Periodic Solutions of Some Nonlinear Autonomous Functional Differential Equations.

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Summary. – *We develop here some new fixed point theorems and apply them to the question of existence of nontrivial periodic solutions of nonlinear, autonomous functional differential equations. We prove that the standard results of G. S. Jones and R. B. Grafton can be obtained by our methods, and we prove periodicity results for some equations, for instance a neutral functional differential equation, which appear inaccessible by previous techniques.*

Introduction.

In 1962 G. S. JONES [17], drawing extensively on earlier work of E. M. WRIGHT [26], proved that the equation $x'(t) = -\alpha x(t-1)(1 + x(t))$ has nontrivial periodic solutions for $\alpha > \pi/2$. Subsequently JONES applied his methods to a number of other equations and proved periodic behaviour. Jone's basic technique was to apply certain fixed point theorems in which the existence of a nontrivial fixed point of a compact map F was guaranteed; these fixed points corresponded to nontrivial periodic solutions of a functional differential equation.

In 1969 R. B. GRAFTON attempted to simplify and generalize Jones's methods. GRAFTON avoided the use of fixed point theorems and gave an abstract result based on a Krasnoselskii theorem concerning eigenvalues of compact maps of a cone into itself. GRAFTON derived as consequences Jones's basic examples and also proved that the van der Pol equation with time lag $x''(t) - \varepsilon x'(t)(1 - x^2(t)) + x(t-r) = 0$ has nontrivial periodic solutions for $\varepsilon > 0$ and $r > 0$.

Both the Jones and Grafton theorems have obvious drawbacks. They can never apply to neutral FDE's because they are restricted to compact maps and the operation of translation along trajectories is never compact in that case. Also, for technical reasons they do not apply to some retarded FDE's for which periodicity results should hold.

In this paper we proceed in the general spirit of Jones's ideas. In Section 1 we prove a new fixed point theorem (Theorem 1.1) which generalizes theorems of Browder and Jones. As we show Theorem 1.1 is directly applicable to a large number of auto-

(*) Partially supported by NSFGP 20228 and a Rutgers Research Council Faculty Fellowship.

(**) Entrata in Redazione il 10 gennaio 1973.

nomous FDE's, though we should remark that there are cases (*e.g.*, Section 4) in which more general fixed point theorems are needed.

Section 2 is basically designed just to show that the standard examples of existence theorems for periodic solutions follow from Theorem 1.1, though at least some of the results (see, for example, Corollary 2.1) appear new.

In Section 3 we prove that the seemingly innocent generalization of Grafton's equation given by $x''(t)-\varepsilon x'(t)\big(1-x^2(t)\big)+x(t-r)-kx(t)$ has a nontrivial periodic solution of period greater than $2r$ if $-k_0<k<1$, where $k_0=\min(\varepsilon/r, 2/r^2)$, $r>0$ and $\varepsilon>0$. If $k<0$ this result appears inaccessible by Grafton's methods. Roughly the same arguments also give Grafton's theorems in [8], though we have not carried this through here. A technical point that may be of interest here is the simplified treatment of the characteristic equation for the linearized FDE.

In Section 4 we prove that the equation $x'(t)=-\alpha x\big(t-1-|x(t)|\big)\big(1-x^2(t)\big)$ has nontrivial periodic solutions for all $\alpha>\pi/2$. We are forced here to use a fixed point theorem different from Theorem 1.1, and in fact the interest of the equation stems from the nonstandard techniques it requires.

In Section 5 we prove that the neutral FDE

$$x'(t)=\left[-\alpha x(t-1)+\frac{k}{m+1}\frac{d}{dt}x^{m+1}(t-1)\right][1-x^2(t)]$$

has nontrivial periodic solutions if $m\geqslant 1$, $\alpha>\pi/2$ and

$$|k|\leqslant\frac{m+1}{4}\left(1+\frac{2}{m-1}\right)^{(m-1)/2}\left(|k|\leqslant\tfrac{1}{2}\text{ if } m=1\right).$$

Even this seemingly simple equation raises a number of unanswered questions.

In Sections 3, 4 and 5 we have not striven for the greatest possible generality; and a number of mechanical generalizations can be carried through. Instead we have tried to pose the simplest examples of equations which pose substantial technical difficulties.

1. – Our goal in this section is to establish some fixed point theorems which will be directly applicable to most of the examples we will consider. We begin by recalling some basic ideas. If (X, ϱ) is a metric space and A is a bounded subset of X, KURATOWSKI [16] has defined $\gamma(A)$, the measure of noncompactness of A, to be $\inf\{d>0\colon A$ has a finite covering by sets of diameter less than or equal to $d\}$. Of course if A is compact, $\gamma(A)=0$. If (X_1, ϱ_1) and (X_2, ϱ_2) are metric spaces and $f\colon X_1\to X_2$ is a continuous map, we shall call f a «k-set-contraction» if for every bounded set $A\subset X_1$, $f(A)$ is bounded and $\gamma_2\big(f(A)\big)\leqslant k\gamma_1(A)$; this idea is also due to KURATOWSKI [16]. Examples of k-set-contractions are given in [22], Section A.

The basic properties of the measure of noncompactness have been established by KURATOWSKI [16] and DARBO [6]. If X is a complete metric space and $\{A_n\}$ is a decreasing sequence of closed, bounded subsets of X such that $\gamma(A_n)\to 0$, KURATOWSKI proved that $A_\infty \equiv \bigcap_{n\leqslant 1} A_n$ is nonempty and compact and A_n approaches A_∞ in the Hausdorff metric. If X is a Banach space and $\overline{\text{co}}(A)$ denotes the convex closure of a bounded subset A of X, DARDO proved that $\gamma(\overline{\text{co}}(A))=\gamma(A)$. If A and B are any bounded subsets of X and $A+B=\{a+b\colon a\in A,\, b\in B\}$, DARDO also proved that $\gamma(A+B)\leqslant\gamma(A)+\gamma(B)$.

There is nothing sacred about the particular measure of noncompactness defined above. Generally, suppose that X is a Banach space and suppose that there is a real-valued function μ which assigns to each bounded set $A\subset X$ a nonnegative real number $\mu(A)$. Suppose that (1) there exists constants $m>0$, $M>0$ such that $m\mu(A)\leqslant\gamma(A)\leqslant M\mu(A)$ for every bounded set A, (2) $\mu(\overline{\text{co}}(A))=\mu(A)$ for every bounded set $A\subset X$, (3) if $A\subset B$, $\mu(A)\leqslant\mu(B)$, (4) $\mu(A\cup B)=\max(\mu(A)),\mu(B))$ and (5) $\mu(A+B)\leqslant\mu(A)+\mu(B)$. If μ satisfied 1-5 above, we shall call μ a generalized measure of noncompactness; obviously many other generalizations are possible. If $D\subset X$ and $f\colon D\to X$ is a continuous map such that $f(A)$ is bounded for every set $A\subset D$, we shall say f is a « k-set-contraction with respect to μ » if $\mu(f(A))\leqslant k\mu(A)$ for every bounded set $A\subset D$.

We mention as an example one generalized measure of noncompactness which will prove useful in Section 5. Let I denote a compact subinterval of $\boldsymbol{R}$ and let $C(I,\boldsymbol{R}^n)$ denote the space of continuous maps from I to $\boldsymbol{R}^n$. For $x\in C(I,\boldsymbol{R}^n)$, define $\|x\|=\sup_{t\in I}|x(t)|$, where $|\cdot|$ denotes a fixed norm on $\boldsymbol{R}^n$. If A is a bounded subset of $C(I,\boldsymbol{R}^n)$ and $\delta>0$ we write $\omega(\delta;A)=\sup\{|x(t)-x(s)|\colon x\in A,\, t,s\in I,\ |t-s|\leqslant\delta\}$, and we define $\omega(A)$, the modulus of continuity of A, to be $\lim_{\delta\to 0}\omega(\delta;A)$. It is proved in [24] that $\frac{1}{2}\omega(A)\leqslant\gamma(A)\leqslant\omega(A)$, and the other properties for ω to be a generalized measure of noncompactness are immediate. If $I=[a,b]$ and $C^1(I,\boldsymbol{R}^n)$ denotes the space of continuously differentiable maps from I to $\boldsymbol{R}^n$ with the norm $\|x\|=$ $=|x(a)+\sup_{t\in I}|x'(t)|$, then it is proved in [25] that if A is a bounded set in $C^1[I,\boldsymbol{R}^n)$, $\gamma(A)=\gamma_1(A')$, where γ_1 denotes the measure of noncompactness in $C(I,\boldsymbol{R}^n)$ and A' denotes the set of derivatives of functions in A. It follows that if we define $\mu(A)=\omega(A')$ for bounded sets $A\subset C^1(I,\boldsymbol{R}^n)$, then μ is a generalized measure of noncompactness.

We shall also need some results related to the so-called fixed point index. If A is a compact, metric space, recall that A is called a compact, metric ANR if given any metric space M, any closed subset B of M and any continuous map $f\colon B\to A$, then f has a continuous extension $\tilde{f}\colon U\to A$, defined on some open neighborhood U of A. If A is a subset of a Banach space X and $A=\bigcup_{i=1}^{n} C_i$, where C_i are compact, convex subsets of X, then A is known to be a compact, metric ANR. If G is an open subset of a compact metrix ANR A and $f\colon G\to A$ is a continuous map which has a compact (possibly empty) set of fixed points in G, then there is defined an integer $i_A(f,G)$, the fixed point index of f over G. This fixed point index can be thought of as an algebraic count of the number of fixed points of f in G. A summary of the properties of the fixed point index is given in [22], Section C; and a somewhat more complete development can be found in [27]. We shall summarize here only those properties of the index we shall immediately need. If $i_A(f, G)$ is defined and nonzero, f has a fixed point in G; and if G_1 and G_2 are open subsets of A, $i_A(f, G)$ and $i_A(f, G_2)$ are defined, and f has no fixed points in $G_1 \cap G_2$, then $i_A(f, G_1) + i_A(f, G_2) = i_A(f, G_1 \cup G_2)$ (this is the additivity property). If h is a homeomorphism of A onto a compact, metric ANR B, then $i_A(f, G) = i_B(hfh^{-1}, h(G))$ (this is a special case of the so-called commutativity property). Finally, we have the normalization property: $i_A(f, A) = \Lambda(f)$, where $\Lambda(f)$ denotes the Lefschetz number of f. Recall that $\Lambda(f) = \sum_{i\geq 0} (-1)^i \operatorname{tr}(f_{*,i})$, where $\operatorname{tr}(f_{*,i})$ denotes the trace of $f_{*,i}$, the induced map of f on $H_i(A)$, the i-th homology group of A with coefficients in the rationals.

In [22], Section D, a generalized fixed point index is defined for k-set-contractions with respect to μ, μ a generalized measure of noncompactness, $k < 1$, and some other classes of maps defined in certain (noncompact) metric ANR's. The usefulness of this generalization derives from the fact that all properties of the classical fixed point index generalize. Here, however, we shall only need the definition of the generalized index for maps defined on closed, convex sets. Suppose that A is a closed, convex subset of a Banach space X and U is a bounded, open subset of A. Let $f: \overline{U} \to A$ be a k-set-contraction with respect to μ, $k < 1$, and assume that $f(x) \neq x$ for $x \in \overline{U} - U$. Define $K_1 = K_1(f, U) = \overline{\text{co}}\, f(U)$ and generally define $K_n = K_n(f, U) = \overline{\text{co}}\, f(U \cap K_{n-1})$. If one sets $K_\infty = K_\infty(f, U) = \bigcap_{n\geq 1} K_n$, it is not hard to verify that K_∞ is compact and convex $\big($since $\mu(K_n) \leq k^n \mu(U)\big)$ and that $f(U \cap K_\infty) \subset K_\infty$. Now let K be any compact, convex set such that $K \supset K_\infty$ and $f(U \cap K) \subset K$; K_∞ itself is such a set, so the set of such K is nonempty. We define $i_A(f, U)$, the generalized fixed point index of f on U, to be $i_K(f, U \cap K)$ if K_∞ is nonempty and 0 if K_∞ is empty. It follows from Lemma 1, page 239, in [22] that $i_K(f, U \cap K)$ is independent of the particular K as above. Furthermore, it is proved in [22] that $i_A(f, U)$ agrees with the ordinary fixed point index if A is compact and simply equals the Leray-Schauder degree of $I - f$ if $A = X$ and f is a compact map.

For those who are familiar with Leray-Schauder degree, the above definition can be phrased differently. Let ϱ be any retraction of X onto K, where K is as above (such retractions are known to exist). Then $f \circ \varrho: \varrho^{-1}(\overline{U} \cap K) \to K$ is a compact map and $\deg_{LS}\big(I - f \circ \varrho, \varrho^{-1}(U \cap K)\big)$ is defined, and one can prove that it equals $i_A(f, U)$ (in particular, it is independent of the retraction ϱ).

For our subsequent work we also need to recall some geometrical results.

LEMMA 1.1 (KLEE [15]). - Let C be a compact, convex, infinite dimensional subset of a Banach space. Then C is homeomorphic to the Hilbert parallelotope. Furthermore, if x_0 is any prescribed point in C, the homeomorphism h may be chosen to take x_0 into any prescribed point in the Hilbert parallelotope.

Using Lemma 1.1 Browder establishes the following result in [3]:

LEMMA 1.2 (BROWDER [3]). – If C is an infinite dimensional compact, convex subset of a Banach space X and $\{x_1, x_2, ..., x_n\}$ is a finite subset of C, then there exist arbitrarily small neighborhoods U_j of x_j in C, $1 \leq j \leq r$, such that $C - \bigcup_{j=1}^{n} U_j$ is homeomorphic to the Hilbert parallelotope.

BROWDER also introduced in [3] the notion of an ejective fixed point of a map f. With a view to the later applications we have to weaken his definition somewhat. If C is a topological space, $x_0 \in C$, W is an open neighborhood of x_0 and $f: W - \{x_0\} \to C$ is a continuous map, we shall say that x_0 is an «ejective point» of f if there exists an open neighborhood U of x_0 such that for every $x \in U - \{x_0\}$ there is a positive integer $m = m(x)$ such that $f^m(x)$ is defined and $f^m(x) \notin U$. If f is defined and continuous at x_0 and $f(x_0) = x_0$, this definition agrees with Browder's, but for some examples f cannot be continuously defined at x_0.

Our next lemma is proved by BROWDER for ejective fixed points, and the same argument carries over to ejective points.

LEMMA 1.3 (BROWDER [3]). – Let C be a compact, Hausdorff space, $x_0 \in C$, and $f: C - \{x_0\} \to C - \{x_0\}$ a continuous map such that x_0 is an ejective point of f. Then there exists an open neighborhood U of x_0 such that for any open neighborhood V of x_0, there is an integer $m(V)$ such that $f^m(C - V) \subset C - U$ for $m \geq m(V)$.

The following lemma is intuitively obvious, but we include a proof for completeness.

LEMMA 1.4. – Let G be a closed, bounded convex infinite dimensional subset of a Banach space X. Then there exists a compact, convex infinite dimensional set $K \subset G$.

