A local property of Fourier series.

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To Enrico Bompiani on his scientific Jubilee

Summary. - See the theorems of the n. 1 of the text.

1. Let $f(x)$ be integrable in the LEBESGUE sense and periodic with period 2π . Let its FOURIER series be

$$(1) \quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum A_n(x).$$

Write

$$\varphi = \varphi_0(t) = \frac{1}{2} \{ f(x+t) + f(x-t) - 2s \}$$

and define the fractional integral of order ν of $\varphi(t)$ as

$$\varphi_\nu(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} \varphi(u) du \quad (\nu > 0).$$

It is known that [2, p. 273] the summability $(C, -\alpha)$ ($0 < \alpha < 1$) of the series (1) is not a local property of $f(t)$ near the point $t=x$ but depends on the behaviours of $f(t)$ in the whole interval $(0, 2\pi)$. BOSANQUET and OFFORD [2] proved that if $A_n(x) = O(n^{-\alpha})$, then the $(C, -\alpha)$ summability of the series (1) at x depends on the local properties of $f(t)$ near the point $t=x$. They have established the following

THEOREM A. - *Necessary and sufficient conditions for the Fourier series (1) to be summable $(C, -\alpha)$ ($0 < \alpha < 1$) to the sum s are that (A) $A_n(x) = O(n^{-\alpha})$, (B) $\varphi_1(t) = o(t)$ ($t \rightarrow 0$) and (C)*

$$(2) \quad n^\alpha \int_0^\delta \varphi(t) \left(1 - \frac{t}{\delta}\right) \frac{\sin(n; t)}{t^{1-\alpha}} dt = o(1)$$

as $n \rightarrow \infty$, where $0 < \delta < \pi$ being an arbitrary quantity and

$$(n; t) = \left(n + \frac{1}{2} - \frac{1}{2}\alpha\right)t + \frac{1}{2}\alpha\pi.$$

It is well-known that [4], (B) is a necessary condition for the FOURIER series (1) to be summable by the CESÀRO means of negative order at x . In this note, we improve Theorem A by proving that (2) is still a necessary and sufficient condition for the $(C, -\alpha)$ summability of the series (1) at x , if we replace (B) by some weaker conditions. Such conditions are of course the necessary conditions for the series (1) to be summable $(C, -\alpha)$ at x . We prove the following

THEOREM. - If (i) $A_n(x) = O(n^{-\alpha})$ ($0 < \alpha < 1$) as $n \rightarrow \infty$, (ii) $\varphi_{1+\beta}(t) = o(t^{1+\beta})$ ($0 < \beta < 1$) and (iii) $\varphi_1(t) = o(t^{1-\alpha})$ as $t \rightarrow 0$, then the necessary and sufficient condition for the series (1) to be summable $(C, -\alpha)$ at x is (2).

2. The following lemmas are required.

LEMMA 1. - If $0 < \alpha < 1$, $\beta > \alpha$, then

$$(3) \quad \varphi_\beta(t) = \frac{1}{\Gamma(\beta - \alpha)} \int_0^t (t-u)^{\beta-\alpha-1} d\varphi_{1-\alpha}(u)$$

for almost all t .

This is an immediate consequence of a known result [1, p. 114].

In the lemmas below, A denotes an absolute constant independent of n and t and different in different occurrences.

LEMMA 2. - Let $\sigma_n^{-\alpha}(x)$ be the n -th Cesàro mean of order $-\alpha$ of the series (1), then

$$(4) \quad \sigma_n^{-\alpha}(x) - s = \int_0^\pi \varphi(t) K_n^{-\alpha}(t) dt,$$

