Generalized Minkowski Formulas for Closed Hypersuperfaces in Riemann space.

By YOSHIE KATSURADA (a Sapporo, Japan)

To Enrico Bompiani on his scientific Jubilee

Summary. - *This paper generalized* MINKOWSKI *formulas for a closed orientable hypersurface in a* RIEMANN *space with constant curvature which have been introduced by* C. C. HSIUNG, *and studies on some properties of the hypersurface whose the* ν*-th mean curvature is constant, on making use of the generalized formulas.*

Introduction. - We consider an ovaloid F in a Euclidean space E^3 of three dimensions, and let H and K be the mean curvature and the GAUSS curvature at a point P of F, then as well-known formula of MINKOWSKI we have

$$\iint_F (Kp + H)dA = 0, \tag{0.1}$$

where p denotes the oriented distance from a fixed point 0 in E^3 to the tangent space of F at P and dA is the area element of F at P.

As generalization of this formula for a closed orientable hypersurface, C. C. HSIUNG proved the following three theorems.

THEOREM 0.1. - Let V^m be a closed orientable hypersurface twice differentiably imbedded in a Euclidean space E^{m+1} of $m+1(\geqq 3)$ dimensions, then

$$\int_{V^m}\!\!\cdots\!\int H_{\nu+1}p\,dA + \int_{V^m}\!\!\cdots\!\int H_\nu dA = 0 \qquad \text{for } \nu = 0, \dots, m-1, \tag{0.2}$$

where $k_1, k_2, \dots k_m$ being the m principal curvatures at a point P of V^m, H_ν is the ν-th mean curvature of V^m at the point P which is defined to be the ν-th elementary symmetric function of $k_1, \dots, k_m$ divided by the nnmber of terms, that is,

$$\binom{m}{\nu} H_\nu = \Sigma\, k_1 k_2, \ldots k_\nu \qquad \nu = 1, \ldots, m,$$

and $H_0 = 1$ ([1]) [1].

THEOREM 0.2. - Let V^m be a closed orientable hypersurface twice differentiably imbedded in a RIEMANN space R^{m+1} of $m+1(\geqq 3)$ dimensions, then

$$\int_{V^m}\!\!\ldots\!\int H_1 p\, dA + \int_{V^m}\!\!\ldots\!\int dA = 0, \tag{0.3}$$

where p is the scalar product of the unit normal vector of the hypersurface V^m at the point P and the position vector of the point P with respect to any orthogonal frame in the space R^{m+1} ([2]).

THEOREM 0.3. - Let V^m be a closed orientable hypersurface of class C^3 imbedded in an $(m+1)$-dimensional Riemann space R^{m+1} of constant RIEMANN curvature, then

$$\int_{V^m}\!\!\ldots\!\int H_m p\, dA + \int_{V^m}\!\!\ldots\!\int H_{m-1} dA = 0 \qquad ([2]). \tag{0.4}$$

The purpose of this paper is to generalize more these formulas of HSIÚNG, in a RIEMANN space. In § 1 generalized MINKOWSKI formulas are expressed and the special cases which inclose these formulas of HSIUNG are discussed in § 2, and in § 3 some properties for the closed orientable hyperusrface are obtained on making use of the generalized MINKOWSKI formulas.

§ 1. **On a generalization of Minkowski formulas.** - We suppose an $(m+1)$-dimensional RIEMANN space $R^{m+1}(m+1 \geqq 3)$ of class $C^\nu(\nu \geqq 3)$ which admits an infinitesimal point transformation

$$\bar{x}^i = x^i + \xi^i(x)\delta\tau \tag{1.1}$$

and assume that the paths of the infinitesimal transformations cover simply R^{m+1}. Let us choose a coordinate system such that the path of the infinitesimal

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(1) Numbers in brackets refer to the references at the end of the paper.

transformation is a new x^1-coordinate curve, that is, the coordinate system in which the vector ξ^i has the components δ^i_1, where a symbol δ^i_j denotes the KRONECKER's delta, then (1.1) becomes as follows

$$x^{\bar{i}} = x^i + \delta^i_1 \delta\tau \tag{1.1$'$}$$

and R^{m+1} admits a one-parameter continuous group G of transformations given by

$$\bar{x}^i = x^i + \delta^i_1 \tau \tag{1.2}$$

in the new special coordinate system. If the vector ξ^i is a KILLING vector, a homothetic KILLING, a conformal KILLING etc., then the one-parameter continuous group G of transformations has been called isometric, homothetic, conformal etc. respectively ([3]).