PROOF. – Let ε_n be a sequence of positive real numbers which approach 0. Let x_1 be any fixed nonzero point in G, and assume we have selected points $x_2, x_3, ..., x_n \in G$ such that $\{x_1, x_2, ..., x_n\}$ is a linearly independent set and $\|x_n - x_1\| \leq \varepsilon_n$ for $n > 1$. Let F_n denote the linear subspace spanned by $0, x_1, x_2, ..., x_n$ and let $G_n = F_n \cap G$. Since G is not finite dimensional, there exists $y_{n+1} \in G - G_n$. If we define $x_{n+1} = (1 - t)x_1 + ty_{n+1}$, where $t > 0$ is chosen so small that $\|x_{n+1} - x_1\| \leq \varepsilon_{n+1}$, then it is also easy to see that $x_{n+1} \in G$ and $\{x_1, x_2, ..., x_{n+1}\}$ is linearly independent.

We define $K = \overline{\text{co}}\,\{x_n : n \geq 1\}$. By our construction K is closed, bounded, convex and infinite dimensional. It only remains to show that K is compact, and since K is a complete metric space, compactness will follow if $\gamma(K) = 0$. By the properties of the measu e of noncompactness, $\gamma(K) = \gamma(\{x_n : n \geq 1\})$. If we take $\varepsilon > 0$ and select N so large that $\varepsilon_n < \varepsilon/2$ for $n \geq N$, we have that $\{x_n : n \geq 1\} = \bigcup_{n=1}^{N} x_n \cup \bigcup_{n \geq N+1} x_n$. The set $\{x_n : n \geq N\}$ has diameter less than ε, so $\gamma(\{x_n : n \geq 1\}) \leq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the lemma is proved. Q.E.D.

Our next lemma is a very special case of Theorem 1 in [21]. It can also be established directly in a few pages of reasoning without the elaborate apparatus used in [21].

LEMMA 1.5 (See Theorem 1 in [21]). – Let C be a closed, convex subset of a Banach space X, let V be a bounded open subset of C and suppose that $f\colon V\to V$ is a continuous, compact map. Assume that there is a compact set $K\subset V$ such that K is homotopic in itself to a point and such that $f^n(V)\subset K$ for n greater than or equal to some integer m. It then follows that $i_C(f, V)=1$.

We are finally ready to state our main theorem. The following result is a generalization of work of G. S. JONES [17,19] and F. E. BROWDER [2, 3].

THEOREM 1.1. – Let G be a closed, bounded, convex, infinite dimensional subset of a Banach space X, μ a generalized measure of noncompactness on X, $x_0\in G$, and $f\colon G-\{x_0\}\to G$ a continuous map which is a k-set-contraction with respect to μ, $k<1$. Then if x_0 is an ejective point of f and U is an open neighborhood of x_0 such that $f(x)\neq x$ for $x\in\overline{U}-\{x_0\}$, $i_G(f, G-\overline{U})=1$ and f has a fixed point in $G-\overline{U}$. If G is finite dimensional (not equal to a point) and x_0 is an extreme point of G, then the same conclusion holds.

PROOF. – By Lemma 1.4 (if G is infinite dimensional) there exists a compact, convex infinite dimensional subset K of G. Define $C_1=\overline{\mathrm{co}}\big(K\cup f(G-\{x_0\})\big)$ $C_n=\overline{\mathrm{co}}\big(K\cup f(C_{n-1}-\{x_0\})\big)$ for $n>1$, and $C=\bigcap\limits_{n\geqslant 1} C_n$. It is not hard to see that C is compact, convex and infinite dimensional and that $f\big((G-\{x_0\})\cap C\big)\subset C$ and $C\supset K_\infty\big(f, G-\{x_0\}\big)$. It follows by our definition that $i_G(f, G-\overline{U})=i_C(f, (G-\overline{U})\cap C)$, so if we define $W=G-\overline{U}$, it sufficies to prove $i_C(f, W\cap C)=1$.

By the Krein-Millman theorem C has an extreme point x_1, and by Lemma 1.1 there exists a homeomorphism h of C onto C such that $h(x_0)=x_1$. By one of the previously mentioned properties of the fixed point index, $i_C(f, W\cap C)=$ $=i_C\big(hfh^{-1}, h(W\cap C)\big)$. It follows that for the purposes of our theorem we may as well assume originally that x_0 is an extreme point of C. If G originally is finite dimentional, and not equal to a point, we take $C=G$ originally, and by assumption x_0 is an extreme point of G.

We now use a trick from [3] and construct an auxiliary function f_ε. Our purpose is to avoid the technically bothersome possibility that $f(x)=x_0$ for some $x\in G-\{x_0\}$. By the definition of ejectivity there exists an open neighborhood U_1 of x_0 in C, $U_1\subset U\cap C$, such that for each $x\in\overline{U}_1-\{x_0\}$ there is an integer $m=m(x)$ such that $f^m(x)$ is defined and $f^m(x)\notin U_1$. For $x\in C$, define $\varrho(x)=d(x,\overline{U}_1)=$ the distance of x to $\overline{U}_1$. Let x_2 be any point in C unequal to x_0 and define $f_\varepsilon(x)=\big(1-\varepsilon\varrho(x)\big)f(x)+$ $+\varepsilon\varrho(x)x_2$ for $x\in C-\{x_0\}$ and positive ε so small that $\varepsilon\varrho(x)<1$ for all $x\in C$. Since f has no fixed points in $W\cap C-(C-\overline{U}_1)$, the additivity property of the fixed point index implies that $i_C(f, W\cap C)=i_C(f, C-\overline{U}_1)$; and the so-called homotopy property of the fixed point index implies that for ε small enough $i_C(f, C-\overline{U}_1)=$ $=i_C(f_\varepsilon, C-\overline{U}_1)$. Thus it suffices to prove that $i_C(f_\varepsilon, C-\overline{U}_1)=1$. It is clear that

$f_\varepsilon: C-\{x_0\} \to C$ and in fact $f_\varepsilon: C-\{x_0\} \to C-\{x_0\}$; for $x \in \overline{U}_1$, $f_\varepsilon(x)=f(x)$, so certainly $f(x) \neq x_0$ for $x \in \overline{U}_1$ and for $x \notin \overline{U}_1$, $f_\varepsilon(x) \neq x_0$ because x_0 is an extreme point.

It follows according to Lemma 1.3 (since x_0 is certainly an ejective point of f_ε) that there exists an open neighborhood U_2 of x_0 in C with the following property: given any open neighborhood V of x_0 in C, there exists an integer $m(V)$ such that $f_\varepsilon^m(C-V) \subset C-U_2$ for $m \geqslant m(V)$. If C is infinite dimensional, then by Lemma 1.2 there exists an open neighborhood U_3 of x_0 such that $\overline{U}_3 \subset U_1 \cap U_2$ and $C-U_3$ is homeomorphic to the Hilbert parallelotope. If $C=G$ is finite dimensional such a U_3 exists such that $C-U_3$ is homeomorphic to C. In either case $C-U_3 \equiv K_1$ is certainly contractible in itself to a point. By definition there exists a positive integer m such that $f_\varepsilon^n(C-U_3) \subset C-U_2 \subset C-\overline{U}_3$ for $n \geqslant m$. If we define $B_j = f_\varepsilon^j(C-U_3) =$ a compact set and $V_m = C-\overline{U}_3$, there is an open neighborhood V_{m-1} of B_{m-1} such that $f_\varepsilon(V_{m-1}) \subset V_m$ and $x_0 \notin \overline{V}_{m-1}$. Proceeding by finite induction, there exist open neighborhoods V_j of B_j for $0 \leqslant j \leqslant m-1$ such that $x_0 \notin \overline{V}_j$ for $0 \leqslant j \leqslant m$, $f_\varepsilon(V_j) \subset V_{j+1}$ for $0 \leqslant j \leqslant m-1$ and $f_\varepsilon(V_m) \subset B_0 \subset V_0$. It follows that $V = \bigcup V_j$ is an open set such that $x_0 \notin \overline{V}$, $f_\varepsilon(V) \subset V$ and $K_1 = C-U_3 \subset V$. Since $x_0 \notin \overline{V}$, there exists a positive integer m_1 such that $f_\varepsilon^n(V) \subset K_1$ for $n \geqslant m_1$. It follows from Lemma 1.5 that $i_C(f_\varepsilon, V)=1$, and by the additivity property $i_C(f_\varepsilon, C-\overline{U}_1) = i_C(f_\varepsilon, V)$.

Q.E.D.

If f happens to be defined and continuous at an ejective point x_0, so that $i_C(f, U)$ is defined, then somewhat more detailed information can be obtained.

COROLLARY 1.1. – Let G be a closed, bounded convex and infinite dimensional subset of a Banach space X, μ a genralized measure of noncompactness, and $f: G \to G$ a continuous map which is a k-set-contraction with respect to μ, $k<1$. Then if x_0 is an ejective fixed point of f and U is a neighborhood of x_0 such that $f(x) \neq x$ for $x \in \overline{U}-\{x_0\}$, $i_G(f, U)=0$. Furthermore, f has a fixed point in G which is not ejective.

PROOF. – If C is as in Theorem 1.1, then it is easy to check by our definition that $i_G(f, U) = i_C(f, U \cap C)$ and $i_G(f, G-\overline{U}) = i_C(f, C \cap (G-\overline{U}))$. According to Theorem 1.1, $i_G(f, G-\overline{U})=1$ and by the additivity property $i_G(f, U) + i_G(f, G-\overline{U}) = i_C(f, C)$. Thus it suffices to prove that $i_C(f, C)=1$. But we know that $i_C(f, C) = \Lambda(f|C)$, and $\Lambda(f|C)=1$ because C is homologically trivial.

If S denotes the set of fixed points of f, S clearly must lie in C, so S is compact. If f has no non-ejective fixed points, then since each ejective fixed point is isolated, $S=\{x_1, x_2, \dots, x_n\}$ is a finite set. Let U_j be an open neighborhhood of x_j such that $f(x) \neq x$ for $x \in \overline{U}_j - \{x_j\}$ and define $U = \bigcup_{j=1}^{n} U_j$: Then by the additivity property $i_G(f, U) = \sum_{j=1}^{n} i_G(f, U_j) = 0$. It follows that $i_G(f, G-\overline{U}) = i_G(f, G) = 1$, so f has a fixed point in $G-\overline{U}$, contradicting the assumption that f has no nonejective fixed point.

Q.E.D.

COROLLARY 1.2. – Let K be a closed, convex infinite dimensional subset of a Banach space such that $0 \in K$. For some $R > 0$ let $G = \{x \in K: \|x\| \leqslant R\}$ and let $f: G - \{0\} \to K$ be a k-set-contraction, $k < 1$, with respect to μ, μ a generalized measure of noncompactness. Then if $f(x) \neq tx$ for $x \in K$, $\|x\| = R$ and $t \geqslant 1$ and if 0 is an ejective point of f, f has a fixed point in $G - \{0\}$.

PROOF. – Let ϱ be the radial retraction of K onto G: $\varrho(x) = x$ if $\|x\| \leqslant R$ and $\varrho(x) = R(x/\|x\|)$ if $\|x\| \geqslant R$. The same proof given in [22], page 224, shows that ϱ is a 1-set-contraction with respect to μ. It follows easily that if $g(x) = \varrho(f(x))$, $g(G - \{x_0\}) \subset G$ and g is a k-set-contraction with respect to μ. It is also clear that 0 is also an ehective point of g and that G is closed, bounded, convex and infinite dimensional. It follows from Theorem 1.1 that g has a fixed point x_1 in $G - \{x_0\}$. If $\varrho(f(x_1)) = f(x_1)$, we are done. But if $\varrho(f(x_1)) = x_1 \neq f(x_1)$, we must have $\|x_1\| = R$ and $f(x_1) = tx_1$, $t \geqslant 1$, a contradiction. Q.E.D.

2. – In this section we wish to consider the retarded functional differential equation

$$\text{(2.1)} \qquad \begin{aligned} &y'(t) = -f(y(t-1)) \quad \text{for} \quad t \geqslant 0 \\ &y(t) \ = \varphi(t) \quad \text{for} \quad -1 \leqslant t \leqslant 0 . \end{aligned}$$

In (2.1) φ denotes a given continuous function and f a continuous real-valued function. We shall see later that (2.1) contains some more general examples than one might expect. Our goal is to place enough further conditions on f to guarantee that (2.1) has a nonzero periodic solution.

Before stating our first lemma, we recall some standard notation. We shall denote the Banach space X of real valued, continuous functions on $[-1, 0]$ by $C([-1, 0])$; if $\varphi \in C([-1, 0])$, $\|\varphi\| = \sup |\varphi(t)|$. We shall denote by $y(t; \varphi)$ (or $y(t)$, if φ need not be emphasized) the solution of (2.1) which equals φ on $[-1, 0]$. If $x(t)$ is a continuous function defined for $-1 \leqslant t < \infty$, then for $0 \leqslant t < \infty$, we define $x_t \in C([-1, 0])$ by $x_t(s) = x(t+s)$ for $-1 \leqslant s \leqslant 0$.

LEMMA 2.1. – Assume that $f: \boldsymbol{R} \to \boldsymbol{R}$ is a continuous function and that $\varphi \in C([-1, 0])$. Then there is a unique solution $y(t)$ of (2.1) defined for all $t \geqslant -1$. Furthermore, $y(t; \varphi)$ depends continuously on φ in the following sense: given $\varphi_0 \in C([-1, 0])$, $T > 0$, and $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\varphi \in C([-1, 0])$ with $\|\varphi - \varphi_0\| < \delta$, $\sup\limits_{0 \leqslant t \leqslant T} |y(t; \varphi) - y(t; \varphi_0)| < \varepsilon$.

PROOF. – If $y(t)$ satisfies (2.1), then on $[0, 1]$ we must have $y(t) = y(0) - \int\limits_{-1}^{t-1} f(y(s))\, ds$, and conversely, such a y satisfies (2.1) on $[0, 1]$. On the interval $[1, 2]$, we have $y(t) = y(1) - \int\limits_{0}^{t-1} f(y(s))\, ds$. Continuing in this way it is clear that y is uniquely defined for $-1\leqslant t<\infty$: The continuity statement of the lemma follows easily from the continuity of f and the explicit formula given. Q.E.D.

Lemma 2.2. – Assume that f is a continuous function and that $yf(y)>0$ for all $y\neq 0$. Assume that there exists a positive constant A such that either (a) $f(y)>-A$ for all y or (b) $|f(y)|\leqslant A$ if $|y|\leqslant A$. Then if $\varphi\in C([-1,0])$ satisfies $0\leqslant\varphi(t)\leqslant A$ for $-1\leqslant t\leqslant 0$ and if condition (a) on f holds, it follows that $y(t;\varphi)\leqslant A$ for all $t\geqslant 0$; if condition (b) on f holds, $|y(t;\varphi)|\leqslant A$ for all $t\geqslant 0$.

