The Parabolic Equation as a Limiting Case of a Certain Elliptic Equation.

MILOŠ ZLÁMAL (Brno, Cecoslovacchia)

To Enrico Bompiani on his scientific Jubilee.

Summary. - *We consider the Dirichlet problem for the equation* $L_\varepsilon(u) \equiv u_{xx} + \varepsilon u_{yy} + A(x, y)u_x - B(y)u_y + C(x, y)u = F(x, y)$ *where* $B(y) > 0$ *and* ε *is a small positive parameter. An asymptotic formula is proved, from which it follows that in a suitable part of the domain of definition* $u(x, y, \varepsilon) \to U(x, y)$ *as* $\varepsilon \to 0+$, *where* $U(x, y)$ *is the solution of the corresponding boundary-value problem for the reduced equation* $L_0(U) \equiv U_{xx} + A(x, y)U_x - B(y)U + C(x, y)U = F(x, y)$.

1. We shall deal with the first boundary-value problem for the elliptic equation

$$L_\varepsilon(u) \equiv u_{xx} + \varepsilon u_{yy} + A(x, y)u_x - B(y)u_y + C(x, y)u = F(x, y), \tag{1}$$

where ε is a small positive parameter and $B(y) > 0$. Let the boundary ∂R of the closed region $\bar{R}$, in which the equation (1) is considered, consist of a part of the line $y = y_1$, of two continuous curves $x = \nu_1(y)$, $x = \nu_2(y)$ with $\nu_1(y) < \nu_2(y)$ for $y_1 \leq y < y_2$ and, if $\nu_1(y_2) < \nu_2(y_2)$, of a part of the line $y = y_2$. The aim of the present paper is to examine the asymptotic form of the solution $u(x, y, \varepsilon)$ of (1) which satisfies the boundary condition

$$u \mid_{\partial R} = \varphi. \tag{2}$$

It is to be expected that in a suitable part of the region $\bar{R}$, $u(x, y, \varepsilon)$ converges, as $\varepsilon \to 0+$, to a suitable solution of the parabolic equation

$$L_0(U) \equiv U_{xx} + A(x, y)U_x - B(y)U_y + C(x, y)U = F(x, y). \tag{3}$$

Since we cannot prescribe the complete boundary condition (2) for the solution of the equation (3) the problem is of singular perturbation type; therefore the so-called boundary layer terms will appear in the asymptotic formula for $u(x, y, \varepsilon)$ (see [1]).

The solution $U(x, y)$ is determined in the following manner: We choose y_0 such that $y_0 > y_1$ and $y_0 < y_2$ if $\nu_1(y_2) = \nu_2(y_2)$, $y_0 = y_2$ if $\nu_1(y_2) < \nu_2(y_2)$. Let $\bar{\Omega}$

denote the part of the region $\bar{R}$ which lies in the half-plane $y \leq y_0$ and Γ the boundary of Ω from which we extract the line segment $\{(x, y) \mid \nu_1(y_0) < x < \nu_2(y_0), y = y_0\}$. Then $U(x, y)$ is uniquely determined by the boundary condition $U \mid_\Gamma = \varphi$; in greater detail

$$(4) \qquad U(x, y_1) = \psi(x), \quad U(\nu_1(y), y) = \varphi_1(y), \quad U(\nu_2(y), y) = \varphi_2(y).$$

Equation (1) was not chosen in the most general form: a more general equation than (1) would be represented by

$$a(x, y)u_{xx} + \varepsilon u_{yy} + A(x, y)u_x - B(x, y)u_y + C(x, y)u = F(x, y).$$

It is not important that we restricted ourselves to the case $a(x, y) = 1$, since the case with an arbitrary $a(x, y)$ can be treated by the same method which is described in this paper, and the results are the same. However, the assumption that the coefficient B depends on y only is more essential for our method, as we shall see later.

2. We assume the following properties of the coefficients and the right-hand side of (1) in the region $\bar{R}$ and of the functions $\nu_1(y)$, $\nu_2(y)$, $\psi(x)$, $\varphi_1(y)$, $\varphi_2(y)$:

a) $B(y) > 0$.

b) $B(y)$ has a derivative of the third order, $A(x, y)$ has partial derivatives of the third order, $C(x, y)$ and $F(x, y$ of the second order, and all these derivatives satify a Lipschitz condition with respect to all their variables.

c) $\nu_1(y)$, $\nu_2(y)$ and $\psi(x)$ have derivatives of the fourth order and $\varphi_1(y)$, $\varphi_2(y)$ derivatives of the third order and all these derivatives satisfy a Lipschitz condition.