We now consider a closed orientable hypersuperface V^m of class C^3 imbedded in R^{m+1} which does not pass through any singular point of a tangent vector field of the paths, written in the expression

$$x^i = x^i(u^\alpha) \qquad \begin{array}{l} i = 1, 2, \dots, m+1, \\ \alpha = 1, 2, \dots, m. \end{array}$$

We shall henceforth confine ourselves to that LATIN indices run from 1 to $m+1$ and GREEK indices from 1 to m.

Let us consider a differential form of $m-1$ degree at a point P of the hypersurface V^m, defined by

$$\begin{aligned} ((n, \delta_1, \underbrace{dx, \dots, dx}_{m-1})) &= \sqrt{g}(n, \delta_1, dx, \dots, dx) \\ &= \sqrt{g}\left(n, \delta_1, \frac{\partial x}{\partial u^{\alpha_1}}, \dots, \frac{\partial x}{\partial u^{\alpha_{m-1}}}\right) du_{\alpha_1} \wedge \dots \wedge du^{\alpha_{m-1}} \end{aligned} \tag{1.3}$$

where n^i is a unit normal vector at the point P of the hypersuperface V^m and dx^k a displacement along the hypersurface V^m, i. e., $dx^k = \frac{\partial x^k}{\partial u^\alpha} du^\alpha$, and g denotes the determinant of a metric tensor g_{ij} of R^{m+1}, and let $g_{\alpha\beta}$ and $b_{\alpha\beta}$ be the first fundamental metric tensor and the second fundamental tensor of the hypersuperface V^m respectively, and b^β_α means $b_{\alpha\gamma} g^{\gamma\beta}$. Then the exterior differential of the differential form (1.3) divided by $m!$ becomes as follows, (on making use of the formula for the covariant differential of the unit normal vector along the hypersurface: $\delta n^i = -b_\alpha^\beta \frac{\partial x^i}{\partial u^\beta} du^\alpha$)

$$\begin{aligned} &\frac{1}{m!} d((n, \delta_1, dx, \dots, dx)) \\ (1.4) \qquad &= (-1)^m \left\{ H_1 n_i \delta_1^i dA + \frac{1}{2m} g^{*lk} \underset{\xi}{\mathfrak{L}} g_{lk} dA \right\}, \end{aligned}$$

where $\mathfrak{L} g_{lk}$ is the LIE detivative of the tensor g_{lk} with respect to the infinitesimal point transformation (1.1), and $g^{*lk} = g^{lk} - n^l n^k$.

Integrating both members of (1.4) over the whole hypersurface V^m and applying the STOKES' theorem, we obtain the formula

$$\frac{1}{m!} \int\limits_{\partial V^m} ((n, \delta_1, dx, \dots, dx)) = (-1)^m \left\{ \int\limits_{V^m} \cdots \int H_1 n_i \delta_1^i dA + \frac{1}{2m} \int\limits_{V^m} \cdots \int g^{*lk} \underset{\xi}{\mathfrak{L}} g^{lk} dA \right\},$$

where ∂V^m means the boundary of V^m. On making use of that the hypersurface V^m is closed, it follows that

$$(1.5) \qquad \int\limits_{V^m} \cdots \int H_1 n_i \delta_1^i dA + \frac{1}{2m} \int\limits_{V^m} \cdots \int g^{*lk} \underset{\xi}{\mathfrak{L}} g_{lk} dA = 0.$$