Proof. – Since $f(y)>0$ for $y>0$, $y'(t)\leqslant 0$ on $[0, z_1+1]$, where z_1 denotes the first zero of y on $[0,\infty)$ and we take $z_1=\infty$ if y has no zero. If $z_1=\infty$, we are done, so suppose $z_1<\infty$. If condition (a) holds there are two possibilities: either $y(t)<0$ for all $t\geqslant z_1+1$, in which case we are done, or y has a first zero $z_2\in[z_1+1,\infty)$. Since $y'(t)\geqslant 0$ on $[z_1+1, z_2+1]$, the maximum of $y(t)$ on $[z_1+1, z_2+1]$ is achieved at z_2+1. But we have that $y(z_2+1)=-\int\limits_{z_2-1}^{z_2} f(y(s))\,ds<A$, by our assumption on f. However, at this point we are back in our original situation with y_{z_2+1} serving as φ. Thus if we repeat the argument (formally use induction) we find that $y(t;\varphi)\leqslant A$ for all $t\geqslant 0$.

If condition (b) holds, we find that $|y(z_1+1)|=\int\limits_{z_1-1}^{z_1} f(y(s))\,ds\leqslant A$. If $y(t)<0$ for $t\geqslant z_1+1$, we are done; otherwise $y(t)$ has a first zero z_2 on $[z_1+1,\infty)$. Just as above y is increasing on $[z_1+1, z_2+1]$ and $|y(z_2+1)|\leqslant A$. Continuing in this way we obtain the result. Q.E.D.

Lemma 2.3. – Let f be a continuous function such that $yf(y)>0$ for $y\neq 0$. Assume there exist constants $\varepsilon>0$ and $c>1$ such that $|f(y)|>c|y|$ for $y\in[-\varepsilon,\varepsilon]$. for $y\in[-\varepsilon,\varepsilon]$. Suppose that $\varphi\in C([-1,0])$ is a nonnegative function with $\varphi(0)>0$. Then if we define $z_1(\varphi)=\inf\{z\geqslant 0: y(z;\varphi)=0\}$ and $z_n(\varphi)=\inf\{z\geqslant z_{n-1}(\varphi)+1$: each $z_n(\varphi)$ is defined and finite and $y(t;\varphi)$ is monotonic decreasing from 0 to $z_1(\varphi)+1$, monotonic increasing from $z_1(\varphi)+1$ to $z_2(\varphi)+1$, and so on. Furthermore, if $M>0$, there exists a constant $C(M)$ such that $z_2(\varphi)\leqslant C(M)$ for all φ as above with $\|\varphi\|\leqslant M$.

Proof. – Suppose that $\|\varphi\|\leqslant M$. Since $f(y)>0$ for $y>0$, define a positive number $d=\min\{f(y)\colon \varepsilon\leqslant y\leqslant M\}$. Since $y'(t)\leqslant -d$ as long as $y(t,>\varepsilon$, if $y(0)>\varepsilon$, there exists some first time $t_1>0$ such that $y(t_1)=\varepsilon$ and $t_1\leqslant M/d$. If $y(0)\leqslant\varepsilon$, we define $t_1=0$. If $y(z)=0$ for some number z, $t_1<z\leqslant t_1+1$, we have $z_1(\varphi)\leqslant M/d+1$. Otherwise, by the assumptions on f we have $y'(t)=-f(y(t-1))\leqslant -c(y(t-1))\leqslant$ $\leqslant -cy(t_1+1)$ for $t_1+1\leqslant t\leqslant t_1+2$. This implies that y must have a zero on the interval $[t_1+1, t_1+1+c^{-1}]$. Thus in any event we find that $z_1(\varphi)\leqslant M/d+2=M_1$. It is obvious from our assumptions that y is monotonic decreasing on $[0, z_1(\varphi)+1]$. If we define $M'=\max\{f(x)\colon 0\leqslant y\leqslant M\}$, it follows that $y(z_1+1)\geqslant -M'$; and now if we set $d'=\min\{-f(y)\colon M'\leqslant y\leqslant -\varepsilon\}$, the same proof as before shows that y has a zero z_2 and $z_2-(z_1+1)\leqslant M'/d'+2$. Q.E.D.

LEMMA 2.4. – Let $K \subset C([-1, 0])$ denote the cone of monotonic increasing functions φ such that $\varphi(-1) = 0$. Assume that f satisfies the conditions of Lemma 2.3. Then the map $\varphi \to z_2(\varphi)$ is continuous from $K - \{0\}$ to $\boldsymbol{R}$. The map $F: K \to K$ defined by $F(\varphi)(t) = y(z_2(\varphi) + 1 + t; \varphi)$, $-1 \leqslant t \leqslant 0$, for $\varphi \neq 0$, and $F(0) = 0$ is a continuous compact map of K into K.

PROOF. – Take $\varphi \in K - \{0\}$ and let z_1 denote the first zero of $y(t; \varphi)$ on $[0, \infty)$; $z_1 > 0$ since $\varphi(0) > 0$. If $z_1 \geqslant 1$, we have $y'(z_1) = -f(y(z_1 - 1)) < 0$, because $y(z_1 - 1) > 0$. If $0 < z_1 < 1$ we also must have $y'(z_1) < 0$. Otherwise, $y(z_1 - 1) = 0 = \varphi(z_1 - 1)$, and since φ is monotonic increasing this implies that $\varphi(t) = 0$ for $-1 \leqslant t \leqslant z_1 - 1$. It would then follow that $y'(t) = 0$ for $0 \leqslant t \leqslant z_1$, which would imply that $y(z_1) = y(0) > 0$, a contradiction. The same proof shows that $y'(z_n) \neq 0$ for all $n \geqslant 1$. It is now easy to see (using Lemma 2.1) that $\varphi \to z_2(\varphi)$ is continuous on $K - \{0\}$.

The fact that F is continuous on $K - \{0\}$ is now immediate. To show that F is continuous at 0, recall that by Lemma 2.3, there exists a constant C such hat $z_2(\varphi) \leqslant C$ if $\varphi \in K$ and $\|\varphi\| \leqslant 1$. By Lemma 2.1, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\varphi\| < \delta$, $|y(t; \varphi)| < \varepsilon$ for $0 \leqslant t \leqslant C + 1$. But this implies F is continuous at 0.

Now suppose that A is a bounded subset of K, $\|\varphi\| \leqslant M$ for $\varphi \in A$. The proof of Lemma 2.3 shows that $|y(t; \varphi)| \leqslant M' = \max\{f(y): 0 \leqslant y \leqslant M\}$ for $z_1(\varphi) \leqslant t \leqslant z_2(\varphi)$, and the same proof shows that $|y(t; \varphi)| \leqslant M'' = \max\{f(y): 0 \leqslant y \leqslant M'\}$ for $z_2(\varphi) \leqslant t \leqslant z_3(\varphi)$. It follows that $\|F(\varphi)\| \leqslant M''$ for $\varphi \in A$ and $|y'(t; \varphi)| \leqslant M''$ for $z_2(\varphi) \leqslant t \leqslant z_2(\varphi) + 1$. The Ascoli-Arzela theorem now implies that $F(A)$ has compact closure in K. Q.E.D.

It is clear that fixed points of F correspond to periodic solutions of (2.1). Our goal from here on will be to impose further conditions on f which, with the aid of the fixed point theorems of Section 1, will guarantee that F has a nonzero fixed point.

Our main lemma is a fragment of Theorem 5 of E. M. WRIGHT's article [26]. It can also easily be proved directly.

LEMMA 2.5 (see [26]). – If $\alpha > \pi/2$, the equation $\lambda + \alpha e^{-\lambda} = 0$ has a complex root λ such that $\operatorname{Re}(\lambda) > 0$ and $0 < \operatorname{Im}(\lambda) < \pi$.

The next two lemmas comprise the heart of our proof of the existence of nonzero periodic solutions of (2.1). The basic idea for the proof of the following lemma seems to be due to E. M. WRIGHT (see Theorem 4 of [26] and the argument on p. 76).

LEMMA 2.6 (see [26]). – Let f be a continuous, real-valued function such that $yf(y) > 0$ for all $y = 0$. Assume that f is continuously differentiable on some open neighborhood of the origin and $\alpha = f'(0) > \pi/2$. Then there exists a positive constant a (independent of φ) such that for any $\varphi \in K$ with $\varphi(0) > 0$ (K as in Lemma 2.4), $\overline{\lim\limits_{t \to \infty}} |y(t; \varphi)| \geqslant a$.

PROOF. – Take $\varphi \in K$ with $\varphi(0) > 0$, set $y(t) = y(t; \varphi)$, and define $z_n = z_n(\varphi)$ ($z_n(\varphi)$ is well defined by Lemma 2.3). We can assume that $|y(t)|$ is bounded, or we are done. Furthermore, it is easy to see that if $y(t) = 0$ for t in an interval of length 1, then φ must have been identically zero. Therefore for any $T\geqslant 0$ we have $\sup_{t\geqslant T}|y(t)|>0$.

Let $\lambda=\mu+i\nu$ be the complex solution of $\lambda+\alpha\exp[-\lambda]=0$ which is guaranteed by Lemma 2.5 and take a positive number ε such that $\varepsilon<\frac{1}{2}\mu\cos\nu/2$. Since f is continuously differentiable on a neighborhood of the origin, there exists a positive number a such that $|f(y)-\alpha y|\leqslant\varepsilon|y|$ if $|y|\leqslant a$, where $\alpha=f'(0)$. To prove the lemma it suffices to show that $\sup_{t\geqslant z_n}|y(t)|>a$ for any zero z_n of y. We prove this by contradiction. Thus suppose that for some zero z of y, $\sup_{t\geqslant z}|y(t)|=\delta<a$. Since y achieves its local maxima and minima on $[z,\infty)$ at the points z_n+1, $z_n\geqslant z$, there exists a zero $z_n\geqslant z$ such that $|y(z_n+1)|\geqslant\delta/2$. For notational convenience we set $T=z_n+1$, and we assume that $y(T)>0$, since the proof in the case $y(T)<0$ is analogous. If we integrate $\int_T^\infty y'(t)\exp[-\lambda t]\,dt$ by parts we obtain

$$\int_T^\infty y'(t)\exp[-\lambda t]\,dt=-y(T)\exp[-\lambda T]+\lambda\int_T^\infty y(t)\exp[-\lambda t]\,dt\,. \tag{2.2}$$

On the other hand, if we set

$$y'(t)=-f\big(y(t-1)\big)=-\alpha y(t-1)+\big[-f\big(y(t-1)\big)+\alpha y(t-1)\big]\,,$$

we obtain

$$\int_T^\infty y'(t)\exp[-\lambda t]\,dt=-\alpha\int_T^\infty y(t-1)\exp[-\lambda t]\,dt+\\+\int_T^\infty\big[-f\big(y(t-1)\big)+\alpha y(t-1)\big]\exp[-\lambda t]\,dt\,.$$

If we change variables in the latter equation we obtain

$$\int_T^\infty y'(t)\exp[-\lambda t]\,dt=-\alpha\exp[-\lambda]\int_T^\infty y(t)\exp[-\lambda t]\,dt-\alpha\int_T^{T+1}y(t-1)\exp[-\lambda t]\,dt+\\+\int_T^\infty\big[-f\big(y(t-1)\big)+\alpha y(t-1)\big]\exp[-\lambda t]\,dt\,. \tag{2.3}$$

Setting (2.2) equal to (2.3) and using the fact that $\lambda+\alpha\exp[-\lambda]=0$ we find that

$$-y(T)\exp[-\lambda T]+\alpha\exp[-\lambda]\int_{T-1}^{T}y(t)\exp[-\lambda t]\,dt=\\=\int_T^\infty\big[-f\big(y(t-1)\big)+\alpha y(t-1)\big)\exp[-\lambda t]\,dt\,. \tag{2.4}$$

If one integrates the left hand of (2.4) by parts and multiplies both sides by $\exp[\lambda(T-\frac{1}{2})]$ one obtains:

$$(2.5)\qquad -\int_{T-1}^{T} y'(t)\exp[-\lambda(t-T+\tfrac{1}{2})]\,dt = \int_{T}^{\infty}[-f(y(t-1))+ay(t-1)]\exp[-\lambda(t-T+\tfrac{1}{2})]\,dt\,.$$

Taking the real part of the left hand side of the above equation, one finds that

$$\left|\int_{T-1}^{T} y'(t)\exp[-\lambda(t-T+\tfrac{1}{2})]\,dt\right| \geqslant \bigl(\exp[-\mu/2]\cos\nu/2\bigr)\,y(T);$$

and one can trivially see that the modulus of the right side is less than $\varepsilon\delta\exp\cdot[-\mu/2](\mu^{-1})$. Since we are assuming that $\delta/2 \leqslant y(T)$, this implies that $\frac{1}{2}\cos\nu/2 \leqslant \varepsilon/\mu$, a contradiction. Q.E.D.

The conclusion of Lemma 2.6 can also be obtained under different hypotheses on f, as the following lemma shows:

LEMMA 2.7. – Let f be a continuous, real-valued function such that $yf(y)>0$ for all $y\neq 0$. Assume that $\lim_{y\to 0} f(y)/y = +\infty$ and that f is monotonic increasing on some open neighborhood of 0. Then there exists a positive constant a (independent of φ) such that for any $\varphi\in K$ with $\varphi(0)>0$, $\lim_{t\to\infty}\sup|y(t;\varphi)|\geqslant a$.

PROOF. – Just as before, if we let $y(t)=y(t;\varphi)$ and $z_n=z_n(\varphi)$, it suffices to prove that $\sup_{t\geqslant z_n}|y(t)|\geqslant a$ for some a. Take c to be a positive number such that $k=\min\{\frac{1}{2}c(1-1/c)^2, \sqrt{2c}-1\}>1$. Since $\lim_{y\to 0} f(y)/y=\infty$, let $a>0$ be a positive. number such that $f(y)/y>c$ if $|y|\leqslant a$ and $f(y)$ is monotonic increasing on $[-a, a]$

If for every zero z of y, $\sup_{t\geqslant z}|y(t)|\geqslant a$, we are done. Thus we assume that for some zero z $\sup_{t\geqslant z}|y(t)|=\delta<a$, and we try to get a contradiction. It follows as before that there exists a zero $z_n\geqslant z$ such that $|y(z_n+1)|>(1/k)\delta$, and we may as well assume that $y(z_n+1)>0$.

Our first claim is that $z_{n+1}\in(z_n+1, z_n+2+c^{-1}]$. This will certainly be true if $z_{n+1}\in(z_n+1, z_n+2]$, so assume that $z_{n+1}>z_n+2$. Equation 2.1 immediately implies that y is concave downward and decreasing on $[z_n+1, z_n+2]$. It follows that $y'(t)=-f(y(t-1))\leqslant -cy(t-1)\leqslant -cy(z_n+2)$ for all $t\in[z_n+2, z_n+3]$, and this immediately implies the claim.

