dense sequence in $\boldsymbol{R}^N$. In this case, since Ω is arbitrary, we get from (4.5) $g(x, \xi_k) = g(x + y, \xi_k)$ and, by (4.2),

$$g(x, \xi) = g(x + y, \xi)\,, \qquad \forall \xi \in \boldsymbol{R}^N\,.$$

This proves that $g(\cdot, \xi)$ is constant with the exception of a set of measure zero.

Now we consider again the functions γ_ε^0 and γ_ε^1 of the previous section.

LEMMA 4.3. – *Let* γ_ε^0 *and* γ_ε^1 *be defined by* (3.3), (3.4); *let* g *and* ε_{h_r} *be as in Theorem* 4.1. *We have*

$$\lim_{r\to+\infty} \gamma^0_{\varepsilon_{h_r}}(\xi) = \lim_{r\to+\infty} \gamma^1_{\varepsilon_{h_r}}(\xi) = g(\xi)\,, \qquad \forall \xi \in \boldsymbol{R}^N.$$

PROOF. – Using Lemma 4.2 and (iii) of Theorem 4.1 with $V = H_0^{1,p}(Y)$, $w(x) = (\xi, x)$ and $\varphi = 0$, we have

$$\lim_{r\to+\infty} \gamma^0_{\varepsilon_{h_r}}(\xi) = \lim_{r\to+\infty} \min\left\{\frac{1}{\operatorname{meas}(Y)}\int_\Omega f\left(\frac{x}{\varepsilon_{h_r}}, Dv + \xi\right)dx : v \in H_0^{1,p}(Y)\right\} = \tag{4.6}$$

$$= \lim_{r\to+\infty} \min\left\{\frac{1}{\operatorname{meas}(Y)}\int_\Omega f\left(\frac{x}{\varepsilon_{h_r}}, Dv\right)dx : v \in H_0^{1,p}(Y) + (\xi, x)\right\} =$$

$$= \min\left\{\frac{1}{\operatorname{meas}(Y)}\int_\Omega g(Dv)\,dx : v \in H_0^{1,p}(Y) + (\xi, x)\right\}.$$

As g is convex, we can use Jensen's inequality:

$$\frac{1}{\operatorname{meas}(Y)}\int_Y g(Dv)\,dx \geqslant g\left(\frac{1}{\operatorname{meas}(Y)}\int_Y Dv\,dx\right) = g(\xi) \tag{4.7}$$

for all $v \in H_0^{1,p}(Y) + (\xi, x)$, since $\int_Y Dv\,dx = \int_Y \xi\,dx$. As in (4.7) equality holds if $Dv = \xi$, then $u(x) = (\xi, x)$ realizes the minimum in the right-hand side of (4.6) and the minimum value is $g(\xi)$.

The proof for $\gamma^1_{\varepsilon_{h_r}}(\xi)$ is analogous.

Now we can prove the homogenization theorem.

THEOREM 4.4. – *We suppose f as in (3.1), Ω an open bounded set in $\mathbf{R}^N$, V a closed subspace of $H^{1,p}(\Omega)$ containing $H_0^{1,p}(\Omega)$, $w \in H^{1,p}(\Omega)$, $\varphi \in L^{p'}(\Omega)$ $(p^{-1} + p'^{-1} = 1)$. Let us denote by*

$$g(\xi) = \min\left\{\frac{1}{\operatorname{meas}(Y)}\int_Y f(x, Dv + \xi)\,dx : v \in W(Y)\right\}. \tag{4.8}$$

Then g is a real convex function on $\mathbf{R}^N$ satisfying (4.2) such that

$$\lim_{\varepsilon\to 0} \min\left\{\int_\Omega \left\{f\left(\frac{x}{\varepsilon}, Dv\right) - \varphi(x)\,v\right\}dx : v \in V + w\right\} = \tag{4.9}$$

$$= \min\left\{\int_\Omega \{g(Dv) - \varphi(x)\,v\}\,dx : v \in V + w\right\}.$$

Moreover if f is strictly convex also g is strictly convex. In this case if $\{v \in V : Dv = 0\} = \{0\}$ the function u_ε which minimizes $\int_\Omega \{f(x/\varepsilon, Dv) - \varphi(x)v\}\,dx$ on $V + w$ as $\varepsilon \to 0$ converges in $L^p(\Omega)$ to the function u which minimizes $\int_\Omega \{g(Dv) - \varphi(x)v\}\,dx$ on $V + w$.

PROOF. – Let ε_h be a sequence of positive real numbers converging to zero. By (iii) of Theorem 4.1 and Lemma 4.2 we find a convex function $g = g(\xi)$ and a subsequence ε_{h_r} of ε_h for which (4.9) holds with ε_{h_r} instead of ε.