Our aim is to prove the following.

THEOREM. - *Let there be satisfied the assumptions a), b), c). Then in the region* $\bar{\Omega}$ *we have*

$$(5) \qquad u(x, y, \varepsilon) = U(x, y) + h(x, y, \varepsilon)\, e^{-\frac{\alpha(y)}{\varepsilon}} + O\left(\varepsilon^{\frac{1}{2}}\right),$$

where $h(x, y, \varepsilon) = O(1)$ in $\bar{\Omega}$ and $\alpha(y) = \int_y^{y_0} B(s)ds$.

The term $h(x, y, \varepsilon)\, e^{-\frac{\alpha(y)}{\varepsilon}}$ is a boundary layer term since it becomes important only near the line $y = y_0$ where it equalises different values of $u(x,y, \varepsilon)$ and $U(x, y)$.

In this paragraph we shall only sketch the proof. It will be apparent why we are forced to restrict ourselves to the case of B depending on y only. First of all we may suppose $y_1 = 0$, $y_0 = 1$, $\nu_1(0) = 0$, $\nu_2(0) = 1$. Further we may assume $\psi(x) = \varphi_1(y) = \varphi_2(y) = 0$. For it is sufficient to put $u = v + \tau(x, y)$ where

$$\tau(x, y) = \psi\left[\frac{x - \nu_1(y)}{\nu_2(y) - \nu_1(y)}\right] + \left[\varphi_1(y) - \varphi_1(0)\right]\left[1 - \frac{x - \nu_1(y)}{\nu_2(y) - \nu_1(y)}\right]$$

$$+ \left[\varphi_2(y) - \varphi_2(0)\right]\frac{x - \nu_1(y)}{\nu_2(y) - \nu_1(y)},$$

and it is seen that $v \mid_\Gamma = 0$. Also, $L_\varepsilon(v) = F(x, y) - L_0(\tau(x, y)) - \varepsilon\tau_{yy}(x, y)$. However, the right-hand side of this equation depends on ε. But putting $v = v_1 + \varepsilon v_2$ and taking v_2 such that $L_\varepsilon(v_2) = -\tau_{yy}$ and $v_2 \mid_{\partial\Omega} = 0$, we can easily prove by means of the maximum principle that $v_2 = O(1)$. Concerning v_1 we obtain $L_\varepsilon(v_1) = F(x, y) - L_0(\tau(x, y))$ and, as $v \mid_\Gamma = v_2 \mid_{\partial\Omega} = 0$, $v_1 \mid_\Gamma = 0$. By our assumptions, the function $L_0(\tau(x, y))$ has the same derivability properties as $F(x, y)$, and the function $\tau_{yy}(x, y)$ is certainly continuous. In the sequel no new notation is used for the right-hand side $F(x, y) - L_0(\tau(x, y))$; the equation will have the form (1) and instead of (4) we shall have

$$U(x, 0) = U(\nu_1(y), y) = U(\nu_2(y), y) = 0. \tag{6}$$

Now, following [1], let us seek the solution $u(x, y, \varepsilon)$ in the form

$$u(x, y, \varepsilon) = U(x, y) + h(x, y, \varepsilon)\, e^{-\frac{g(x, y)}{\varepsilon}} + \varepsilon z(x, y, \varepsilon).$$

Substituing into (1), the term of highest order on the left is $\frac{1}{\varepsilon^2} g_x^2 e^{-\frac{g}{\varepsilon}}$. If we want it to be zero we must choose g such that $g_x \equiv 0$, i. e. $g = \alpha(y)$. By this choice of the exponent the term of next highest order is $\frac{1}{\varepsilon} h\alpha'(B + \alpha')e^{-\frac{\alpha}{\varepsilon}}$. As $h \not\equiv 0$, $\alpha \neq$ konst, we must set $\alpha'(y) = -B$ and consequently B can depend on y only. In order to obtain a boundary layer term near the line $y = 1$, we choose $\alpha(y) = \int_y^1 B(s)ds$. Then the equation for $z(x, y, \varepsilon)$ reduces to

$$L_\varepsilon(z) = -U_{yy}(x, y) - \frac{1}{\varepsilon} e^{-\frac{\alpha(y)}{\varepsilon}} L_\varepsilon^*(h),$$

where

$$L_\varepsilon^*(h) \equiv h_{xx} + \varepsilon h_{yy} + A(x, y)h_x + B(y)h_y + [B'(y) + C(x, y)]h.$$