For notational convenience we shall write $u=z_n$ and $v=z_{n+1}$ from here on. If $v\in(u+1, u+2]$ we have

$$(2.6)\qquad y(u+1) = -\int_{u+1}^{v} y'(s)\,ds = \int_{u}^{v-1} f(y(s))\,ds \leqslant (v-1-u)\,f(y(v-1))\,.$$

Since $f(y(s))$ is monotonic increasing on $[u, u+1]$, Equation (2.6) implies

$$\int_{v-1}^{u+1} f(y(s))\,ds \geqslant (u-v+2)f(y(v-1)) \geqslant \frac{(u-v+2)}{(v-1-u)}\,y(u+1)\,. \tag{2.7}$$

Since y is concave downward on $[u+1, u+2]$, we must have $y(s) \geqslant (v-u-1)^{-1}\cdot$ $\cdot(v-s)y(u+1)$ for $u+1 \leqslant s \leqslant v$. This implies that

$$\int_{u+1}^{v} f(y(s))ds \geqslant c\int_{u+1}^{v} y(s)\,ds \geqslant \frac{c}{v-u-1}\,y(u+1)\int_{u+1}^{v}(v-s)\,ds = \frac{1}{2}c(v-u-1)y(u+1). \tag{2.8}$$

Since we know that $|y(v+1)| = \int_{v-1}^{v} f(y(s))\,ds$, Equations (2.7) and (2.8) imply that

$$|y(v+1)| \geqslant \frac{(1-x)}{x}\,y(u+1) + \frac{1}{2}\,cxy(u+1) \tag{2.9}$$

where $x = v-1-u$ and $0 < x \leqslant 1$. It is easy to check that $\min\,(1-x)/x +$ $+ \frac{1}{2}cx = \sqrt{2c}-1$, so $|y(v+1)| > ky(u+1) > \delta$, a contradiction.

If $u+2 < v \leqslant u+2+c^{-1}$, we observe that

$$|y(1+v)| = \int_{v-1}^{v} f(y(s))\,ds \geqslant \int_{v-1}^{u+2} f(y(s))\,ds\,. \tag{2.10}$$

Since $y(s)$ is concave downward on $[u+1, u+2]$ and positive at $u+2$, we have that $f(y(s)) \geqslant cy(s) \geqslant c(u+2-s)y(u+1)$ for $u+1 \leqslant s \leqslant u+2$. Using this estimate in equation (2.10) we find that

$$\begin{aligned} |y(1+v)| &\geqslant \tfrac{1}{2}cy(u+1)(u+2-v+1)^2 \\ &\geqslant \tfrac{1}{2}c(1-c^{-1})^2 y(u+1) \\ &> \delta\,. \end{aligned} \tag{2.11}$$

Thus we also obtain a contradiction in this case. Q.E.D.

We are now almost ready to prove our main theorem. We need one more lemma.

LEMMA 2.8. – Let f be a continuous map which either satisfies the conditions of Lemma 2.7 or Lemma 2.6. Then if F and K are as in Lemma 2.4, 0 is an ejective fixed point of F.

PROOF. – If $\varphi \in K$ and $\varphi \neq 0$, so $\varphi(0) > 0$, denote $y(t; \varphi)$ by $y(t)$ and $z_n(\varphi)$ by z_n. According to Lemma 2.6 or 2.7 there exists a positive constant a such that

$|y(z_n+1)| > a/2$ for infinitely many integers n. Let b be a positive number such that $b < a/2$ and $\max\{f(y): 0 \leqslant y \leqslant b\} < a/2$. If $y(z_{2n}+1) \leqslant b$ for every integer $n \geqslant N$, then the standard argument shows that $|y(z_{2n+1}+1)| < a/2$ for every integer $n \geqslant N$. It follows that there must be a positive integer n such that $y(z_{2n}+1) = \|F^n(\varphi)\| > b$. Q.E.D.

Our next theorem gives a generalization (by weakening the conditions on f) of the existence part of Theorem 2.1 in [14]. We should emphasize, however, that the real importance of YORKE and KAPLAN's work is their results on stability of the periodic solution.

THEOREM 2.1 (compare Theorem 2.1 of [14]). – Let f be a continuous real-valued map such that f is monotonic increasing on an open neighborhood of 0 and $yf(y) > 0$ for all nonzero y. Assume that there exists a positive constant A such that either $f(y) > -A$ for all y or $|f(y)| \leqslant A$ if $|y| \leqslant A$. Finally, suppose either that f is continously differentiable on some open neighborhood of 0 and $f'(0) = \alpha > \pi/2$ or that $\lim_{y\to 0} f(y)/y = +\infty$. Then Equation (2.1) has a nonzero periodic solution y such that $y|[-1, 0]$ is monotonic increasing. Furthermore, if F and K are as in Lemma 2.4, $G = \{\varphi \in K: \|\varphi\| \leqslant A\}$ and U is an open neighborhood of the origin in G such that F has no fixed points except 0 in $\overline{U}$, $i_G(F, U) = 0$.

PROOF. – If F and K are as in Lemma 2.4, then according to Lemma 2.8, 0 is an ejective fixed point of F. Lemma 2.4 implies that F is a continuous, compact map (hence a 0-set-contraction), and Lemma 2.2 implies that $F(G) \subset G$. It is clear that G infinite dimensional, so Corollary 1.1 implies that F has a non-ejective fixed point and that $i_G(F, U) = 0$. We have already noted that fixed points of F give periodic solutions of (2.1). Q.E.D.

As an example, we mention the following apparently new result, which follows trivially from Theorem 2.1.

COROLLARY 2.1. – The equation $y'(t) = -(\exp[y(t-1)] - 1)^{\frac{1}{3}}$ has a nontrivial periodic solution y such that $y(-1) = 0$ and $y|[-1, 0]$ is monotonic increasing.

Our next lemma follows by standard arguments in ordinary differential equations, and we leave it to the reader. The only novelty is that one does not have uniqueness in the differential equation below.

LEMMA 2.9. – Let $H: \boldsymbol{R} \to \boldsymbol{R}$ be a continuous map. Assume that there exists a number z_1, $-\infty < z_1 < 0$ such that $H(z_1) = 0$ and suppose there exists a number z_2, $0 < z_2 \leqslant +\infty$ such that $N(z_2) = 0$ if $z_2 \neq +\infty$. Suppose that $N(z) > 0$ for $z_1 < z < z_2$ and $N(z) \leqslant A|z| + B$ for some constants A and B and every $z \in (z_1, z_2)$. Then the equation $f'(u) = N(f(u))$, $f(0) = 0$, has a continuously differentiable solution $f(u)$ defined for all real u and such that $z_1 \leqslant f(u) \leqslant z_2$ for all u (the inequalities are strict if N is Lipschitz).

COROLLARY 2.2. – If N is a continuous function as in Lemma 2.9 and if $\alpha N(0) > \frac{1}{2}\pi$, then the equation $x'(t) = -\alpha x(t-1) N(x(t))$ has a nontrivial periodic solution $x(t)$ such that $x(-1) = 0$ and $x|[-1, 0]$ is monotonic increasing.

PROOF. – Let $f(u)$ be as in Lemma 2.9 and consider the equation $u'(t) = -\alpha f(u(t-1))$. Since $f'(u) = N(f(u))$ and $f(0) = 0$, f is monotonic increasing and strictly monotonic increasing on some open neighborhood of 0 and $\alpha f'(0) = \alpha N(0) > \pi/2$. It follows from Theorem 2.1 that there exists a nontrivial periodic function u such that $u'(t) = -\alpha f(u(t-1))$, $u(-1) = 0$ and u is monotonic increasing on $[-1, 0]$. If we define $x(t) = f(u(t))$, $x'(t) = f'(u(t))\, u'(t) = \alpha x(t-1) N(x(t))$, and we are done. Q.E.D.

It is clear that Corollary 2.2 implies the standard results in the literature (see [17] or [19]) concerning periodic solutions of $x'(t) = -\alpha x(t-1)[1 + x(t)]$ or $x'(t) = -\alpha x(t-1)(a - x(t))(x(t) + b)$, $a, b > 0$.

3. – Suppose that r_1, r_2 and r are nonnegative numbers and that $\varrho = \max\{r_1, r_2, r\}$ is positive. Assume also that ε is a positive number and that k is a real number. In this section we wish to consider the following equations:

$$\begin{aligned} &x'(t) = y(t) + \varepsilon\big(x(t-r_1) - x^3(t)/3\big)\,, \qquad t \geqslant 0 \\ (3.1)\qquad &y'(t) = -x(t-r) + kx(t-r_2)\,, \qquad t \geqslant 0 \\ &x|[-\varrho, 0] = \varphi = a \text{ continuous function}, \qquad y(0) = y_0\,. \end{aligned}$$

Equations (3.1) constitute a generalization of equations considered by GRAFTON in [7]. We wish to extend Grafton's periodicity result in [7] and show that it can be obtained as a consequence of our Theorem 1.1. Grafton's original proof was along different lines. Actually, Grafton's periodicity theorem in [8] can also be shown to follow from Corollary 1.1—assuming Grafton's results on the qualitative behaviour of his equations—, but we shall restrict ourselves to (3.1).

It is a straightforward exercise (which we leave to the reader) to show that given φ and y_0, there exists a unique solution $(x(t), y(t))$ of (3.1) defined for all $t \geqslant 0$. Our first lemma shows that for a wide range of the parameters, $x(t)$ oscillates and has infinitely many zeros.

LEMMA 3.1. – If $(x(t), y(t))$ denotes the solution of (3.1) and if $k < 1$ and $r_2 \leqslant r$ when $k > 0$, then if $\varphi(0) > 0$ there exists a number $T_1 > 0$ such that $x(T_1) < 0$.

PROOF. – We shall assume that $x(t) \geqslant 0$ for $t \geqslant 0$ and obtain a contradiction. Define $M_0 = \max\limits_{-\varrho \leqslant t \leqslant 0} |\varphi(t)|$ and $k_1 = \max(k, 0)$ and denote by ξ the largest positive root of

$$(3.2)\qquad y(0) + (1 + |k|)\,\varrho M_0 + k_1 \varrho \xi + \varepsilon \xi - \varepsilon \xi^3/3 = 0$$

(We take $\xi = 0$ if (3.2) has no positive solution ξ).

If $M > \max(M_0, \xi)$, our first claim is that $x(t) < M$ for all $t \geqslant 0$. To see this, suppose the contrary and let $T > 0$ be the first time such that $x(T) = M$. It follows by taking the left-hand derivative at T that $x'(T)\geqslant 0$. On the other hand we have that

$$y(T)=y(0)-\int_{-r}^{T-r}x(s)\,ds+k\int_{-r_2}^{T-r_2}x(s)\,ds \tag{3.3}$$

$$\leqslant y(0)+\big(1+|k|\big)\varrho M_0+(k-1)\int_0^{T-\varrho}x(s)\,ds+k_1\varrho M$$

$$\leqslant y(0)+\big(1+|k|\big)\varrho M_0+k_1\varrho M\,.$$

It follows that we have

$$x'(T)=y(T)+\varepsilon[x(T-r_1)-x^3(T)/3] \tag{3.4}$$

$$\leqslant y(0)+\big(1+|k|\big)\varrho M_0+k_1\varrho M+\varepsilon M-\varepsilon M^3/3$$

$$<0\,.$$

This is a contradiction, so we must have $x(t)<M$ for all $t\geqslant 0$.

For $t\geqslant 2\varrho$ we see that

$$y(t)=y(\varrho)-\int_{\varrho-r}^{t-r}x(s)\,ds+k\int_{\varrho-r_2}^{t-r_2}x(s)\,ds \tag{3.5}$$

$$\leqslant y(\varrho)+A-(1-k_1)\int_{\varrho}^{t-\varrho}x(s)\,ds-\int_{t-\varrho}^{t-r}x(s)\,ds+k\int_{t-\varrho}^{t-r_2}x(s)\,ds$$

$$\leqslant y(\varrho)+A+k\varrho M-(1-k_1)\int_{\varrho}^{t-\varrho}x(s)\,ds\,,$$

where $A=-\int_{\varrho-r}^{\varrho}x(s)\,ds+k\int_{\varrho-r_2}^{\varrho}x(s)\,ds$. It follows that unless $x(t)$ is integrable on $[0,\infty)$, $y(t)$ approaches $-\infty$. However, if $y(t)$ approaches $-\infty$, $x'(t)$ approaches $-\infty$ as t approaches ∞, and this would contradict the assumption that $x(t)$ is always nonnegative. It follows that $x(t)$ is integrable. However, this immediately implies that $y(t)$ approaches $y(0)-\int_{-r}^{0}x(s)\,ds+k\int_{-r_2}^{0}x(s)\,ds-(1-k)\int_0^{\infty}x(s)\,ds$ as t approaches ∞. Since $x'(t)=y(t)+\varepsilon[x(t-r_1)-x^3(t)/3]$, we see that $x'(t)$ is bounded and since $x(t)$ is nonnegative and integrable on $[0,\infty)$, we must have that $\lim_{t\to\infty}x(t)=0$. If we set $B=\lim_{t\to\infty}y(t)$, $\lim_{t\to\infty}x'(t)=B$, so we must have that $B=0$.

If $k>0$ (so that $r_2\leqslant r$) we see that

$$y(t)=y(0)-\int_{-r}^{0}x(s)\,ds+k\int_{-r_2}^{0}x(s)\,ds-(1-k)\int_{0}^{t-r}x(s)\,ds+k\int_{t-r}^{t-r_2}x(s)\,ds\,,$$

and if we compare this to the expression for $\lim_{t\to\infty}y(t)=0$, we see that $y(t)\geqslant 0$ for $t\geqslant r$. If $k<0$, $y(t)$ is monotonic decreasing for $t\geqslant\varrho$, so we must certainly have $y(t)\geqslant 0$ for $t\geqslant\varrho$.

We are now almost done. Since $\lim_{t\to\infty}x(t)=0$, let $s>\varrho$ be such that $x(s)>0$ and $x(t)<\sqrt{3}$ for $t\geqslant s$. For $t\geqslant s$ we then have

$$\begin{aligned}(3.6)\qquad x(t)&=x(s)+\int_{s}^{t}x'(u)\,du\\&=x(s)+\int_{s}^{t}y(u)\,du+\varepsilon\int_{s-r_1}^{t-r_1}x(u)\,du-\varepsilon/3\int_{s}^{t}x^3(u)\,du\\&\geqslant x(s)+\varepsilon\int_{s}^{t-r_1}\big(x(u)-x^3(u)/3\big)\,du-\varepsilon/3\int_{t-r_1}^{t}x^3(u)\,du\\&\geqslant x(s)-\varepsilon/3\int_{t-r_1}^{t}x^3(u)\,du\,.\end{aligned}$$

Since $\lim_{t\to\infty}\int_{t-r_1}^{t}x^3(u)\,du=0$, equation (3.6) implies that for t large enough $x(t)>\frac{1}{2}x(s)$, a contradiction. Q.E.D.

Of course Lemma 3.1 implies that x has infinitely many zeros, since the same proof implies that there exists a number $T_2>T_1$ such that $x(T_2)>0$, and so on.