If the space R^{m+1} assumes of constant RIEMANN curvature which includes a Euclidean space, we consider the following differential form of $m-1$ degree

$$((n, \delta_1, \underbrace{\delta n, \dots, \delta n}_{\nu}, \underbrace{dx, \dots dx}_{m-\nu-1})) \overset{\text{def}}{=} \sqrt{g} (n, \delta_1, \underbrace{\delta n, \dots, \delta n}_{\nu}, \underbrace{dx, \dots, dx}_{m-\nu-1})$$

for a fixed integer ν satisfyng $m-1 \geqq \nu \geqq 1$, and differentiating exteriorly, we have

$$\begin{aligned} &d((n, \delta_1, \delta n, \dots, \delta n, dx, \dots, dx)) \\ (1.6) \qquad &= ((\delta n, \delta_1, \delta n, \dots, \delta n, dx, \dots, dx)) \\ &+ ((n, \delta(\delta_1), \delta n, \dots, \delta n, dx, \dots, dx)) \end{aligned}$$

because of $\delta\delta n^i = 0$ for the space of constant RIEMANN curvature. On substituting $\delta n^i = -b^\beta_\alpha \frac{\partial x^i}{\partial u^\beta} du^\alpha$ into the first term of the right-hand member of (1.6) we obtain

$$((\delta n, \delta_1, \delta n, \dots, \delta n, dx, \dots, dx)) = m\,!\,(-1)^{m-\nu} H_{\nu+1} n_i \delta^i_1 dA.$$

And from that the vector $n \times \underbrace{\delta n \times \dots \times \delta n}_{\nu} \times \underbrace{dx \times \dots \times dx}_{m-\nu-1}$ is orthogonal to the normal vector n and $\delta u^i = -b^\beta_\alpha \frac{\partial x^i}{\partial u^\beta} du^\alpha$, the second term of the right-hand member of (1.6) becomes as follows

$$((n, \delta(\delta_1), \delta n, \dots, \delta n, dx, \dots, dx)) = m\,!\,(-1)^{m-\nu} \frac{1}{2m} H^{\alpha\beta}_\nu \underset{\xi}{\mathfrak{L}} g_{\alpha\beta},$$

where $\varepsilon^{\alpha_1 \cdots \alpha_m}$ being the ε-symbol of the hypersurface V^m,

$$H^{\alpha\beta}_\nu \overset{\text{def}}{=} \frac{1}{(m-1)} \varepsilon^{\alpha\alpha_1 \cdots \alpha_{m-1}} \varepsilon^{\beta\beta_1 \cdots \beta_{m-1}} b_{\alpha_1\beta_1} \cdots b_{\alpha_\nu\beta_\nu} g_{\alpha_{\nu+1}\beta_{\nu+1}} \cdots g_{\alpha_{m-1}\beta_{m-1}}$$

and

$$\underset{\xi}{\mathfrak{L}} g_{\alpha\beta} \overset{\text{def}}{=} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta} \underset{\xi}{\mathfrak{L}} g_{ij}.$$

Accordingly we have

$$\frac{1}{m\,!} d\,((n, \delta_1, \delta n, \dots, \delta n, dx, \dots, dx)) = (-1)^{m-\nu} \left\{ H_{\nu+1} n_i \delta^i_1 dA + \frac{1}{2m} H^{\alpha\beta}_\nu \underset{\xi}{\mathfrak{L}} g_{\alpha\beta} dA \right\}.$$

Integrating the above expression over the whole hypersurface V^m and applying the STOKES' theorem, it follows that

$$\frac{1}{m\,!} \int_{\partial V^m} ((n,\delta_1,\delta n,\dots,\delta n,dx,\dots,dx)) = (-1)^{m-\nu} \left\{ \int_{V^m}\!\!\cdots\!\int H_{\nu+1} n_i \delta^i_1 dA + \frac{1}{m\,!} \int_{V^m}\!\!\cdots\!\int H^{\alpha\beta}_\nu \underset{\xi}{\mathfrak{L}} g_{\alpha\beta} dA \right\}.$$

Thus we have

$$\int_{V^m}\!\!\cdots\!\int H_{\nu+1} n_i \delta^i_1 dA + \frac{1}{2m} \int_{V^m}\!\!\cdots\!\int H^{\alpha\beta}_\nu \underset{\xi}{\mathfrak{L}} g_{\alpha\beta} dA = 0 \tag{1.7}$$

by virtue of that the hypersurface V^m is closed.