where $K_n^{-\alpha}$ is the n -th Cesàro mean of order $-\alpha$ of the series

$$\frac{1}{\pi} + \frac{2}{\pi} \sum_{v=0}^{\infty} \cos vt$$

and for any $0 < t < \pi$ [3, pp. 264-267],

$$(5) \quad |K_n^{-\alpha}(t)| \leq An,$$

$$(6) \quad \left| \frac{d}{dt} K_n^{-\alpha}(t) \right| \leq An^2,$$

$$(7) \quad \left| \frac{d^2}{dt^2} K_n^{-\alpha}(t) \right| \leq An^3,$$

$$(8) \quad |K_n^{-\alpha}(t) - K(n, t)| \leq An^{-1}t^{-2},$$

Moreover, for $t \geq 1/n$,

$$(9) \quad \left| \frac{d}{dt} \{K_n^{-\alpha}(t) - K(n, t)\} \right| \leq An^{-1}t^{-3},$$

where

$$K(n, t) = \frac{2}{\pi} \frac{\Gamma(n+1)\Gamma(1-\alpha)}{\Gamma(n+1-\alpha)} \cdot \frac{\sin(n; t)}{\left(2 \sin \frac{t}{2}\right)^{1-\alpha}}.$$

LEMMA 3. - We have, for $0 < t < \pi$,

$$(10) \quad \left| \frac{d^2}{dt^2} K(n, t) \right| \leq An^{2+\alpha}t^{\alpha-1} + An^{1+\alpha}t^{\alpha-2} + An^{\alpha}t^{\alpha-3},$$

and for $t \geq 1/n$,

$$(11) \quad \left| \frac{d^2}{dt^2} \{K_n^{-\alpha}(t) - K(n, t)\} \right| \leq An^{-1}t^{-4}.$$

The first inequality can easily be obtained by direct differentiation twice $K(n, t)$. In order to prove the second, let us consider

$$\begin{aligned} K_n^{-\alpha}(t) &= C_n \left(2 \sin \frac{t}{2}\right)^{-\alpha} + K(n, t) + \Im \left\{ \frac{e^{i\left(n+\frac{3}{2}\right)t}}{A_n^{-\alpha} \left(2 \sin \frac{t}{2}\right)^{\alpha}} \sum_{k=n+1}^{\infty} A_n^{-(\alpha+k)} e^{-ikt} \right\} \\ &= P + Q + R \quad [5, \text{p. 56}], \end{aligned}$$

where

$$\begin{aligned} A_n^{-\alpha} &= \frac{\Gamma(n+1)\Gamma(1-\alpha)}{\Gamma(n+1-\alpha)}, \\ C_n &= \frac{2}{\pi} \frac{1}{A_n^{-\alpha}} \{ A_n^{-(\alpha+1)} + A_n^{-(\alpha+2)}/2 \} \\ &= O(n^{-1}). \end{aligned}$$

Denote

$$\begin{aligned} S &= \frac{e^{i(n+\frac{3}{2})t}}{A_n^{-\alpha} \left(2 \sin \frac{t}{2} \right)^3}, \\ T &= \sum_{k=n+1}^{\infty} A_k^{-(\alpha+3)} e^{-ikt} \end{aligned}$$

and notice that, for $t \geq 1/n$, $S = O(n^{\alpha}t^{-3})$,

$$\begin{aligned} |S'| &\leq An^{1+\alpha}t^{-3} + An^{\alpha}t^{-4} \\ &= O(n^{1+\alpha}t^{-3}), \\ |S''| &\leq An^{2+\alpha}t^{-3} + An^{1+\alpha}t^{-4} + An^{\alpha}t^{-5} + An^{\alpha}t^{-5} \\ &= O(n^{2+\alpha}t^{-3}), \\ |T| &\leq 4 |A_{n+1}^{-(\alpha+3)}| |1 - e^{-it}| \\ &= O(n^{-(\alpha+3)}t^{-1}), \\ |T'| &\leq 4(n+1) |A_{n+1}^{-(\alpha+3)}| |1 - e^{-it}| \\ &= O(n^{-(\alpha+2)}t^{-1}), \\ |T''| &\leq 4(n+1)^2 |A_{n+1}^{-(\alpha+3)}| |1 - e^{-it}| \\ &= O(n^{-(\alpha+1)}t^{-1}). \end{aligned}$$