The equations (3.1) come from the second order equation $x''(t)=-x(t-r)+$ $+kx(t-r_2)+\varepsilon x'(t-r_1)-\varepsilon x^2(t)x'(t)$. If one linearizes this equation and searches for solutions of the form $\exp[\lambda t]$, one is led to the so-called characteristic equation:

$$(3.7)\qquad \lambda^2-\varepsilon\lambda\exp[-r_1\lambda]+\exp[-r\lambda]-k\exp[-r_2\lambda]=0\,.$$

Our next lemma analyzes (3.7) for the case $r_1=k=0$. In [7] Grafton outlines a different proof from the one given here, and a detailed version of that proof can be found in [11], pages 169-172. We believe our proof is considerably simpler.

LEMMA 3.2 (compare [7] and [11], pages 169-172). – If $r>0$ and if $\varepsilon\gamma_0>-\sin r\gamma_0$ for every γ_0 such that $\gamma_0^2=\cos r\gamma_0$ and $0<\gamma_0<\pi/r$ (in particular, if $\varepsilon\geqslant 0$), then the equation $\lambda^2-\varepsilon\lambda+\exp[-r\lambda]=0$ has precisely two solutions λ (counted algebraically) such that $\operatorname{Re}(\lambda)>0$ and $-\pi/r<\operatorname{Im}(\lambda)<\pi/r$.

PROOF. – First we show that if $\varepsilon' \geqslant \varepsilon$, $\lambda^2 - \varepsilon'\lambda + \exp[-r\lambda] \neq 0$ for $\lambda = i\lambda$, $0 \leqslant v \leqslant \pi/r$ or $\lambda = \mu + i\pi/r$, $\mu \geqslant 0$. If $\lambda = iv$, $0 \leqslant v \leqslant \pi/r$, and $\lambda^2 - \varepsilon'\lambda + \exp[-r\lambda] = 0$, then by taking real and imaginary parts we find that $-v^2 + \cos rv = 0$ and $-\varepsilon' v - \sin rv = 0$. It follows that we have $0 < v < \pi/r$ and $v^2 = \cos rv$, so by the assumptions $\varepsilon' v \geqslant \varepsilon v > -\sin rv$ a contradiction. If $\lambda = \mu + i\pi/r$ and $\lambda^2 - \varepsilon'\lambda + \exp[-r\lambda] = 0$, then by taking real and imaginary parts we find that $\mu^2 - (\pi/r)^2 - \varepsilon'\mu - \exp[-r\mu] = 0$ and $2\mu(\pi/r) - \varepsilon'(\pi/r) = 0$. The second equation implies that $\mu = \frac{1}{2}\varepsilon'$, but setting $\mu = \frac{1}{2}\varepsilon'$ in the first equation gives a negative number. It follows by taking complex conjugates that $\lambda^2 - \varepsilon'\lambda + \exp[-r\lambda] \neq 0$ for $\lambda = -iv$, $0 \leqslant v \leqslant \pi/r$ and $\lambda = \mu - i\pi/r$, $\mu \geqslant 0$.

Let ε' be a positive number, $\varepsilon' \geqslant \varepsilon$, and let $R_1 = (1 + \varepsilon')$. It follows by trivial estimates that if $\varepsilon \leqslant \varepsilon_1 \leqslant \varepsilon'$ and $\operatorname{Re}(\lambda) \geqslant R_1$, then $\lambda^2 - \varepsilon_1\lambda + \exp[-r\lambda] \neq 0$. For any $R \geqslant R_1$, define $G_R = \{\lambda \in \mathbf{C} : 0 < \operatorname{Re}(\lambda) < R$ and $|\operatorname{Im}(\lambda)| < \pi/r\}$. If we define $f_t(\lambda) = \lambda^2 - (1-t)\varepsilon\lambda - t\varepsilon'\lambda + \exp[-r\lambda]$, $0 \leqslant t \leqslant 1$, then our calculations show that $f_t(\lambda) \neq 0$ for $\lambda \in \partial G_R$, $0 \leqslant t \leqslant 1$. It follows by Rouche's theorem that $f_1(\lambda) = 0$ has the same algebraic number of solutions in G_R $(R \geqslant R_1)$ as $f_0(\lambda) = 0$.

Now we are almost done. Let b be a positive number such that $(\varepsilon')^2/4 < b < (\varepsilon')^2/4 + (\pi^2)/r^2$ and select $R_2 > \max\{1 + \varepsilon', b + \varepsilon'\}$. It is easy to check that if $\operatorname{Re}(\lambda) \geqslant R_2$, then $\lambda^2 - \varepsilon'\lambda + (1-s)\exp[-r\lambda] + sb \neq 0$ for $0 \leqslant s \leqslant 1$ and $\operatorname{Re}(\lambda) \geqslant R_2$ and $\lambda^2 - \varepsilon'\lambda + b$ has precisely two roots (algebraically) in G_{R_2}. To complete the proof, it suffices by Rouche's theorem to show that $\lambda^2 - \varepsilon'\lambda + (1-s)\exp[-r\lambda] + sb \neq 0$ for $0 \leqslant s \leqslant 1$ and $\lambda = iv$, $0 \leqslant v \leqslant \pi/r$ or $\lambda = \mu + i(\pi/r)$, $\mu \geqslant 0$. If $\lambda^2 - \varepsilon'\lambda + (1-s)\exp[-r\lambda] + sb = 0$ for $\lambda = iv$, $0 \leqslant v \leqslant \pi/r$, the same computation as before shows that $-v^2 + (1-s)\cos rv + sb = 0$ and $-\varepsilon' v - (1-s)\sin rv = 0$. The second equation implies that $v = 0$ (since $\varepsilon' > 0$), and this is impossible. If $\lambda = \mu + i\pi/r$ we find that $\mu^2 - (\pi/r)^2 - \varepsilon'\mu - (1-s) + sb = 0$ and $2\mu(\pi/r) - \varepsilon'(\pi/r) = 0$. The second equation implies that $\mu = \frac{1}{2}\varepsilon'$, and substituting this value in the first equation we find that $-\frac{1}{4}(\varepsilon')^2 - (\pi/r)^2 - (1-s) + sb = 0$. However, since we chose b so that $b < (\varepsilon')^2/r + (\pi/r)^2$, this is impossible. Q.E.D.

Our next lemma generalizes Lemma 3.2 and indicates the advantages of using Rouche's theorem in this context.

LEMMA 3.3. – If $\varepsilon > 0$, $r > 0$, $0 \leqslant r_1 \leqslant r$, $0 \leqslant r_2 \leqslant \frac{1}{2}r$ and $0 \leqslant k < 1$, the equation $\lambda^2 - \varepsilon\lambda\exp[-r_1\lambda] + \exp[-r\lambda] - k\exp[-r_2\lambda] = 0$ has precisely two solutions λ such that $\operatorname{Re}(\lambda) > 0$ and $-\pi/r < \operatorname{Im}(\lambda) < \pi/r$. If $\varepsilon \geqslant 0$, $r > 0$, $0 \leqslant r_1 \leqslant r$ and $-(\pi/2r)^2 < k < 0$, the equation $\lambda^2 - \varepsilon\lambda\exp[-r_1\lambda] + \exp[-r\lambda] - k = 0$ also has precisely two solutions λ such that $\operatorname{Re}(\lambda) > 0$ and $-\pi/r < \operatorname{Im}(\lambda) < \pi/r$.

PROOF. – Let the parameters be as in the first case above and notice that if $\operatorname{Re}(\lambda) \geqslant R_1 = (1 + \varepsilon + k)$, then for any nonnegative numbers r_1', r', r_2' and any nonnegative $k' \leqslant k$,

$$|\lambda^2 - \varepsilon'\lambda\exp[-r_1'\lambda] + \exp[-r'\lambda] - k'| \geqslant |\lambda||\lambda - \varepsilon'\exp[-r_1'\lambda]| - 1 - k$$
$$\geqslant (1 + \varepsilon + k)(1 + k) - 1 - k > 0\,.$$

If $G_{R_1} = \{\lambda: 0 < \operatorname{Re}\lambda < R_1, \ -\pi/r < \operatorname{Im}(\lambda) < \pi/r\}$, then according to Lemma 3.2 $\lambda^2 - \varepsilon\lambda + \exp[-r\lambda]$ has precisely two zeros in G_{R_1}. By Rouche's theorem it suffices to show that $\lambda^2 - \varepsilon\lambda\exp[-r_1\lambda] + \exp[-r\lambda] - k\exp[-r_2\lambda] = 0$ can be deformed into $\lambda^2 - \varepsilon\lambda + \exp[-r\lambda]$ without introducing zeros on ∂G_{R_1}. Thus, if we define $f_t(\lambda) = \lambda^2 - \varepsilon\lambda\exp[-tr_1\lambda] + \exp[-r\lambda] - tk\exp[-r_2\lambda]$, it suffices to show that $f_t(\lambda) \neq 0$ for $0 \leqslant t \leqslant 1$ and $\lambda \in \partial G_{R_1}$. By the above remarks $f_t(\lambda) \neq 0$ for λ such that $\operatorname{Re}(\lambda) \geqslant R_1$, and by taking complex conjugates we only have to show that $f_t(\lambda) \neq 0$ for $\lambda = i\nu$, $0 \leqslant \nu \leqslant \pi/r$ or $\lambda = \mu + i\pi/r$, $\mu \geqslant 0$. If $\lambda = i\nu$ and $f_t(\lambda) = 0$ we are led to the equations $-\nu^2 - \varepsilon\nu\sin tr_1\nu + \cos r\nu - tk\cos r_2\nu = 0$ and $-\varepsilon\nu\cos tr_1\nu - \sin r\nu + k\sin r_2\nu = 0$. If $\nu = 0$, the first equation reduces to $1 - tk > 0$. If $0 < \nu \leqslant \pi/2r$, $\sin r\nu > \sin r_2\nu$ and $\cos tr_1\nu \geqslant 0$ so that $-\varepsilon\nu\cos tr_1\nu - \sin r\nu + k\sin r_2\nu < 0$. If $\pi/2r \leqslant \nu < \pi/r$, we have $\cos r_2\nu \geqslant 0$ (since $0 \leqslant r_2 \leqslant \frac{1}{2}r$), $\cos r\nu \leqslant 0$, and $-\nu^2 - \varepsilon\nu\sin tr_1\nu + \cos r\nu - tk\cos r_2\nu < 0$. If $\lambda = \mu + i\pi/r$, we are led to the equations $[\mu^2 - (\pi/r)^2] - \varepsilon\exp[-tr_1\mu][\mu\cos tr_1\pi/r + \pi/r\sin tr_1\pi/r] - \exp[-r\mu] - kt\exp[-r_2\mu]\cos r_2\pi/r = 0$ and $2\mu(\pi/r) - \varepsilon\exp[-tr_1\mu]\,[\pi/r\cos tr_1\pi/r - \mu\sin r_1\pi/r] + kt\exp[-r_2\mu]\sin r_2\pi/r = 0$. If $0 \leqslant tr_1 \leqslant r/2$, then the first equation implies that $\mu^2 > \varepsilon\mu\exp[-tr_1\mu]\cos tr_1\pi/r$ or (assuming $\mu > 0$) $\mu > \varepsilon\exp[-tr_1\mu]\cos tr_1\pi/r$. Applying this estimate in the second equation, we see that $2\mu(\pi/r) - (\varepsilon\exp[-tr_1\mu])(\pi/r\cos tr_1\pi/r)$ is positive, so the second equation is positive. If $r/2 < tr_1 \leqslant r$, then we see immediately that $\cos tr_1\pi/r \leqslant 0$ so the second equation is positive. This shows that $f_t(\lambda) \neq 0$ for $\lambda \in \partial G_{R_1}$ and completes the proof of the first part of the lemma.

To avoid repetition we shall be sketchier in proving the second part of the lemma. Just as before, it suffices to show that $g_t(\lambda) = \lambda^2 - \varepsilon\lambda\exp[-tr_1\lambda] + \exp[-r\lambda] - tk \neq 0$ for $\lambda = i\nu$, $0 \leqslant \nu \leqslant \pi/r$, $\lambda = \mu + i\pi/r$, $\mu \geqslant 0$, and $0 \leqslant t \leqslant 1$. If $g_t(i\nu) = 0$, we obtain the equations $-\nu^2 - \varepsilon\nu\sin tr_1\nu + \cos r\nu - tk = 0$ and $-\varepsilon\nu\cos tr_1\nu - \sin r\nu = 0$. The same proof as before shows these equations cannot be satisfied by ν such that $0 \leqslant \nu \leqslant \pi/2r$. If $\pi/2r < \nu \leqslant \pi/r$, the first equation is negative, since $k > -(\pi/2r)^2$. If $g_t(\mu + i\pi/r) = 0$, we are led to the equations $[\mu^2 - (\pi/r)^2] - \varepsilon\exp[-tr_1\mu][\mu\cos tr_1\pi/r + \pi/r\sin tr_1\pi/r] - \exp[-r\mu] - kt = 0$ and $2\mu(\pi/r) - \varepsilon\exp[-tr_1\mu][\pi/r\cos tr_1\pi/r - \mu\sin tr_1\pi/r] = 0$. If $0 \leqslant tr_1 \leqslant r/2$, then since $-(\pi/r)^2 - kt < 0$, the first equation implies as before that $\mu > \varepsilon\exp[-tr_1\mu]\cos tr_1\pi/r$, and using this estimate we find that the left-hand side of the second equation is positive (assuming $\mu > 0$). If $r/2 < tr_1 \leqslant r$, we immediately see that the left-hand side of the second equation is positive. Q.E.D.

Unfortunately, Lemmas 3.1 and 3.3 constitute about all we can say about equations (3.1) in the stated generality. Computer numerical studies strongly suggest that for a wide range of the parameters ε, r_1, r_2, r and k (r_1, r_2 and r commensurable), (3.1) has a non-zero periodic solution, but our techniques seem to break down in this generality. To obtain further results we shall have to restrict ourselves to the case

$r_1 = r_2 = 0$, i.e. to the following equations:

$$
\begin{aligned}
&x'(t) = y(t) + \varepsilon[x(t) - x^3(t)/3]\,, \qquad t \geqslant 0 \\
(3.8) \qquad &y'(t) = -x(t-r) + kx(t)\,, \qquad t \geqslant 0 \\
&x|[-r, 0] = \varphi = a \text{ given continuous function}\,, \qquad y(0) = y_0
\end{aligned}
$$

Even in this seemingly innocent generalization of Grafton's original equation in [7], there are serious technical problems (especially for $k < 0$) in trying to prove existence of periodic solutions. In fact our results will partially answer a question raised by GRAFTON in [8], page 526, and which seems inaccessible by his techniques.

The case $k \geqslant 0$ and $k < 0$ in (3.8) seem to demand different techniques. In our next few lemmas we study the case $0 \leqslant k < 1$.