We can see easily that in the general coordinate system, the formulas (1.5) and (1.7) become as follows

$$\text{(I)} \qquad \int_{V^m}\!\!\cdots\!\int H_1 n_i \xi^i dA + \frac{1}{2m}\int_{V^m}\!\!\cdots\!\int g^{*ij} \underset{\xi}{\mathfrak{L}} g_{ij} dA = 0,$$

and

$$\text{(II)} \qquad \int_{V^m}\!\!\cdots\!\int H_{\nu+1} n_i \xi^i dA + \frac{1}{2m}\int_{V^m}\!\!\cdots\!\int H_{\nu}^{\alpha\beta} \underset{\xi}{\mathfrak{L}} g_{\alpha\beta} dA = 0 \qquad (m-1 \geqq \nu \geqq 1).$$

We call such the formulas (I) and (II) the generalized MINKOSKI formulas for the closed orientable hypersurface V^m in R^{m+1}.

§ 2. **The Minkowski formulas concerning some special transformations.** In this section, we shall discuss the formulas (I) and (II) for a special infinitesimal point transformation. Let the group G of transformations be conformal, that is, ξ^i satisfies an equation: $\underset{\xi}{\mathfrak{L}} g_{ij} \equiv \xi_{i;j} + \xi_{j;i} = 2\Phi g_{ij}$ ([32]), p. 32), then we obtain

$$g^{*ij} \underset{\xi}{\mathfrak{L}} g_{ij} = 2m\Phi, \quad H_{\nu}^{\alpha\beta} \underset{\xi}{\mathfrak{L}} g_{\alpha\beta} = 2m\Phi H_{\nu}.$$

Therefore (I) and (II) are rewritten in the following forms

$$\text{(I)}_c \qquad \int_{V^m}\!\!\cdots\!\int H_1 n_i \xi^i dA + \int_{V^m}\!\!\cdots\!\int \Phi dA = 0,$$

$$\text{(II)}_c \qquad \int_{V^m}\!\!\cdots\!\int H_{\nu+1} n_i \xi^i dA + \int_{V^m}\!\!\cdots\!\int \Phi H_{\nu} dA = 0 \qquad (m-1 \geqq \nu \geqq 1),$$

and we can see

$$\text{(I)}_h \qquad \int_{V^m}\!\!\cdots\!\int H_1 n_i \xi^i dA + c\int_{V^m}\!\!\cdots\!\int dA = 0,$$

$$\text{(II)}_h \qquad \int_{V^m}\!\!\cdots\!\int H_{\nu+1} n_i \xi^i dA + c\int_{V^m}\!\!\cdots\!\int H_{\nu} dA = 0 \qquad (m-1 \geqq \nu \geqq 1)$$

in case of $\Phi = \text{constant}$ $(\equiv C)$ (G being homothetic), and

$$\text{(I)}_i \qquad \int_{V^m}\cdots\int H_1 n_i \xi^i dA = 0,$$

$$\text{(II)}_i \qquad \int_{V^m}\cdots\int H_{\nu+1} n_i \xi^i dA = 0 \qquad (m-1 \geqq \nu \geqq 1)$$

in case of $\Phi = 0$ (G being isometric).

Especially if our space R^{m+1} is a Euclidean space E^{m+1} and if the path of the infinitesimal point transformation (1.1) is the straight line which pass through a fixed point 0, x^i being the coordinate of a point P in E^{m+1} with respect to a Cartesian coordinate system with the point 0 as the origin, let the position vector x^i take as the vector ξ^i, then we have

$$\underset{\xi}{\mathcal{L}} g_{ij} = 2 g_{ij}$$

according to $g_{ij} = \text{constant}$ and from the formulas (I)_h and (II)_h in the case $C = 1$, we have

$$\int_{V^m}\cdots\int H_1 p\, dA + \int_{V^m}\cdots\int dA = 0,$$

$$\int_{V^m}\cdots\int H_{\nu+1} p\, dA + \int_{V^m}\cdots\int H_\nu dA = 0 \qquad (m-1 \geqq \nu \geqq 1).$$

The above results are nothing but the formulas (0.2) given by HSIUNG.