The lemma follows from

$$\begin{aligned} |R''| &\leq |ST''| + 2|S'T'| + |S''T| \\ &= O(n^{-1}t^{-4}) \end{aligned}$$

and

$$\left| \frac{d^2}{dt^2} \{ K_n^{-\alpha}(t) - K(n, t) \} \right| \leq |P''| + |R'|.$$

LEMMA 4. - If $0 < u \leq 1/n$. Write

$$G(n, u) = \int_u^{u+1/n} \left\{ \frac{d^2}{dt^2} (K_n^{-\alpha}(t) - K(n, t)) \right\} (t-u)^{-\beta} dt,$$

$$H(n, u) = \int_{u+1/n}^{\pi} \left\{ \frac{d^2}{dt^2} (K_n^{-\alpha}(t) - K(n, t)) \right\} (t-u)^{-\beta} dt.$$

Then,

$$(12) \quad G(n, u) = O(n^{2+\beta}) + O(n^{\alpha+\beta+1}u^{\alpha-1}) + O(n^{\alpha+\beta}u^{\alpha-2}) + O(n^{\alpha+\beta-1}u^{\alpha-3}),$$

$$(13) \quad H(n, u) = O(n^{2+\beta}).$$

We have, by (7) and (10),

$$\begin{aligned} G(n, u) &= \int_u^{u+1/n} \left(\frac{d^2}{dt^2} K_n^{-\alpha}(t) \right) (t-u)^{-\beta} dt - \int_u^{u+1/n} \left(\frac{d^2}{dt^2} K(n, t) \right) (t-u)^{-\beta} dt \\ &= O\left(n^3 \int_u^{u+1/n} (t-u)^{-\beta} dt\right) + O\left(n^{2+\alpha} \int_u^{u+1/n} t^{\alpha-1} (t-u)^{-\beta} dt\right) \\ &\quad + O\left(n^{1+\alpha} \int_u^{u+1/n} t^{\alpha-2} (t-u)^{-\beta} dt\right) + O\left(n \int_u^{u+1/n} t^{\alpha-3} (t-u)^{-\beta} dt\right). \end{aligned}$$

By applying the second mean-value theorem to the last three integrals, (12) follows.

$$\begin{aligned} H(n, u) &= \int_{u+1/n}^{\pi} \left\{ \frac{d^2}{dt^2} (K_n^{-\alpha}(t) - K(n, t)) \right\} (t-u)^{-\beta} dt \\ &= n^{\beta} \int_{u+1/n}^{\pi} \frac{d^2}{dt^2} \{ K_n^{-\alpha}(t) - K(n, t) \} dt \\ &= n^{\beta} \left\{ \frac{d}{dt} (K_n^{-\alpha}(t) - K(n, t)) \right\}_{u+1/n}^{\pi} \\ &= O(n^{2+\beta}) \end{aligned}$$

by (9) of Lemma 2. The lemma thus follows.

LEMMA 5. - If $u \geq 1/n$. Write

$$G'(n, u) = \int_u^{u+1/n} \left\{ \frac{d^2}{dt^2} (K_n^{-\alpha}(t) - K(n, t)) \right\} (t-u)^{-\beta} dt,$$

$$H'(n, u) = \int_{u+1/n}^{\pi} \left\{ \frac{d^2}{dt^2} (K_n^{-\alpha}(t) - K(n, t)) \right\} (t-u)^{-\beta} dt.$$