LEMMA 3.4. – In equations (3.8) assume that $0 \leqslant k < 1$, φ is monotonic increasing, $\varphi(-r) = 0$, $\varphi(0) \geqslant 0$, $y_0 \geqslant 0$, $\max\{\varphi(0), y(0)\} > 0$ and $y_0 + \varepsilon[\varphi(0) - (\varphi(0))^3/3] \geqslant 0$. Define $T_1 = \sup\{t \geqslant 0 : x'(s) \geqslant 0 \text{ for } 0 \leqslant s \leqslant t\}$ and $z_1 = \inf\{t > 0 : x(t) = 0\}$ (we know by Lemma 3.1 that both T_1 and z_1 are finite). It then follows that z_1 is an isolated zero of x and $x'(t) \leqslant 0$ and $y'(t) \leqslant 0$ for $T_1 \leqslant t \leqslant z_1 + r$.

PROOF. – First we show that $x'(t) \leqslant 0$ and $y'(t) \leqslant 0$ for $T_1 \leqslant t \leqslant T_1 + r$. Obviously we must have $x'(0) = 0$ (if $T_1 = 0$, one uses the assumption that $x'_+(0) \geqslant 0$ to guarantee this) and $x''(T_1) = y'(T_1) \leqslant 0$. There are technical problems if $y'(T_1) = 0$, so we use a technical device. Take $\eta > 0$ and define a new functional differential equation for functions $x_\eta(t)$ and $y_\eta(t)$ as follows:

$$
\begin{aligned}
&x'_\eta(t) = y_\eta(t) + \varepsilon[x_\eta(t) - x_\eta^3(t)/3] - \eta(t - T_1)\,, \qquad t \geqslant T_1 \\
(3.9) \qquad &y'_\eta(t) = -x_\eta(t-r) + kx_\eta(t)\,, \qquad t \geqslant T_1 \\
&x_\eta|[T_1 - r, T_1] = x|[T_1 - r, T_1]\,, \qquad y_\eta(T_1) = y(T_1)\,.
\end{aligned}
$$

It is clear that $\lim_{\eta \to 0} x_\eta(t) = x(t)$ and $\lim_{\eta \to 0} y_\eta(t) = y(t)$ for $T_1 \leqslant t \leqslant T_1 + r$, so it suffices to show that x_η and y_η are monotonic decreasing on $[T_1, T_1 + r]$. If we define $t_1 = \sup\{t : T_1 \leqslant t \leqslant T_1 + r \text{ and } x'_\eta(s) \leqslant 0 \text{ for } T_1 \leqslant s \leqslant t\}$, then the introduction of η guarantees that $x''_\eta(T_1) \leqslant -\eta$ (derivatives are taken from the right at T_1) so that $t_1 > T_1$ and $x_\eta(t_1) < x_\eta(T_1)$. If $t_1 < T_1 + r$, then we must have $x'_\eta(t_1) = 0$; however we see that $x''_\eta(t_1) = -x_\eta(t_1 - r) + kx_\eta(t_1) - \eta \leqslant -x_\eta(T_1 - r) + kx_\eta(T_1) - \eta \leqslant -\eta < 0$, and this contradicts the selection of t_1. It follows that x_η is decreasing (strictly) on $[T_1, T_1 + r]$. Therefore, we see that $y'_\theta(t) = -x_\eta(t - r) + kx_\eta(t) \leqslant -x_\eta(T_1 - r) + kx({}_\eta T_1) \leqslant 0$ for $T_1 \leqslant t \leqslant T_1 + r$, so y_η is decreasing (in fact strictly decreasing) on $[T_1, T_1 + r]$.

With the aid of the above result, our lemma now easily follows. Let $t_2 = \sup\{t \geqslant T_1 : x'(s) \leqslant 0 \text{ for } T_1 \leqslant s \leqslant t_2\}$. We have shown that $t_2 \geqslant T_1 + r$; we assume that $t_2 < z_1 + r$ and obtain a contradiction. By definition of t_2 we have that $x'(t_2) = 0$, so it follows that $x''(t_2) = y'(t_2) = -x(t_2 - r) + kx(t_2) \leqslant (k-1)x(t_2 - r) < 0$. This contradicts the selection of t_2.

It only remains to prove that z_1 is isolated. By our work above we have that $x'(z_1) = y(z_1) \leqslant 0$. If $x'(z_1) < 0$, z_1 is isolated, so we assume $x'(z_1) = 0$. If $x''(z_1) = y'(z_1) < 0$, z_1 is again isolated, so we assume $y'(z_1) = 0$. We now obtain a contradiction from the assumptions that $x'(z_1) = y'(z_1) = 0$. By definition we know that $y'(z_1) = -x(z_1 - r)$, so that if $y'(z_1) = 0$ we have $z_1 - r < 0$ and $\varphi(z_1 - r) = 0$. Since φ is monotonic increasing and $\varphi(-r) = 0$, we see that $\varphi(t) = 0$ for $-r \leqslant t \leqslant z_1 - r$. It follows that for this range of t, $y(t) = y(0) + k\int_0^t x(s)ds$ and, in particular, $y(z_1) > 0$ unless $y(0) = 0$ and $k = 0$. Thus we must assume that $y_0 = k = 0$, and (3.9) reduces to the ordinary differential equation $x'(t) = \varepsilon[x(t) - x^3(t)/3]$, $x(0) = \varphi(0)$, for $0 \leqslant t \leqslant z_1$. However it is easy to see that if $x(0) < \sqrt{3}$, $x(t)$ is increasing and $x(t) < \sqrt{3}$ for $0 \leqslant t \leqslant z_1$; if $x(0) = \sqrt{3}$, $x(t) = \sqrt{3}$ for $0 \leqslant t \leqslant z_1$; and if $x(0) > \sqrt{3}$, $x(t)$ is decreasing and $x(t) > \sqrt{3}$ for $0 \leqslant t \leqslant z_1$: In any event, we find that $x(z_1) > 0$, a contradiction. Q.E.D.

The same proof now shows that the subsequent zeros z_2, z_3, etc. of x are isolated, that x is either monotonic increasing or monotonic decreasing on $[z_n, z_n + r]$ depending on whether $x(z_n + r)$ is positive or negative, and that y is monotonic increasing or decreasing on $[z_n, z_n + r]$. Furthermore, it is clear that for $n \geqslant 2$, we must have $y'(z_n) > 0$ if $x(z_n + r) > 0$ and $y'(z_n) < 0$ if $x(z_n + r) < 0$.

At this point we can describe our method of proof. Let $G = \{(\varphi, y_0): \varphi$ is a continuous, monotonic increasing function on $[-r, 0]$, $y_0 \geqslant 0$, $\varphi(-r) = 0$ and $y_0 + \varepsilon[\varphi(0) - (\varphi(0))^3/3] \geqslant 0\}$. It is easy to check that G is a closed, convex subset of the Banach space $X = C([-r, 0]) \times \boldsymbol{R}$. For a given $(\varphi, y_0) \in G - \{0\}$ (0 denotes the origin in X), let $(x(t), y(t))$ be the corresponding solution of (3.8) and let $z_1(\varphi)$ be the corresponding first positive zero of x. We define $F(\varphi, y_0) = (-\psi, -y_1)$, where $\psi(t) = x(z_1(\varphi) + r + t)$ for $-r \leqslant t \leqslant 0$ and $y_1 = y(z_1(\varphi) + r)$. Owing to the symmetry properties of (3.9), fixed points of F correspond to periodic solutions of (3.9). Lemma 3.4 implies that $F(G - \{0\}) \subset G - \{0\}$. It is easy to prove that F is continuous on $G - \{0\}$, though in general F is *not* continuous at 0.

In our next lemma we show that F is actually a compact map.

LEMMA 3.5. – The map $F: G - \{0\} \to G - \{0\}$ defined above is a compact map.

PROOF. – Let A be a bounded subset of $G - \{0\}$, say $\|(\varphi, y_0\| \leqslant M_0$ for $(\varphi, y_0) \in A$. To prove compactness we have to show $F(A)$ has compact closure in X. By the Ascoli theorem it suffices to show $F(A)$ is bounded and $\{\psi: (\psi, y_1) \in F(A)$ for some $y_1\}$ is equicontinuous. Since ψ satisfies the differential equation $\psi'(t) = y(t + z_1(\varphi) + r) + \varepsilon[\psi(t) - (\psi(t))^3/3]$, equicontinuity will follow if we prove there exists a constant M_1 such that $\max\{|x(t)|: z_1(\varphi) \leqslant t \leqslant z_1(\varphi) + r\} \leqslant M_1$ and $\max\{|y(t)|: z_1(\varphi) \leqslant t \leqslant z_1(\varphi) + r\} \leqslant M_1$, where x and y are solutions of (3.8) corresponding to $(\varphi, y_0) \in A$.

If M is defined as in Lemma 3.1 and $(x(t), y(t))$ is a solution of (3.8) corresponding to $(\varphi, y_0)\in A$, then the same argument used in Lemma 3.1 implies $x(t)< M$ for $0\leqslant t\leqslant z_1(\varphi)$. Suppose that $y(t_0)=(-kr-\varepsilon-1/r)M$ for some $t_0\in(0, z_1(\varphi)]$. Then since $y'(t)\leqslant kx(t)$, $0\leqslant t\leqslant z_1(\varphi)+r$ we must have $y(t)\leqslant(-\varepsilon-1/r)M$ for $t_0\leqslant t\leqslant t_0+r$. This implies that $x'(t)\leqslant(-\varepsilon-1/r)M+\varepsilon x(t)\leqslant -M/r$ as long as $t_0\leqslant t\leqslant t_0+r$ and $x(t)\geqslant 0$; and since $x(t_0)\leqslant M$, it follows that $t_0\leqslant z_1(\varphi)\leqslant t_0+r$. Since $y'(t)\geqslant-(1+k)M$ for $0\leqslant t\leqslant z_1(\varphi)$, it follows that $y(t)\geqslant-(kr+\varepsilon+1/r)M-r(1+k)M$ for $0\leqslant t\leqslant z_1(\varphi)$. Of course if $y(t)>-(kr+\varepsilon+1/r)M$ for $0\leqslant t\leqslant z_1(\varphi)$, we have an even better estimate. In either case, there exists a positive constant N such $y(t)>-N$ for $0\leqslant t\leqslant z_1(\varphi)$.

Obvious estimates now imply that $y(t)>-N-rM+krx(t)$ for $z_1(\varphi)\leqslant t\leqslant z_1(\varphi)+r$. If $x(t)>-\sqrt{3(\varepsilon+kr)/\varepsilon}$ for $z_1(\varphi)\leqslant t\leqslant z_1(\varphi)+r$, we are done; otherwise $x(t_1)=-\sqrt{3(\varepsilon+kr)/\varepsilon}$, and for $t_1\leqslant t\leqslant z_1(\varphi)+r$, we obtain that $x'(t)\geqslant-(N+rM)$. It follows that in any event $x(t)\geqslant-\sqrt{3(\varepsilon+kr)/\varepsilon}-r(N+rM)$ for $z_1(\varphi)\leqslant t\leqslant z_1(\varphi)+r$. Q.E.D.

Our previous lemma showed that F takes bounded sets to bounded sets. In our next lemma, we wish to show that if $\|(\varphi, y_0)\|$ is large enough, $\|F(\varphi, y_0)\|<\|(\varphi, y_0)\|$.

LEMMA 3.6. – There exists a positive constant R such that if $(\varphi, y_0)\in G-\{0\}$ and $\|(\varphi, y_0)\|\geqslant R$, then $\|F(\varphi, y_0)\|<\|(\varphi, y_0)\|$.

PROOF. – Given a positive constant R, let ξ be the largest positive solution of $R+kr\xi+\varepsilon(\xi-\xi^3/3)=0$. If we write $\xi=\delta(R)R$, it is clear that $\lim_{R\to\infty}\delta(R)=0$. Select R_1 such that $\delta(R)<1$ for $R\geqslant R_1$. If $(\varphi, y_0)\in G$ and $\|(\varphi, y_0)\|=\max(\varphi(0), y_0)\geqslant R_1$, then we must have $\varphi(0)<y_0$, since it is assumed that $y_0+\varepsilon(\varphi(0)-(\varphi(0))^3/3)\geqslant 0$. If we write $y_0=R\geqslant R_1$, then the same argument used in Lemma 3.1 shows that $x(t)\leqslant\xi=\delta(R)R$ for $0\leqslant t\leqslant z_1(\varphi)$.

Define $M=\max\{x(t): 0\leqslant t\leqslant z_1(\varphi)\}$. The same argument used in the proof of Lemma 3.5 shows that $y(t)\geqslant-cM$ for $0\leqslant t\leqslant z_1(\varphi)$, where $c=2kr+\varepsilon+1/r+r$. Also the same proof shows that $y(z_1(\varphi)+r)\geqslant-(c+r)M+krx(z_1(\varphi)+r)=N$ and $x(z_1(\varphi)+r)\geqslant\sqrt{3(\varepsilon+kr)/\varepsilon}-r(cM+rM)$. It follows that $\|F(\varphi, y_0)\|\leqslant KM$, where K is a constant independent of $R\geqslant R_1$; and if R_2 is chosen so large that $\delta(R)K<1$ for $R\geqslant R_2$, then $\|F(\varphi, y_0)\|<R=\|(\varphi, y_0)\|$ for $R\geqslant R_2$: Q.E.D.

In our next lemma we establish the crucial step: that every nonzero solution of (3.8), no matter how small φ and y_0, grows to an a priori size. The idea is to use the direct analogue of a previously mentioned trick of E. M. WRIGHT (see Lemma 2.6). We consider an integral of the form $\int_T^\infty X'(t)\cdot\exp[-\lambda t]b\,dt$, where $X'(t)=(x'(t), y'(t))$, λ is an appropriate root of the characteristic equation, b is an appropriate vector, and the inner product of the two vectors is taken. It is not hard to see that Wright's trick is closely related to later work of HALE [11] and HALE-PERELLO [12].

LEMMA 3.7. – Assume that $(\varphi, y_0)\in G-\{0\}$ and let $(x(t), y(t))$ be the corresponding solution of (3.8). Then there exists a positive constant a, independent of (φ, y_0), such that $\lim_{t\to\infty} \sup |x(t)| \geqslant a$.

PROOF. - An examination of the argument in Lemma 3.6 shows that it also implies that both $|x(t)|$ and $|y(t)|$ are bounded for all $t\geqslant 0$. According to Lemma 3.3 there exist precisely two solutions λ_1 and λ_2 of $\lambda^2 - \varepsilon\lambda + \exp[-r\lambda] - k = 0$ such that $\operatorname{Re}(\lambda_j) > 0$ and $-\pi/r < \operatorname{Im}(\lambda_j) < \pi/r$. There are three possibilities: (1) $\operatorname{Im}(\lambda_1) > 0$ and $\lambda_2 = \bar{\lambda}_1$, (2) λ_1 and λ_2 are real and $\lambda_1 < \lambda_2$, and (3) $\lambda_1 = \lambda_2$, λ_1 is real, and λ_1 is a double root of $\lambda^2 - \varepsilon\lambda + \exp[-r\lambda] - k = 0$.