Next, let our RIEMANN space R^{m+1} have more an assumption to admit a special coordinate system y, in which the components of the vector ξ^i are equal to the coordinate of the point P, that is, $\xi^i(y) = y^i$. Since the quantity $g_{ij}(x)\xi^i(x)\xi^j(x)$ is the homogeneous function of two degree with respect to ξ^i, it is requested that $g_{ij}(y)y^i y^j$ is also the homogeneous function of two degree with respect to y^i. Therefore $g_{ij}(y)$ must be the homogeneous function of zero degree with respect to y^i, that is,

$$\text{(2.1)} \qquad \frac{\partial g_{ij}(y)}{\partial y^k} y^k = 0.$$

On making use of the above relation (2.1), we can observe

$$\underset{\xi}{\mathfrak{L}} g_{ij}(y) = 2 g_{ij}(y)$$

in such the special coordinate system y, the formulas (I) and (II) are written in the form

$$(2.2)\qquad \begin{aligned} &\int_{V^m} \!\!\cdots\!\! \int H_1 n_i y^i dA + \int_{V^m} \!\!\cdots\!\! \int dA = 0, \\ &\int_{V^m} \!\!\cdots\!\! \int H_{\nu+1} n_i y^i dA + \int_{V^m} \!\!\cdots\!\! \int H_\nu dA = 0 \qquad (m-1 \geqq \nu \geqq 1) \end{aligned}$$

The formulas (0.3) and (0.4) of HSIUNG are nothing but a special case of (2.2)

§. 3. Some properties of a closed orientable hypersurface. - In this section we suppose again that the group G of tranformation (1.1) is conformal, then we can show the following four fheorems for a closed orientable hypersurface V^m in a RIEMANN space R^{m+1} of constant RIEMANN curvature.

THEOREM 3.1. - If in R^{m+1}, there exists such a group of conformal transformations as $\tilde{p}$ is positive (or negative) at each point of V^m and if H_1 is constant, then every point of V^m is umbilic, where $\tilde{p}$ denotes $n_i \xi^i$.

PROOF. - Multiplying the formula $(\mathrm{I})_c$ in § 2 by $H_1 = \text{const.}$, we have

$$\int_{V^m} \!\!\cdots\!\! \int H_1^2 \tilde{p} dA + \int_{V^m} \!\!\cdots\!\! \int \Phi H_1 dA = 0$$

and from the formula $(\mathrm{II})_c$ in § 2

$$\int_{V^m} \!\!\cdots\!\! \int H_2 \tilde{p} dA + \int_{V^m} \!\!\cdots\!\! \int \Phi H_1 dA = 0.$$

Consequently it follows that

$$(3.1)\qquad \int_{V^m} \!\!\cdots\!\! \int (H_1^2 - H_2) \tilde{p} dA = 0$$

which holds if and only if $H_1^2 - H_2 = 0$, since

$$H_1^2 - H_2 = \frac{1}{m^2}(\Sigma k_i)^2 - \frac{2}{m(m-1)}\Sigma k_{i_1}k_{i_2}$$

(3.2)

$$= \frac{1}{m^2(m-1)}\Sigma (k_{i_1} - k_{i_2})^2 \geqq 0,$$

where i_1, i_2 are distinct and run from 1 to m. From (3.1) and (3.2) we obtain

$$k_1 = k_2, \dots = k_m$$

at each point of V^m. Accordingly every point of the hypersurface V^m is umbilic.

We can see easily that if every point of a hypersurface in a RIEMANN space of constant RIEMANN curvature is umbilic, then the hypersurface V^m is also an m-dimensional RIEMANN space of constant RIEMANN curvature.

THEOREM 3.2. - If in R^{m+1}, there exists such a group G of conformal transformations as $\tilde{p}$ is positive (or negative) at each point of V^m, and if the principal curvature $k_1, k_2, \dots, k_m$ at each point of the hypersurface V^m are positive and H_ν is constant for any $\nu(m-1 \geqq \nu \geqq 1)$, then every point of V^m is nmbilic.