Then,

$$(14) \quad G'(n, u) = O(n^{\beta-1}u^{-3}),$$

$$(15) \quad H'(n, u) = O(n^{\beta-1}u^{-3}).$$

By (11) of Lemma 3,

$$\begin{aligned} |G'(n, u)| &\leq A \left\{ n^{-1} \int_u^{u+1/n} t^{-4} (t-u)^{-\beta} dt \right\} \\ &\leq A \left\{ n^{-1} u^{-4} \int_u^{\tau} (t-u)^{-\beta} dt \right\} \\ &\leq A \left\{ u^{-3} \int_u^{\tau} (t-u)^{-\beta} dt \right\} \\ &= O(n^{\beta-1}u^{-3}) \end{aligned}$$

and by (9) of Lemma 2,

$$\begin{aligned} H'(n, u) &= \int_{u+1/n}^{\pi} \left\{ \frac{d^2}{dt^2} (K_n^{-\alpha}(t) - K(n, t)) \right\} (t-u)^{-\beta} dt \\ &= n^{\beta} \int_{u+1/n}^{\tau} \frac{d^2}{dt^2} (K_n^{-\alpha}(t) - K(n, t)) dt \\ &= n^{\beta} \left\{ \frac{d}{dt} (K_n^{-\alpha}(t) - K(n, t)) \right\}_{n+1/n}^{\tau} \\ &= O(n^{\beta-1}u^{-3}). \end{aligned}$$

The lemma is thus proved.

LEMMA 6 [2]. - Under the condition (i) of the theorem,

$$\int_0^\pi \varphi(t) K(n, t) dt = n^\alpha \int_0^\pi \varphi(t) \frac{\sin(n; t)}{t^{1-\alpha}} dt + o(1).$$

LEMMA 7 [2]. - Under (i) of the theorem,

$$n^\alpha \int_0^{\hat{\delta}} \varphi(t) \frac{t}{\hat{\delta}} \frac{\sin(n; t)}{t^{1-\alpha}} dt + n^\alpha \int_0^\pi \varphi(t) \frac{\sin(n; t)}{t^{1-\alpha}} dt + o(1).$$

as $n \rightarrow \infty$, where $0 < \hat{\delta} < \pi$ being an arbitrary quantity.

LEMMA 8. - Under (ii) and (iii) of the theorem, the integral

$$\int_0^\pi \varphi(t) K_n^{-\alpha}(t) dt$$

can be replaced by the integral

$$\int_0^\pi \varphi(t) K(n, t) dt.$$

We have

$$\begin{aligned} \int_0^\pi \varphi(t) \{ K_n^{-\alpha}(t) - K(n, t) \} dt &= \left\{ \varphi_1(t) (K_n^{-\alpha}(t) - K(n, t)) \right\}_0^\pi \\ &\quad - \int_0^\pi \varphi_1(t) \frac{d}{dt} \{ K_n^{-\alpha}(t) - K(n, t) \} dt. \end{aligned}$$

Using (5), (8) and (iii) of the theorem and noticing that

$$| K(n, t) | \leq A n^\alpha t^{\alpha-1},$$

we see that the integrated term is $o(1)$. Let the above integral be denoted by J_0 . Integrating it by parts and using (3) of Lemma 1, we find

$$\begin{aligned} J_0 &= \left[\varphi_2(t) \frac{d}{dt} \{ K_n^{-\alpha}(t) - K(n, t) \} \right]_0^\pi \\ &\quad + \frac{1}{\Gamma(2-\beta)} \int_0^\pi \left[\frac{d^2}{dt^2} \{ K_n^{-\alpha}(t) - K(n, t) \} \right] dt \int_0^t (t-u)^{1-\beta} d\varphi_{1+\beta}(u). \end{aligned}$$

Since the condition (ii) of the theorem implies $\varphi_2(t) = o(t^2)$, the integrated term is $o(1)$ by (6), (9) and

$$\left| \frac{d}{dt} K(n, t) \right| \leq An^2 t^{2-\alpha} + An^{\alpha+1} t^{\alpha-1}.$$