Now it seems obvious to choose the function h in the following manner

$$L_\varepsilon^*(h) = \varepsilon,\ h \mid_\Gamma = 0,\ h(x, 1) = u(x, 1, \varepsilon) - U(x, 1).$$

By means of the maximum principle we can easily prove that $h(x, y, \varepsilon) = 0(1)$. The function $z(x, y, \varepsilon)$ satisfies the equation $L_\varepsilon(z) = -U_{yy} - e^{-\frac{\alpha}{\varepsilon}}$ and the boundary condition $z \mid_{\partial\Omega} = 0$. If $U_{yy}(x, y)$ were a bounded function, we should have, again from the maximum principle, $z(x, y, \varepsilon) = 0(1)$ so that the final result would be better than formula (5), namely $u(x, y, \varepsilon) = U(x, y) + h(x, y, \varepsilon)e^{-\frac{\alpha(y)}{\varepsilon}} + 0(\varepsilon)$. But $U_{yy}(x, y)$ need not be bounded even if we choose the boundary values arbitrarily smooth. For the boundedness it is necessary that at the points (0, 0), (1, 0) certain equations of conformity should be satisfied. Of course, such an assumption does not correspond to the nature of the original Dirichlet problem. To do without it, we shall apply a device used in the investigation of hyperbolic equations with a small parameter (see [2]). We shall replace the right-hand side $F(x, y)$ by the function $\overline{F}(x, y)$ which differs from it for $0 \leq y \leq \delta$ only, where δ is a new parameter, and such that the solution $\overline{U}(x, y)$ of $L_0(\overline{U}) = \overline{F}$, $\overline{U} \mid_\Gamma = 0$ has a bounded $\overline{U}_{yy}(x, y)$. We will then be able to prove $\overline{U}_{yy} = 0(\delta^{-1})$. Denote $F(x, y) - \overline{F}(x, y)$ by $\Phi(x, y)$ and construct on Ω the solutions w, $\overline{u}$, W of the following equations: $L_\varepsilon(w) = \Phi(x, y)$ and $w \mid_{\partial\Omega} = 0$, $L_\varepsilon(\overline{u}) = \overline{F}(x, y)$ and $\overline{u} \mid_{\partial\Omega} = u \mid_{\partial\Omega}$, $L_0(W) = \Phi(x, y)$ and $W \mid_\Gamma = 0$.

Then we have $u = \overline{u} + w$, $U = \overline{U} + W$. Concernin $\overline{u}$ we can already set $\overline{u} = \overline{U} + he^{-\frac{\alpha}{\varepsilon}} + \varepsilon z$ where $L_\varepsilon^*(h) = \varepsilon$, $h \mid_\Gamma = 0$, $h(x, 1, \varepsilon) = \overline{u}(x, 1, \varepsilon) - \overline{U}(x, 1) = u(x, 1, \varepsilon) - \overline{U}(x, 1)$. The final result is

$$u = U + he^{-\frac{\alpha}{\varepsilon}} - W + \varepsilon z + w. \tag{8}$$

On the basis of the estimate $\overline{U}_{yy} = 0(\delta^{-1})$ we conclude $z = 0(\delta^{-1})$ (¹).

Concerning W, we shall prove $W = O(\delta)$; this is quite reasonable since the righ-hand side Φ differs from zero for $0 \leq y \leq \delta$ only. It then follows from (8) that $u = U + he^{-\frac{\alpha}{\varepsilon}} + O(\delta) + \varepsilon O(\delta^{-1}) + w$.

(¹) This means $\mid z \mid \leq C\delta^{-1}$ where C depends on neither ε nor δ.

Putting $\delta = \varepsilon^{\frac{1}{2}}$ (and this is the best choice) we obtain $u = U + he^{-\frac{\alpha}{\varepsilon}} + O\left(\varepsilon^{\frac{1}{2}}\right) + w$. Now it is sufficient to prove $w = O\left(\varepsilon^{\frac{1}{2}}\right)$.