PROOF. - From the formulas $(\mathrm{I})_c$ and $(\mathrm{II})_c$ in § 2, we obtain

$$\int_{V^m}\!\!\cdots\!\int H_1 H_\nu \tilde{p}\, dA + \int_{V^m}\!\!\cdots\!\int \Phi H_\nu dA = 0,$$

$$\int_{V^m}\!\!\cdots\!\int H_{\nu+1} \tilde{p}\, dA + \int_{V^m}\!\!\cdots\!\int \Phi H_\nu dA = 0$$

because of $H_\nu =$ constant. Therefore we have

$$\int_{V^m}\!\!\cdots\!\int (H_1 H_\nu - H_{\nu+1})\tilde{p}\, dA = 0 \tag{3.3}$$

which holds when and only when $H_1H_\nu - H_{\nu+1} = 0$, since

$$H_1H_\nu - H_{\nu+1} = \frac{\nu\,!\,(m-\nu)\,!}{mm\,!}\left\{\Sigma\, k_i\, \Sigma\, k_{i_1} \dots k_{i_\nu}\right.$$

$$\text{(3.4)} \qquad \left. - \frac{(\nu+1)\,!\,(m-\nu-1)\,!}{m\,!}\,\Sigma\, k_{i_1} \dots k_{i_{\nu+1}}\right\}$$

$$= \frac{\nu\,!\,(m-\nu-1)\,!}{mm\,!}\,\Sigma\, k_{i_1} \dots k_{i_{\nu-1}}(k_{i_\nu} - k_{i_{\nu+1}})^2 \geqq 0,$$

where $i_1,\ i_2, \dots i_{\nu+1}$ are distinct and run from 1 to m, From (3.3) and (3.4) we have

$$k_1 = k_2, \dots, = k_m$$

at each point of V^m. Accordingly every point of V^m is umblic.

THEOREM 3.3. - If in R^{m+1}, there exists such a group G of conformal transformations as $\tilde{p}$ is positive (or negative) at each point of V^m, for which $H_1\tilde{p} + \Phi \geqq o$(or $\leqq 0$) at all points of V^m, then every point of V^m is umbilic.

PROOF. We can see

$$\Phi = -H_1\tilde{p}$$

according to the assumption that $H_1\tilde{p} + \Phi \geqq 0$ (or $\leqq 0$) at all points of V^m and the formula $(\mathrm{I})_c$ in § 2:

$$\int_{V^m}\!\!\cdots\!\int H_1\tilde{p}\,dA + \int_{V^m}\!\!\cdots\!\int \Phi\, dA = 0.$$

On substituting $\Phi = -H_1p$ into the formula $(\mathrm{II})_c$ in § 2, it follows that

$$\int_{V^m}\!\!\cdots\!\int (H_1^2 - H_2)\tilde{p}\,dA = 0.$$

Thus, we have the conclusion from (3.1) and (3.2).

THEOREM 3.4. - If H_1 is positive (or negative) at all points of V^m and if in R^{m+1}, there exists such a group G of conformal transformations as Φ is positive (or negative), for which either $\tilde{p} \leqq \frac{-\Phi}{H_1}$ or $\tilde{p} \geqq \frac{-\Phi}{H_1}$ at all points of V^m, then every point of V^m is umbilic.

PROOF. - The formula $(I)_c$ in § 2 is rewritten as follows

$$\int_{V^m}\cdots\int H_1\left(\tilde{p} + \frac{\Phi}{H_1}\right) dA = 0.$$

By virtue of the assumptions: $H_1 > 0$(or < 0) and $\tilde{p} + \frac{\Phi}{H_1} \geqq 0$(or $\leqq 0$) at all points of V^m, we have the following relation

$$\tilde{p} = \frac{-\Phi}{H_2}$$

On substituting $\tilde{p} = \frac{-\Phi}{H_1}$ into the formula $(II)_c$ in § 2, we obtain

$$\int_{V^m}\cdots\int \frac{\Phi}{H_1}(H_1^2 - H_2) dA = 0$$

which holds if and only if $H_1^2 - H_2 = 0$. Thus we can see the conclusion.

Such the method of calculation referring to a differential form is learned much from the paper ([4]) of H. HOPF and K. VOSS.

The present author wishes to express' to Professor Dr. H. HOPF her very sincere appreciation for his kind suggetion.

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