Our proof will vary depending on which case holds. In any event, if λ is a root of the characteristic equation as above and $T\geqslant 0$, then integration by parts gives

$$(3.10)\quad \int_T^\infty x'(t)(\lambda\exp[-\lambda t])\,dt + \int_T^\infty y'(t)\exp[-\lambda t]\,dt$$
$$= \lambda^2\int_T^\infty x(t)\exp[-\lambda t]\,dt - \lambda x(T)\exp[-\lambda T] - y(T)\exp[-\lambda T] + \lambda\int_T^\infty y(t)\exp[-\lambda t]\,dt .$$

On the other hand, if one substitutes for $x'(t)$ and $y'(t)$ from equation (3.8), one obtains

$$(3.11)\quad \int_T^\infty x'(t)(\lambda\exp[-\lambda t])\,dt + \int_T^\infty y'(t)\exp[-\lambda t]\,dt$$
$$= \int_T^\infty \big(y(t) + \varepsilon x(t)\big)\lambda\exp[-\lambda t]\,dt - \varepsilon\lambda/3\int_T^\infty x^3(t)\exp[-\lambda t]\,dt$$
$$-\int_T^\infty x(t-r)\exp[-\lambda t]\,dt + k\int_T^\infty x(t)\exp[-\lambda t]\,dt .$$

Setting (3.10) equal to (3.11) and using the fact that $\int_T^\infty x(t-r)\exp[-\lambda t]\,dt = \exp[-r\lambda]\int_{T-r}^\infty x(t)\exp[-\lambda t]\,dt$, one finds that

$$(3.12)\quad -\lambda x(T) - y(T) + \int_{T-r}^T x(t)\exp[-\lambda(t-T+r)]\,dt =$$
$$= -\varepsilon\lambda/3\int_T^\infty x^3(t)\exp[-\lambda(t-T)]\,dt .$$

CASE 1. - If $\operatorname{Im}(\lambda) > 0$, so that $\lambda = \mu + i\nu$ with $\mu > 0$ and $0 < \nu < \pi/r$, then by taking the imaginary part of (3.12) we obtain

$$-\nu x(T) - \int_{T-r}^T x(t)\exp[-\mu(t-T+r)]\sin\nu(t-T+r)\,dt$$
$$= -\operatorname{Im}\left(\varepsilon\lambda/3\int_T^\infty x^3(t)\exp[-\lambda(t-T)]\,dt\right) .$$

If z is a zero of x, let $b=\sup_{t\geq z}|x(t)|\geq 0$ and let $z_n\geq z$ be a zero of x such that $\sup_{z_n\leq t\leq z_{n+1}}|x(t)|=c\geq\frac{1}{2}b$. Let T, $z_n<T<z_{n+1}$, be a point such that $|x(T)|=c$; since x is monotonic on $[z_n, z_n+r]$, we know that $T\geq z_n+r$. For definiteness we can assume $x(T)>0$. It follows then that $\int_{T-r}^{T}x(t)\exp[-\mu(t-T+r)]\sin\nu(t-T+r)\,dt>0$, so we obtain the inequality

$$|\nu x(T)|\leq\left|\varepsilon\lambda/3\int_T^\infty x^3(t)\exp[-\lambda(t-T)]\,dt\right|. \tag{3.14}$$

If we estimate the right-hand side, using the fact that $|x^3(t)|\leq b^3$ for $t\geq T$, we find that

$$\tfrac{1}{2}\nu b\leq(\varepsilon/3)|\lambda|b^3(1/\mu). \tag{3.15}$$

Since one can easily prove that $b>0$, (3.15) implies that $b^2>a^2=\frac{3}{2}(\varepsilon|\lambda|)^{-1}\mu\nu$, and the theorem is proved.

CASE 2. – If the characteristic equation has two distinct real roots $\lambda_1<\lambda_2$, then substituting respectively λ_2 and λ_1 for λ in equation (3.12) and subtracting, one obtains

$$\begin{aligned}(3.16)\qquad -(\lambda_2-\lambda_1)x(T)+\int_{T-r}^{T}x(t)\big[\exp[-\lambda_2(t-T+r)]-\exp[-\lambda_1(t-T+r)]\big]\,dt\\ =-\varepsilon\lambda_2/3\int_T^\infty x^3(t)\exp[-\lambda_2(t-T)]\,dt+\varepsilon\lambda_1/3\int_T^\infty x^3(t)\exp[-\lambda_1(t-T)]\,dt.\end{aligned}$$

Now, if z and T are chosen as in Case 1 and b is defined as there, one finds

$$\tfrac{1}{2}(\lambda_2-\lambda_1)b\leq\varepsilon b^3/3+\varepsilon b^3/3. \tag{3.17}$$

Just as before, this implies that b^2 is larger than an a priori positive constant $a^2=3/2\varepsilon(\lambda_2-\lambda_1)$.

CASE 3. – If the characteristic equation has a real root λ of multiplicity 2, then λ must also satisfy the equation $2\lambda-\varepsilon-r\exp[-r\lambda]=0$. If one carries through the calculations preceding (3.12) with a number $\lambda_1>\lambda$ one obtains the following equation:

$$\begin{aligned}(3.18)\qquad -\lambda_1x(T)-y(T)+\int_{T-r}^{T}x(t)\exp[-\lambda_1(t-T+r)]\,dt\\ +\left(\lambda_1^2-\varepsilon\lambda_1+\exp[-r\lambda_1]-k\right)\int_T^\infty x(t)\exp[-\lambda_1(t-T)]\,dt\\ =-(\varepsilon\lambda_1/3)\int_T^\infty x^3(t)\exp[-\lambda_1(t-T)]\,dt.\end{aligned}$$

If one subtracts (3.12) from (3.18), divides by $\lambda_1 - \lambda$, and then lets λ_1 approach λ, one obtains (using Lebesgue dominated convergence)

$$(3.19)\qquad -x(T) - \int_{T-r}^{T} x(t)(t - T + r)\exp[-\lambda(t - T + r)]\,dt = -\varepsilon/3\int_{T}^{\infty} x^3(t)\exp[-\lambda(t - T)] + \varepsilon\lambda/3\int_{T}^{\infty} x^3(t)(t - T)\exp[-\lambda(t - T)]\,dt\,.$$

The proof now proceeds essentially as in the previous cases. Q.E.D.

LEMMA 3.8. – Let $x(t)$ and $y(t)$ be as in Lemma 3.7 and let z_n, $n \geq 1$, denote the zeros of x. Then there exists a positive constant a^*, independent of (φ, y_0), such that $\limsup_{n\to\infty}\big(\max|x(z_n + r)|, |y(z_n + r)|\big) \geq a^*$.

PROOF. – By Lemma 3.7 there exists an a priori positive constant a such that $\limsup_{t\to\infty}|x(t)| \geq a$. Select z_n such that $\max\{|x(t)|: z_n \leq t \leq z_{n+1}\} = \alpha \geq \frac{1}{2}a$ and let $T \in [z_n, z_{n+1}]$ be such that $|x(T)| = \alpha$. We know that $T \geq z_n + r$, and for convenience we can assume that $x(T) > 0$. Select T_1, $T < T_1 < z_{n+1}$, such that $x(T_1) = \min\big(\frac{1}{2}\alpha, \frac{1}{2}\sqrt{3}\big) = \beta$. By our previous work we know that $x'(T_1) = y(T_1) + \varepsilon x(T_1)\cdot[1 - x^2(T_1)/3] \leq 0$, and this implies that $y(T_1) \leq -\frac{1}{2}\varepsilon\beta$. Since y is decreasing on $[T_1, z_{n+1} + r]$, this implies our result. Q.E.D.

THEOREM 3.1. If $\varepsilon > 0$, $r > 0$ and $0 \leq k < 1$, then equation (3.8) has a nonzero periodic solution $\big(x(t), y(t)\big)$ of period greater than $2r$.

PROOF. – Let G and F be as in Lemma 3.5. By Lemma 3.6 there exists a constant R_1 such that $\|F(\varphi, y_0)\| \leq \|(\varphi, y_0)\|$ if $\|(\varphi, y_0)\| \geq R_1$. If we define $R_2 = \sup\{\|F(\varphi, y_0)\|: \|(\varphi, y_0)\| \leq R_1\}$, then it is clear that if we define $G_R = \{(\varphi, y_0) \in G: \|(\varphi, y_0)\| \leq R\}$, $F(G_R - \{0\}) \subset G_R - \{0\}$ for $R \geq \max(R_1, R_2)$. Lemma 3.8 implies that 0 is an ejective point of F, and Lemma 3.5 implies that F is compact. It follows by our fixed point theorem that F has a nonzero fixed point in G, and in fact $i_G(F, G - \overline{U}) = 1$ for an appropriate open neighborhood U of 0. This fixed point corresponds to a nonzero periodic solution of period greater than $2r$. Q.E.D.

We now want to consider equation (3.8) for the case $k < 0$. Numerical studies strongly suggest that even for very large negative values of k, (3.8) has nonzero periodic solutions. However, as $|k|$ increases the period of the periodic solution appears to decrease and become less than $2r$, and our techniques break down in this case. Thus we shall have to restrict the size of $|k|$ in order to guarantee that the zeros of x are at least a distance r apart.

Our first step is to define a new closed, convex set C of starting values. If φ is a continuous real-valued function defined on $[-r, 0]$ and y_0 is a real number, we shall say that $(\varphi, y_0) \in C$ if $\varphi(t) \geq 0$ for $-r \leq t \leq 0$, $\varphi(0) = 0$, and $y_0 \leq 0$. It is clear that C is in fact a closed cone. As before, we shall write $(\varphi, y_0) \neq 0$ if either $\varphi(t) \neq 0$ for some t or $y_0 < 0$.

LEMMA 3.9. – Suppose that $(\varphi, y_0) \in C - \{0\}$ and $(x(t), y(t))$ is the corresponding solution of (3.8). Assume also that $k < 0$ and $-kr < \varepsilon$. Then if $z_1 = z_1(\varphi, y_0) = \sup\{t \geqslant 0 : x(s) \geqslant 0$ for $0 \leqslant s \leqslant t\}$ and $t_1 = \sup\{t \geqslant z_1 : x'(s) \leqslant 0$ for $z_1 \leqslant s \leqslant t\}$, it follows that $z_1 < r$ and either $x(t_1) \leqslant -\delta = -\sqrt{3}\sqrt{1 + (kr/\varepsilon)}$ or $(t_1 - z_1) \geqslant r$.

PROOF. – Given $\eta > 0$, consider the following FDE:

$$
\begin{aligned}
&x'_\eta(t) = y_\eta(t) + \varepsilon[x_\eta(t) - x^3_\eta(t)/3] - \eta t\\
&y'_\eta(t) = -x_\eta(t-r) + kx_\eta(t)\\
&x_\eta|[-r, 0] = \varphi, \qquad y_\eta(0) = y_0 .
\end{aligned}
\tag{3.20}
$$

If $x'_\eta(0) = y_0 < 0$, then $x'_\eta(t) < 0$ for some nonzero length of time; however, if $x'_\eta(0) = 0$, $x''_\eta(0) \leqslant -\eta$ and consequently $x'_\eta(t) \leqslant 0$ for some nonzero length of time. If we define $T_\eta = \sup\{t \geqslant 0 : x'_\eta(s) \leqslant 0$ for $0 \leqslant s \leqslant t\}$, then the above remarks show that $T_\eta > 0$ for $\eta > 0$. We claim that either $T_\eta \geqslant r$ or $x_\eta(T_\eta) \leqslant -\delta$. For suppose not, so that $T_\eta < r$ and $x_\eta(T_\eta) > -\delta$. It is clear from our construction that $x_\eta(t) < 0$ for $0 < t \leqslant T_\eta$ and $y_\eta(t) \leqslant k\int_0^t x_\eta(s)\,ds \leqslant ktx_\eta(t)$ for $0 \leqslant t \leqslant T_\eta$. It follows that $x'_\eta(t) \leqslant krx_\eta(t) + \varepsilon x_\eta(t) - \varepsilon x^3_\eta/3$ for $0 \leqslant t \leqslant T_\eta$, and since it is easy to see that $kru + \varepsilon u - \varepsilon u^3/3 < 0$ for $-\delta < u < 0$, it follows that $x'_\eta(T_\eta) < 0$, a contradiction.

We now suppose that $z_1 \geqslant r$ and obtain a contradiction. If there existed a sequence $\eta_j \to 0$ such that $\min_{0 \leqslant t \leqslant r} x_{\eta_j}(t) \leqslant -\delta$, we would immediately obtain a contradiction. Thus we can assume that x_η is monotonic decreasing on $[0, r]$ for η small enough, and by taking limits, x is monotonic decreasing for $0 \leqslant t \leqslant r$. It follows that $x(t) = 0$ for $0 \leqslant t \leqslant r$ and therefore $x'(t) = y(t) = 0$ for $0 \leqslant t \leqslant r$. Finally, we find that $y'(t) = -x(t-r) = 0$ for $0 \leqslant t \leqslant r$, and this implies $(\varphi, y_0) = 0$, a contradiction.

The remainder of the lemma follows just as above if one considers for $\alpha > 0$ the following FDE:

$$
\begin{aligned}
&x'_\alpha(t) = y_\alpha(t) + \varepsilon[x_\alpha(t) - x^3_\alpha(t)/3] - \alpha(t - z_1)\\
&y'_\alpha(t) = -x_\alpha(t-r) + kx_\alpha(t)\\
&x_\alpha|[z_1 - r, z_1] = x, \qquad y_\alpha(z_1) = y(z_1) .
\end{aligned}
\tag{3.21}
$$

Q.E.D.

Owing to the generality of the starting value (φ, y_0) it may very well happen that $z_1 > 0$.

LEMMA 3.10. – Let notations and assumptions be as in Lemma 3.9 and suppose in addition that $-\frac{1}{2}kr^2 \leqslant 1$. Then if $z_2 = \inf\{z > z_1 : x(z) = 0\}$, $(z_2 - z_1) > r$.

PROOF. – The previous work has already shown that $z_2 > z_1$. We assume that $(z_2 - z_1) \leqslant r$ and obtain a contradiction.

Let t_1 be the last point on $[z_1, z_2]$ where x achieves its minimum.

CASE 1. – Assume that $x(t_1) \geqslant -\sqrt{3}$. Since we are assuming that $(z_2 - z_1) < r$, the same argument as in Lemma 3.9 implies that $y(t_1) \leqslant k(t_1 - z_1)x(t_1)$ and $y(t) \leqslant y(t_1) + k(t - t_1)x(t_1)$ for $t_1 \leqslant t \leqslant z_2$. It follows that for $t_1 \leqslant t \leqslant z_2$, $x'(t) = y(t) + \varepsilon[x(t) - x^3(t)/3] \leqslant y(t) \leqslant k(t_1 - z_1)x(t_1) + k(t - t_1)x(t_1)$. If we integrate both sides of this inequality we obtain

$$|x(t_1)| \leqslant |k|(z_2 - t_1)(t_1 - z_1)|x(t_1)| + \tfrac{1}{2}|k|(z_2 - t_1)^2|x(t_1)| . \tag{3.21}$$

If we write $u = (t_1 - z_1)$ and replace $(z_2 - t_1)$ by the larger $r - u$, we obtain

$$1 \leqslant |k|(r - u)u + \tfrac{1}{2}|k|(r - u)^2 = \tfrac{1}{2}|k|(r^2 - u^2) . \tag{3.22}$$

This contradicts the assumption that $\frac{1}{2}|k|r^2 \leqslant 1$.

