ADDENDUM

Time-Reversal Invariance and Thermodynamics of Electromagnetic Fields in Materials with Memory.

RINALDO BORGHESANI - ANGELO MORRO (Genova) (*)

When our paper [1] was in press, we became aware of two very recent contributions by RIVLIN [2] and by MEIXNER [3], where *two* definitions of time-reversed functions are assumed. In order to avoid possible misinterpretations of some of our results, we show that our main theorems maintain essentially their validity even if the definitions of [2], [3] are adopted.

Following closely the notation given in [1], we recall our unique definition of the time-reversal of a closed path. If $\mathbf{f}(\cdot): (-\infty, \infty) \rightarrow \mathcal{U}$, the time-reversed path $\tilde{\mathbf{f}}(\cdot): (-\infty, \infty) \rightarrow \mathcal{U}$ is defined by

$$\tilde{\mathbf{f}}(t) = \mathbf{f}(-t).$$

According to [2], [3] we are led to lay down two distinct definitions which have only a microscopical justification—see the references quoted in [1]—. More precisely, if $\mathbf{f}_1$ is a *polar* vector, the closed path $\mathbf{f}_1(\cdot): (-\infty, \infty) \rightarrow \mathcal{U}$ has the following time-reversed path $\tilde{\mathbf{f}}_1(\cdot): (-\infty, \infty) \rightarrow \mathcal{U}$ such that

$$\tilde{\mathbf{f}}_1(t) = \mathbf{f}_1(-t).$$

On the contrary, when $\mathbf{f}_2$ is an *axial* vector, the time-reversed $\tilde{\mathbf{f}}_2(\cdot): (-\infty, \infty) \rightarrow \mathcal{U}$ is given by

$$\tilde{\mathbf{f}}_2(t) = -\mathbf{f}_2(-t).$$

If the above definitions are taken into account, the 2nd theorem of [1] suffers very slight modifications. In fact, when $\mathbf{u}(\cdot)$ and $\mathbf{v}(\cdot)$ are derived from a polar and from an axial vector respectively, then our (2.10) reads

$$\mathfrak{S}(\mathbf{u}, \mathbf{v}) - \mathfrak{S}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \int_{-\infty}^{\infty} \int_{-\infty}^t [\mathbf{u}(t) \cdot \mathbf{M}(t-s)\mathbf{v}(s) + \mathbf{v}(t) \cdot \mathbf{M}^T(t-s)\mathbf{u}(s)] ds dt,$$

where $\mathbf{M} = \mathbf{A}_1 + \mathbf{A}_2^T$. Obviously the conclusion $\mathbf{M}(s) = \mathbf{0}$, $\forall s \geq 0$ holds again. Also

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for the 3th theorem of [1] irrelevant changes are necessary. If $\mathbf{u}$ is an axial vector, a direct inspection provides

$$\mathbf{B}(s) = \mathbf{0} \quad \text{and} \quad \mathbf{D}(s) = \mathbf{0} \quad \forall s \geq 0.$$

In conclusion, the only modification in the statement of our main 6th theorem is: $\mathbf{C}_{\text{IV}} = \mathbf{0}$; thus we have

$$e_1 = C_{\text{I}} * \theta + \mathbf{C}_{\text{II}}^0 * \mathbf{g} + \mathbf{C}_{\text{III}}^0 * \mathbf{E},$$

$$\mathbf{q}_1 = \mathbf{K}_{\text{II}} * \mathbf{g} + \mathbf{K}_{\text{III}} * \mathbf{E},$$

$$\mathbf{j}_1 = \mathbf{\Sigma}_{\text{II}} * \mathbf{g} + \mathbf{\Sigma}_{\text{III}} * \mathbf{E},$$

with $\mathbf{K}_{\text{II}} = \mathbf{K}_{\text{II}}^T$, $\mathbf{\Sigma}_{\text{III}} = \mathbf{\Sigma}_{\text{III}}^T$ and $\mathbf{K}_{\text{III}} + \theta_0 \mathbf{\Sigma}_{\text{II}}^T = \mathbf{0}$. Of course the relaxation property for e (7th theorem) yields $\mathbf{C}_{\text{II}}^0 = \mathbf{0}$ and $\mathbf{C}_{\text{III}}^0 = \mathbf{0}$. It is worthwhile to remark that in our analysis, both with a unique and with two distinct definitions of time-reversal, we derive only Onsager-type symmetry relations. The absence of Casimir-type anti-symmetry properties is a consequence of the form of our entropy production which cannot be rearranged in the classical form examined by RIVLIN [2].

REFERENCES

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