3. PROOF OF THE THEOREM: All estimates will be carried through by means of the maximum principle, which, for our purpose we formulate for an equation of the form

$$\mu(x, y)u_{xx} + \nu(x, y)u_{yy} + \alpha(x, y)u_x - \beta(x, y)u_y - \gamma(x, y)u = \psi(x, y) \tag{9}$$

in this way: we suppose $\mu(x, y) > 0$ and either $\nu(x, y) > 0$ or $\nu(x, y) \equiv 0$. In the first case the region R in which the equation (9) is considered can be any bounded domain; in the second case the equation is considered in the domain Ω introduced above, and we suppose $\beta(x, y) > 0$ in $\bar{\Omega}$. Let be $\gamma(x, y) > \frac{1}{\gamma_0} > 0$ in both cases (i. e. either in R or in Ω). Then we have, in $\bar{R}$ or in $\bar{\Omega}$ respectively,

$$| u(x, y) | \leq \max(\gamma_0 M_1, M_2), \tag{10}$$

where either $M_1 = \max\limits_{\bar{R}} | \psi(x, y) |$, $M_2 = \max\limits_{\partial R} | u(x, y) |$ or $M_1 = \max\limits_{\bar{\Omega}} | \psi(x, y) |$, $M_2 = \max\limits_{\Gamma} | u(x, y) |$ respectively.

First, from the maximum principle it follows that the solution $u(x, y, \varepsilon)$ of the Dirichlet problem (1), (2) is bounded with respect to ε, i. e. $u(x, y, \varepsilon) = O(1)$. To prove this it is sufficient to set $u = e^{ky}v$. We obtain

$$v_{xx} + \varepsilon v_{yy} + Av_x + (-B + 2k\varepsilon)v_y - (kB - C - \varepsilon k^2)v = Fe^{-ky}.$$

We choose k such that $kB(y) - C(x, y) \geq 2$ in $\bar{R}$. Then we have $kB(y) - C(x, y) - k\varepsilon^2 \geq 1$ for $\varepsilon \leq \frac{1}{k^2}$, so that following (10) we have $| v | \leq \max\left(\max\limits_{\bar{R}} | F(x, y) \cdot e^{-ky} |, \max\limits_{\partial R} | e^{-ky}\varphi |\right)$, i. e. $v = O(1)$, and thus $u = O(1)$. It is obvious that the same remark holds for the equation $L_\varepsilon^*(h) = \varepsilon$ if the boundary values of the function h are bounded with respect to ε.

Now let us define the function $\bar{F}(x, y)$. Set

$$\bar{F}(x, y) = \omega\left(\frac{y}{\delta}\right) F(x, y), \tag{11}$$

where $\omega(s)$ is a three times continuously differentiable function such that $0 \leq \omega(s) \leq 1$ for $s \geq 0$, $\omega(s) = 0$ for $0 \leq s \leq \frac{1}{2}$, $\omega(s) = 1$ for $s \geq 1$. Obviously we have $\bar{F}(x, y) = 0$ for $y \leq \frac{1}{2}\delta$.

Consider the solution $\bar{U}(x, y)$ of the equation $L_0(\bar{U}) = \bar{F}(x, y)$ with condition $\bar{U} \mid_\Gamma = 0$. We introduce new independent variables $\xi = \frac{x - \nu_1(y)}{\nu_2(y) - \nu_1(y)}$, $\eta = y$. The domain Ω is mapped onto the square Ω^*: $0 < \xi < 1$, $0 < \eta < 1$. If we also introduce a new unknown function V by the relation $\bar{U} = \chi(x, y) \cdot V$ and choose $\chi(x, y)$ conveniently, we can obtain the equation for V in the form

$$V_{\xi\xi} - b(\eta) V_\eta + c(\xi, \eta) V = \bar{f}(\xi, \eta, \delta) \tag{12}$$

with boundary condition

$$V \mid_{\Gamma^*} = 0. \tag{13}$$

where $\Gamma^* = \partial\Omega^* - \{(\xi, \eta) \mid 0 < \xi < 1, \ \eta = 1\}$. The coefficients b, c do not depend on δ, they have derivatives of the second order satisfying a Lipschitz condition with respect to their variables and $b(\eta)$ is positive for $0 \leq \eta \leq 1$. The right-hand side $\bar{f}(\xi, \eta, \delta)$ also has derivatives of the second order satisfying a Lispschitz condition with respect to its variables, and