Denote the above integral by J_1 . Then,

$$\begin{aligned} J_1 &= \int_0^\pi d\varphi_{1+\beta}(u) \int_u^\pi \left\{ \frac{d^2}{dt^2} (K_n^{-\alpha}(t) - K(n, t)) \right\} (t-u)^{1-\beta} dt \\ &= - \left[\varphi_{1+\beta}(u) \int_u^\pi \left\{ \frac{d^2}{dt^2} (K_n^{-\alpha}(t) - K(n, t)) \right\} (t-u)^{1-\beta} dt \right]_{u=0}^\pi \\ &\quad + (1-\beta) \int_0^\pi \varphi_{1+\beta}(u) du \int_u^\pi \left\{ \frac{d^2}{dt^2} (K_n^{-\alpha}(t) - K(n, t)) \right\} (t-u)^{-\beta} dt \\ &= R + (1-\beta)J_2, \end{aligned}$$

say. Denote

$$\begin{aligned} R(u) &= \varphi_{1+\beta}(u) \int_u^\pi \left(\frac{d^2}{dt^2} K_n^{-\alpha}(t) \right) (t-u)^{1-\beta} dt \\ &\quad + \int_u^\pi \left(\frac{d^2}{dt^2} K(n, t) \right) (t-u)^{1-\beta} dt. \end{aligned}$$

Then, by (7) and (10), we obtain

$$\begin{aligned} R(u) &= O \left\{ n^3 \mid \varphi_{1+\beta}(u) \mid \int_u^\pi (t-u)^{-\beta} dt \right\} \\ &\quad + O \left\{ n^{2+\alpha} \mid \varphi_{1+\beta}(u) \mid \int_u^\pi t^{\alpha-1} (t-u)^{1-\beta} dt \right\} \\ &\quad + O \left\{ n^{1+\alpha} \mid \varphi_{1+\beta}(u) \mid \int_u^\pi t^{\alpha-2} (t-u)^{1-\beta} dt \right\} \\ &\quad + O \left\{ n^\alpha \mid \varphi_{1+\beta}(u) \mid \int_u^\pi t^{\alpha-3} (t-u)^{1-\beta} dt \right\} \end{aligned}$$

$$\begin{aligned}
R(u) &= O(n^{\beta} |\varphi_{1-\beta}(u)|) + O\left\{n^{2+\alpha} |\varphi_{1+\beta}(u)| \int_u^{\pi} t^{\alpha-1} dt\right\} \\
&\quad + O\left\{n^{1+\alpha} |\varphi_{1+\beta}(u)| \int_u^{\pi} t^{\alpha-2} dt\right\} \\
&\quad + O\left\{n^{\alpha} |\varphi_{1+\beta}(u)| \int_u^{\pi} t^{\alpha-3} dt\right\} \\
&= o(1) + O\{n^{\alpha} |\varphi_{1+\beta}(u)| u^{\alpha-2}\}
\end{aligned}$$

as $u \rightarrow 0$. Without effecting (ii) of the theorem, we may raise the value of β a little such that $1 - \alpha < \beta < 1$, since $\varphi_{1+\nu} = o(t^{1+\nu})$ for $\nu > \beta$. Hence

$$\begin{aligned}
\varphi_{1+\beta}(u)u^{\alpha-2} &= o(u^{\alpha+\beta-1}) \\
&= o(1)
\end{aligned}$$

as $u \rightarrow 0$. This proves that $R = \lim_{u \rightarrow 0} R(u) = 0$. Write

$$\begin{aligned}
J_2 &= \left(\int_0^{1/n} + \int_{1/n}^{\pi} \right) \varphi_{1+\beta}(u) du \int_u^{\pi} \frac{d^2}{dt^2} (K_n^{-\alpha}(t) - K(n, t)) \{(t-u)^{-\beta} dt \\
&= I_1 + I_2,
\end{aligned}$$

say.