In order to prove that g is given by (4.8), by Lemmas 3.2 and 3.3 we have

$$\limsup_{r\to+\infty} \gamma^1_{\varepsilon_{h_r}}(\xi) \leqslant \mu_1(\xi) \leqslant \liminf_{r\to+\infty} \gamma^0_{\varepsilon_{h_r}}(\xi) ,$$

and therefore, by Lemma 4.3, $g(\xi) = \mu_1(\xi)$. This gives (4.8) (see the definition of μ_1 (3.2)).

For g is independent of the chosed sequence ε_h, with a compactness argument we obtain (4.9).

The last part of theorem follows from (iv) of Theorem 4.1 if we prove that g is strictly convex if f does.

Therefore let us assume f strictly convex and choose $\xi_1 \neq \xi_2$; let u_1, u_2 be the functions in $W(Y)$ which realize respectively the minimum (4.8). Then $Du_1 + \xi_1 \neq Du_2 + \xi_2$ almost everywhere on Y because, if not, we have

$$\xi_1 \operatorname{meas}(Y) = \int_Y (Du_1 + \xi_1)\, dx = \int_Y (Du_2 + \xi_2)\, dx = \xi_2 \operatorname{meas}(Y) ,$$

that is $\xi_1 = \xi_2$. So we can apply strictly convexity of f to infer

$$g(\xi_1) + g(\xi_2) = \frac{1}{\operatorname{meas}(Y)} \int_Y \{f(x, Du_1 + \xi_1) + f(x, Du_2 + \xi_2)\}\, dx >$$
$$> \frac{1}{\operatorname{meas}(Y)} \int_Y 2f\left(x, D\frac{u_1 + u_2}{2} + \frac{\xi_1 + \xi_2}{2}\right) dx \geqslant 2g\left(\frac{\xi_1 + \xi_2}{2}\right).$$

The strictly convexity of g is obtained.

REMARK 4.5. – It is possible to deduce from Theorem 4.4 the convergence in $L^p(\Omega)$ of the minimal functions and the convergence of the minimum values of the obstacle problems

$$\int_\Omega \left\{ f\left(\frac{x}{\varepsilon}, Du_\varepsilon\right) - \varphi(x)\, u_\varepsilon \right\} dx = \text{minimum on } \{v \in V + w : v \geqslant \psi \text{ on } E\}$$

respectively to the minimal function and the minimum value of the problem

$$\int_\Omega \{g(Du) - \varphi(x) u\}\, dx = \text{minimum on } \{v \in V + w : v \geqslant \psi \text{ on } E\} ,$$

where ψ is a fixed $L^\infty(\Omega)$ function and E is a closed subset of Ω or all Ω (we assume that there exists in $V + w$ a v such that $v \geqslant \psi$ on E). It can be proved as in section 6 of [17].

REMARK 4.6. – From Theorem 4.4, or directly from its proof, we can deduce the limit values as $\varepsilon \to 0$ of the functions μ_ε, γ_ε^0, γ_ε^1 defined in (3.2), (3.3), (3.4). We have

$$\mu_\varepsilon(\xi) = \mu_1(\xi) = \lim_{\varepsilon\to 0} \gamma_\varepsilon^0(\xi) = \lim_{\varepsilon\to 0} \gamma_\varepsilon^1(\xi), \qquad \forall \xi \in \mathbf{R}^N. \tag{4.10}$$

As in [25] let us define

$$\gamma_\varepsilon^{\mathrm{per}}(\xi) = \min\left\{\frac{1}{\operatorname{meas}(Y)}\int_Y f\left(\frac{x}{\varepsilon}, Dv + \xi\right) : v \in W(Y)\right\}. \tag{4.11}$$

Since $H_0^{1,p}(Y) \subset W(Y) \subset \{v \in H^{1,p}(Y): \int_Y Dv\,dx = 0\}$ we have $\gamma_\varepsilon^1(\xi) \leq \gamma_\varepsilon^{\mathrm{per}}(\xi) \leq \gamma_\varepsilon^0(\xi)$ for each $\xi \in \mathbf{R}^N$, and therefore, by (4.10), $\lim_{\varepsilon\to 0} \gamma_\varepsilon^{\mathrm{per}}(\xi) = \mu_1(\xi)$.

If we confine ourselves to the case $\varepsilon = 1/k$ with k positive integer, we can prove directly, by using Theorem 2.1, that $\gamma_{1/k}^{\mathrm{per}}(\xi) = \mu_1(\xi)$ for each $\xi \in \mathbf{R}^N$.

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