$$\begin{cases} \bar{f} = 0 \quad \text{for} \quad 0 \leq \eta \leq \frac{1}{2}\delta, \quad \bar{f} = 0(1) \quad \text{on } \bar{\Omega}^*, \\ \bar{f}_\eta = 0(1), \ \bar{f}_{\eta\eta} = 0(1) \ \text{for} \ \eta \geq \delta, \ \bar{f}_\eta = 0(\delta^{-1}), \ \bar{f}_{\eta\eta} = 0(\delta^{-2}) \ \text{for} \ 0 \leq \eta \leq \delta. \end{cases} \tag{14}$$

We are to prove that the solution $V(\xi, \eta)$ has derivatives of the second order on $\bar{\Omega}^*$ and these are $0(\delta^{-1})$.

First, from the maximum principle it follows that $V = 0(1)$ (in the same manner as that used in the proof of boundedness of $u(x, y, \varepsilon)$). Further, from the theorem 2 of [3] it follows that $V(\xi, \eta)$ has continuous derivatives of the first order and a continuous $V_{\xi\xi}$ on $\bar{\Omega}^*$ (note that the conditions of conformity in this theorem are satisfied since $\bar{f}(\xi, 0) = 0$). From the lemma 2 of [4] it follows ([2]) that it is posible to differentiate the equation

([2]) This lemma also concerns the differentiation $\frac{\partial}{\partial\xi}$, $\frac{\partial^2}{\partial\xi^2}$, $\frac{\partial^2}{\partial\eta\partial\xi}$ and $\frac{\partial^3}{\partial\eta\partial\xi^2}$. Therefore the assumptions introduced in it are stronger than those which ensure differentiability $\frac{\partial}{\partial\eta}$. For this, in the case of the equation (12) and boundary condition (13), it suffices to suppose the following: 1, $b(\eta) > 0$, 2, $b(\eta)$ has a derivative satisfying a Lipschitz condition, 3, $c(\xi, \eta)$, $\bar{f}(\xi, \eta, \delta)$ satisfy a Lipschitz condition with respect to both variables, have the derivative of the first order with respect to η satisfying a Lipschitz condition and $c(\xi, 0)$ has a derivative satisfying a Lipschitz condition, 4, $\bar{f}(0, 0) = \bar{f}(1, 0) = 0$.

(12) with respect to η so that V_η satisfies the equation

$$Q_{\xi\xi} - b(\eta)Q_\eta + (c(\xi, \eta) - b'(\eta))Q = f_\eta(\xi, \eta, \delta) - c_\eta(\xi, \eta)V; \tag{15}$$

obviously $V_\eta \mid_{\Gamma_*} = 0$ as it follows from (13), (12) and (14). The assumptions of the theorem 2 of [3] are again fulfilled by the equation (15), so that the derivatives Q_ξ and Q_η, i. e. $V_{\eta\xi}$ and $V_{\eta\eta}$, are continuous on the bounded square $\bar{\Omega}^*$. The right-hand side $P_1(\xi, \eta)$ of (15) is $O(\delta^{-1})$ for $0 \leq \eta \leq \delta$. Put $Q = e^{\frac{\eta}{\delta}} Y$. Then $Y \mid_{\Gamma_*} = 0$ and Y satisfies the equation

$$Y_{\xi\xi} - b(\eta)Y_\eta - \delta^{-1}[b(\eta) - \delta(c(\xi, \eta) - b'(\eta))]Y = e^{-\frac{\eta}{\delta}} P_1(\xi, \eta).$$