$$\begin{aligned}
I_1 &= \int_0^{1/n} \varphi_{1+\beta}(u) du \left(\int_u^{u+1/n} + \int_{u+1/n}^{\pi} \right) \frac{d^2}{dt^2} (K_n^{-\alpha}(t) - K(n, t)) \{(t-u)^{-\beta} dt \\
&= \int_0^{1/n} \varphi_{1+\beta}(u) G(n, u) du + \int_0^{1/n} \varphi_{1+\beta}(u) H(n, u) du \\
&= I_3 + I_4,
\end{aligned}$$

say. By (11),

$$\begin{aligned}
 I_3 &= \int_0^{1/n} \varphi_{1+\beta}(u) \{ O(n^{2+\beta}) + O(n^{1+\beta+\beta}u^{2-1}) + O(n^{2+\beta}u^{2-2}) \\
 &\quad + O(n^{2+\beta-1}u^{2-1}) \} du \\
 &= o\left(n^{2+\beta} \int_0^{1/n} u^{1+\beta} du\right) + o\left(n^{1+\beta+\beta} \int_0^{1/n} u^{2+\beta} du\right) \\
 &\quad + o\left(n^{2+\beta} \int_0^{1/n} u^{2+\beta-1} du\right) + o\left(n^{2+\beta-1} \int_0^{1/n} u^{2+\beta-2} du\right) \\
 &= o(1)
 \end{aligned}$$

as $n \rightarrow \infty$. By (12) of Lemma 4, we have

$$\begin{aligned}
 I_4 &= o\left(n^{2+\beta} \int_0^{1/n} u^{1+\beta} du\right) \\
 &= o(1).
 \end{aligned}$$

Finally, write

$$\begin{aligned}
 I_2 &= \int_{1/n}^{\pi} \varphi_{1+\beta}(u) G'(n, u) du + \int_{1/n}^{\pi} \varphi_{1+\beta}(u) H'(n, u) du \\
 &= I_5 + I_6,
 \end{aligned}$$

say. Then

$$\begin{aligned}
 I_5 &= O\left(n^{\beta-1} \int_{1/n}^{\pi} |\varphi_{1+\beta}(u)| u^{-3} du\right) \\
 &= o\left(n^{\beta-1} \int_{1/n}^{\pi} u^{\beta-2} du\right) \\
 &= o(1)
 \end{aligned}$$

by (3). A similar argument gives $I_6 = o(1)$ by (14). This completes the proof of the lemma.

3. The proof of the theorem is now immediate. The necessary and sufficient condition for the FOURIER series (1) to be summable $(C, -\alpha)$ to the sum s is that the integral in (4) is $o(1)$ as $n \rightarrow \infty$. By Lemmas (8) and (6), this integral can be replaced by

$$\begin{aligned} n^z \int_0^\pi \varphi(t) \frac{\sin(n; t)}{t^{1-z}} dt &= n^z \left(\int_0^\delta + \int_\delta^\pi \right) \\ &= n^z \int_0^\delta \varphi(t) \left(1 - \frac{t}{\delta} \right) \frac{\sin(n; t)}{t^{1-z}} dt + o(1) \end{aligned}$$

by Lemma 7. This completes the proof of the theorem.

REFERENCES

- [1] L. S. BOSANQUET, *Some extensions of Young's criterion for the convergence of a Fourier series*, « Quart. Journ. Math. » (Oxford), 6 (1935), 113-123.
- [2] L. S. BOSANQUET and A. C. OFFORD, *A local property of Fourier series*, « Proc. London Math. Soc. » (2), 40 (1935-36), 273-280.
- [3] J. J. GEBGEN, *Convergence and summability criteria for Fourier series*, « Quart. Journ. Math. » (Oxford), 1 (1930), 251-275.
- [4] G. H. HARDY and J. E. LITTLEWOOD, *On Young's convergence criterion for Fourier series*, « Proc. London Math. Soc. » (2), 28 (1928), 301-311.
- [5] A. ZYGMUND, *Trigonometrical series* (Warsaw), 1935.