Using the maximum principle on the region $0 \leq \xi \leq 1$, $0 \leq \eta \leq \delta$, we see that in this region we have $Y = O(1)$ and accordingly $Q = O(1)$. For $\eta \geq \delta$ there is $P_1(\xi, \eta) = O(1)$ and therefore $Q = O(1)$ holds also for $\eta \geq \delta$. Consequently we have $V_\eta = O(1)$ in $\bar{\Omega}^*$. From (12) we then get $V_{\xi\xi} = P_2$, where $P_2(\xi, \eta) = O(1)$ on $\bar{\Omega}^*$. As $V(0, \eta) = V(1, \eta) = 0$, there exists a $0 < \xi_1(\eta) < 1$ such that $V_\xi(\xi_1, \eta) = 0$; thus $V_\xi = \int_{\xi_1}^{\xi} P_2 d\xi$ and consequently $V_\xi = O(1)$ on $\bar{\Omega}^*$. As to $V_{\eta\eta}$, we again differentiate equation (15) with respect to η (the conditions of conformity are again fulfilled because the right-hand side of this equation is zero for $\eta = 0$). The right-hand side of the resulting equation is $O(\delta^{-2})$ for $0 \leq \eta \leq \delta$, and in the same manner as for V_η we can prove $V_{\eta\eta} = O(\delta^{-1})$ on $\bar{\Omega}^*$. From (15) it follows that $Q_{\xi\xi} = O(\delta^{-1})$, and on integrating $V_{\eta\xi} = Q_\xi = O(\delta^{-1})$.

Finally we must estimate z, W and w. Considering $L_\varepsilon(z) = -\bar{U}_{yy} - e^{-\frac{\alpha}{\varepsilon}} = O(\delta^{-1})$ and $z \mid_{\partial\Omega} = 0$ it follows from the maximum principle (by means of the substitution $z = e^{ky} \cdot v$) that $z = O(\delta^{-1})$ on $\bar{\Omega}$. Concerning W, we have $W \mid_\Gamma = 0$ and $L_0(W) = \Phi(x, y)$, where $\Phi(x, y) = O(1)$ for $0 \leq y \leq \delta$ and $\Phi(x, y) = 0$ for $y \geq \delta$. To prove $W = O(\delta)$ we again make use of the substitution $W = e^{\frac{y}{\delta}} Y$. Then $Y \mid_\Gamma = 0$ and

$$Y_{xx} + A(x, y)Y_x - B(y)Y_y - \delta^{-1}[B(y) - \delta \cdot C(x, y)]Y = e^{-\frac{y}{\delta}} \Phi(x, y).$$

From the maximum principle it follows that $Y = O(\delta)$ and consequently that $W = O(\delta)$ for $0 \leq y \leq \delta$. For $y \geq \delta$ there holds $\Phi = 0$ so that $L_0(W) = 0$ and $W \mid_{y=\delta} = O(\delta)$, $W \mid_{x=\nu_1(y)} = W \mid_{x=\nu_2(y)} = 0$; again by means of the maximum principle we can prove $W = O(\delta)$ for $y \geq \delta$. Consequently $W = O(\delta)$ on $\bar{\Omega}$.

Now we put $\delta = \varepsilon^{\frac{1}{2}}$ and it remains to prove $w = O(\varepsilon^{\frac{1}{2}})$. Setting $w = exp(\varepsilon^{-\frac{1}{2}} y) . Y$, we obtain for Y the equation

$$Y_{xx} + \varepsilon Y_{yy} A(x, y) Y_x + [2\varepsilon^{\frac{1}{2}} - B(y)] Y_y - \varepsilon^{-\frac{1}{2}} [B(y) - \varepsilon^{\frac{1}{2}} (C(x, y) +$$

$$+ 1)] Y = exp(\varepsilon^{-\frac{1}{2}} y) . \Phi(x, y).$$

Further we can proceed in the same manner as in the estimate of W.

BIBLIOGRAPHY

[1] N. LEVINSON: *The first boundary-value problem for* $\varepsilon\Delta u + Au_x + Bu_y + Cu = D$ *for small* ε. «Annals of Math.» 51, 1950, 428-446.

[2] M. ZLÁMAL: *Smešannaya zadača dlya giperboličeskih uravnenii s malym parametrom* Czechoslovak «Math. Journ.» 10 (85), 1960, 83-122.

[3] T. D. VENTCEL': *O primenenii metoda konečnyh raznosteĭ k rešeniyu pervoĭ kraevoĭ zadači dlya uravnenii paraboličeskogo tipa.* «Mat. Sbornik» 40 (82), 1956, 101-122.

[4] O. A. OLEINIK i T. D. VENTCEL': *Pervaya kraevaya zadača i zadača Cauchy dlya kvazilineinyh uravnenii paraboličeskogo tipa.* «Mat. Sbornik» 41 (83), 1957, 105-128.