CASE 2. – Assume that $x(t_1) < -\sqrt{3}$, define $s_1 = \sup\{s : z_1 \leqslant s \leqslant z_2,\ x(s) \leqslant -\sqrt{3}$ and $x'(s) = 0\}$, and define s_2 to be the first time s after s_1 such that $x(s) = -1$. By our construction we must have $x(t) \geqslant -\sqrt{3}$ for $s_2 \leqslant t \leqslant z_2$. Because $u - u^3/3$ is decreasing for $-\infty < u \leqslant -1$, we find that for $s_1 \leqslant t \leqslant s_2$

$$\begin{aligned} x'(t) &= y(t) + \varepsilon[x(t) - x^3(t)/3] \\ &\leqslant y(s_1) + k(t - s_1)x(s_1) + \varepsilon[x(s_1) - x^3(s_1)/3] \\ &\leqslant k(t - s_1)x(s_1) . \end{aligned} \tag{3.23}$$

Integrating inequality (3.23) we obtain

$$-1 - x(s_1) \leqslant \tfrac{1}{2}kx(s_1)(s_2 - s_1)^2 . \tag{3.24}$$

For $s_2 \leqslant t \leqslant z_2$ we find the following estimates:

$$\begin{aligned} x'(t) &= y(t) + \varepsilon[x(t) - x^3(t)/3] \\ &\leqslant y(t) \\ &\leqslant y(s_1) + k(t - s_1)x(s_1) \\ &\leqslant k(t - s_1)x(s_1) . \end{aligned} \tag{3.25}$$

Integrating inequality (3.24) we find

$$1 \leqslant \tfrac{1}{2}kx(s_1)\{(z_2 - s_1)^2 - (s_2 - s_1)^2\} . \tag{3.26}$$

Adding inequalities (3.24) and (3.26) gives

$$|x(s_1)| \leqslant \tfrac{1}{2}|k|\,|x(s_1)|(z_2 - s_1)^2 \tag{3.27}$$

This contradicts the assumptions that $(z_2 - s_1) < r$ and $\frac{1}{2}|k|r^2 \leqslant 1$. Q.E.D.

Notice that z_2 is an isolated zero of x, for if $x'(z_2) = 0$, $x''(z_2) = y'(z_2) = -x(z_2 - r) > 0$.

Assuming $-kr < \varepsilon$ and $-\frac{1}{2}kr < 1$, we can now define a map $F\colon C - \{0\} \to C - \{0\}$ by $F(\varphi, y_0) = (-\psi, -y_1)$, where $\psi(s) = x(z_2 + s)$ for $-r \leqslant s \leqslant 0$ and $y_1 = -y(z_2)$. The fact that F is continuous is not difficult, and we leave it to the reader. It is clear that fixed points of F correspond to nonzero periodic solutions of (3.8) with period greater than $2r$.

We also leave the next lemma to the reader. It follows by the same kinds of arguments used for Lemmas 3.5 and 3.6 (only easier, since $k < 0$).

LEMMA 3.11. – For any $(\varphi, y_0) \in C - \{0\}$, the corresponding solution $(x(t), y(t))$ is bounded for $t \geqslant 0$. The map F defined above is compact, and there exists a constant $R_1 > 0$ such that if $R \geqslant R_1$ and $\|(\varphi, y_0)\| \leqslant R$, then $\|F(\varphi, y_0)\| \leqslant R$.

LEMMA 3.12. – Suppose that $k < 0$, $-kr < \varepsilon$ and $-\frac{1}{2}kr^2 \leqslant 1$. Then if $(\varphi, y_0) \in C - \{0\}$ and $(x(t), y(t))$ is the corresponding solution of (3.8), there exists an a priori positive constant a (independent of (φ, y_0)) such that $\limsup_{t\to\infty} |x(t)| \geqslant a$.

PROOF. – Let δ be as in Lemma 3.9. If $\limsup |x(t)| \geqslant \delta$, we are done. Otherwise, there exists a number u such that $x(t) < \delta$ for $t \geqslant u$. By Lemma 3.9, if $z \geqslant u$ and $x(z) = 0$, $|x(t)|$ is monotonic increasing on $[z, z + r]$, and therefore if z' is the first zero of x after z, $|x(t)|$ achieves its maximum on $[z, z']$ at some time T for which $(T - z) \geqslant r$. Furthermore, the condition that $-\frac{1}{2}kr^2 < 1$ implies that $-(\pi/2r)^2 < k$, so that (by Lemma 3.3) the equation $\lambda^2 - \varepsilon\lambda + \exp[-r\lambda] - k = 0$ has precisely two roots λ such that $\operatorname{Re}(\lambda) > 0$ and $-\pi/r < \operatorname{Im}(\lambda) < \pi/r$. But now exactly the same argument used in Lemma 3.7 implies that there exists an a priori constant a such that $\limsup_{t\to\infty} |x(t)| \geqslant a$. Q.E.D.

LEMMA 3.13. – There exists an a priori positive constant b such that for any $(\varphi, y_0) \in C - \{0\}$, $\limsup_{n\to\infty} \|F^n(\varphi, y_0)\| \geqslant b$.

PROOF. – Let a be as in Lemma 3.12 and let T be a number such that $|x(T)| = \alpha \geqslant \frac{1}{2}a$. We can assume that $z_n < T < z_{n+1}$ (where z_n and z_{n+1} are successive zeros of x) and $x(T) > 0$. If $(z_{n+1} - T) \leqslant r$ we are done. If $(T - z_n) \leqslant r$, define $c = \max\left(\max_{z_n - r \leqslant t \leqslant z_n} |x(t)|, y(z_n)\right)$ and observe that $x'(t) = y(t) + \varepsilon x(t) - \varepsilon x(t)^3/3 \leqslant (1 + r)c + \varepsilon x(t)$ for $z_n \leqslant t \leqslant T$. Using this inequality we find that $\alpha = x(T) \leqslant 1/\varepsilon(\exp[\varepsilon r] - 1)(r + 1)c$, which implies a lower bound on c and gives the result. Thus we can assume $(T - z_n) > r$, or we are done; and it follows that $y'(t) < 0$ for $T \leqslant t \leqslant z_{n+1}$. Define $\beta = \min\left(\sqrt{\tfrac{3}{2}}, \tfrac{1}{2}\alpha\right)$ and $T_1 = \sup\{t \geqslant T\colon x(t) = \beta\}$. Our definition of T_1 implies that $x'(T_1) = y(T_1) + \varepsilon x(T_1)[1 - x^2(T_1)/3] \leqslant 0$, and this implies that $y(T_1) \leqslant -\varepsilon\beta/2$. Since y is decreasing on $[T, z_{n+1}]$, $y(z_{n+1}) \leqslant -\varepsilon\beta/2$, and we are done. Q.E.D.

THEOREM 3.2. – If $k < 0$, $-kr < \varepsilon$ and $-\frac{1}{2}kr^2 \leqslant 1$, equation (3.8) has a nonzero periodic solution of period greater than $2r$.

PROOF. – Let $C_R = \{(\varphi, y_0) \in C : \|(\varphi, y_0)\| \leqslant R\}$. By Lemma 3.11, there exists $R > 0$ such that $F(C_R - \{0\}) \subset C_R - \{0\}$. By Lemma 3.13, 0 is an ejective point of F. It follows that F has a nonzero fixed point. Q.E.D.

4. – In this section we wish to consider the equation $x'(t) = -\alpha x(t - 1 - |x(t)|) \cdot (1 - x^2(t))$, which was mentioned by Halanay and Yorke in [10], page 67. We shall prove below that this equation has a nonzero periodic solution for every $\alpha > \pi/2$. We are less interested in the particular equation than in the nonstandard techniques which seem necessary for its study; in fact the full force of the above periodicity result appears inaccessible by Jones' or Grafton's techniques.

We begin with some existence and uniqueness results for the FDE. Let X denote the Banach space of Lipschitz functions $\varphi : [-2, 0] \to \boldsymbol{R}$; if $\varphi \in X$, the norm is given by

$$\|\varphi\|_1 = \max\left(\max_{-2\leqslant t\leqslant 0} |\varphi(t)|,\ \max_{-2\leqslant u<v\leqslant 0} \frac{|\varphi(v) - \varphi(u)|}{v - u}\right).$$

Given $\varphi \in X$ such that $|\varphi(0)| < 1$ we consider the following FDE:

$$(4.1) \qquad x'(t) = -\alpha x(t - 1 - |x(t)|)(1 - x^2(t)) \qquad x|[-2, 0] = \varphi, \qquad \varphi \in X, \qquad |\varphi(0)| < 1.$$

LEMMA 4.1. – If φ is as above, equation (4.1) has a unique solution $x(t) \equiv x(t; \varphi)$ which is defined for $t \geqslant 0$, continuously differentiable for $t \geqslant 0$, and such that $|x(t)| < 1$ for $t \geqslant 0$. Furthermore, given $\varepsilon > 0$, $N > 0$ and $\varphi \in X$ such that $|\varphi(0)| < 1$, there exists $\delta > 0$ such that if $\psi \in X$ and $\|\psi - \varphi\|_1 < \delta$, then $\sup_{0\leqslant t\leqslant N} |x(t; \varphi) - x(t; \psi)| < \varepsilon$.

PROOF. – For $0 \leqslant t \leqslant 1$, consider the ordinary differential equation $x'(t) = -\alpha\varphi(t - 1 - |x(t)|)(1 - x^2(t))$, $x(0) = \varphi(0)$. Since φ is Lipschitz, this equation has a unique C^1 solution defined on some interval $[0, \eta]$, $\eta > 0$. To show this solution can be extended to a C^1 solution on $[0, 1]$, it suffices to show that if $x(t)$ is a C^1 solution on $[0, a]$, $a \leqslant 1$, then $|x(t)| < 1$ on $[0, a]$. If not, let $t^* \leqslant a$ be the first time t on $[0, a]$ that $|x(t)| = 1$. Dividing both sides of the equation by $1 - x^2(t)$ and integrating from 0 to $t^* - \varepsilon$, $\varepsilon > 0$, gives

$$(4.2) \qquad \frac{1}{2}\log\left(\frac{1 + x(t^* - \varepsilon)}{1 - x(t^* - \varepsilon)}\right) - \frac{1}{2}\log\left(\frac{1 + x(0)}{1 - x(0)}\right) = -\alpha\int_0^{t^*-\varepsilon} \varphi(s - 1 - |x(s)|)\, ds.$$

Since the right hand side of (4.2) is bounded as $\varepsilon \to 0$, and the left hand side approaches $\pm\infty$, we obtain a contradiction. Thus equation (4.1) has a unique C^1 solution on $[0, 1]$, and repeating the procedure gives a unique C^1 solution on $[0, \infty)$ such that $|x(t)| < 1$ for all $t \geq 0$.

The statement about continuous dependence on initial data follows by using continuous dependence on initial data for ordinary differential equations on $[0, 1]$, then $[1, 2]$ and so on. Q.E.D.

Of course the reason for Lemma 4.1 and one of the technical problems in considering equation (4.1) is that (4.1) may not have a unique solution if φ is only continuous.

We now want to investigate the qualitative behaviour of solutions of (4.1) for an appropriate class of starting values φ. Let $Y = C^1([-2, 0], \boldsymbol{R})$ with the usual norm and define $S = \{\varphi \in Y: \varphi(-2) = 0,\ |\varphi(t)| < 1$ for $-2 \leq t \leq 0$, and there exists a number z_0 (depending on φ), $-1 < z_0 \leq 0$, such that $\varphi(t) > 0$ for $-2 < t < z_0$ and $\varphi(t) < 0$ for $z_0 < t \leq 0\}$. If $z_0 = 0$ this definition is meant to imply $\varphi(t) > 0$ for $-2 < t < 0$.

LEMMA 4.2. – If $\alpha > 1$, $\varphi \in S$, and $x(t)$ is the corresponding solution of equation (4.1), then $x(t)$ has infinitely many isolated zeros z_j, $z_j < z_{j+1}$ for $j \geq 1$ ($z_1 = z_0$ if $x(0) \leq 0$), and $x(t) \neq 0$ for $t \notin \{z_j\}$. If $T_j = \sup\{t > z_j: |x(s)|$ is monotonic increasing on $[z_j, t]$ if $z_j \geq 0$ or $|x(s)|$ is monotonic increasing on $[0, t]$ if $z_1 < 0\}$, then $T_j > z_j + 1$ and $|x(t)|$ is monotonic decreasing on $[T_j, z_{j+1}]$. Finally, the derivative of x is nonzero at any zero z_j such that $z_j \geq 0$.

PROOF. – If $\varphi(0) > 0$, then since $\alpha > 1$ essentially the same argument used in Section 1 shows that x has some first zero $z_1 > 0$. Since $x'(z_1) = -\alpha x(z_1 - 1 - |x(z_1)|)$ and $z_1 - 1 - |x(z_1)| > -2$, we have that $x'(z_1) < 0$. If $\varphi(0) \leq 0$, we define $z_0 = z_1$.

It is clear that $x'(t) < 0$ for $0 \leq t \leq z_1 + 1$, so that if we define $T_1 = \sup\{t \geq 0: x'(s) \leq 0$ for $0 \leq s \leq t\}$, $T_1 > z_1 + 1 > 0$. By our construction we have $x'(T_1) = 0$, and therefore we must have $T_1 - 1 - |x(T_1)| = z_1$. Since $(d/dt)(t - 1 - |x(t)|)_{t=T_1} = 1$ and since the derivative is defined for $x(t) \neq 0$, it follows that for $t > T_1$ and t near T_1, $t - 1 - |x(t)| > z_1$. It follows that $x'(t) > 0$ for $t > T_1$ and t near T_1. Once again, since $\alpha > 1$, essentially the same proof used in Section 1 shows that there exists a first zero $z_2 > z_1$ such that $x(z_2) = 0$, and we see that $x'(z_2) = -\alpha x(z_2 - 1) > 0$. If we define $T_2 = \sup\{t \geq T_1: x'(s) \geq 0$ for $T_1 \leq s \leq t\}$, the remarks above show that $T_2 > T_1$. If $T_2 \leq z_2$, then since $x'(T_2) = 0$, we must have $T_2 - 1 - |x(T_2)| = z_1$, and this is impossible, because $T_2 - 1 - |x(T_2)| > T_1 - 1 - |x(T_1)| = z_1$. In fact this remark shows we must have $x'(t) > 0$ for $T_1 < t \leq z_2$: It follows that $T_2 > z_2$, and again because $x'(T_2) = 0$ we must have $T_2 - 1 - |x(T_2)| = z_1$ or $T_2 - 1 - |x(T_2)| = z_2$. If $T_2 - 1 - x(T_2) = z_1$, then because $(d/dt)(t - 1 - |x(t)|)|_{t=T_2} = 1$, we must have $z_1 < t - 1 - |x(t)| < z_2$ for $t > T_2$ and t near T_2. This would imply that $x'(t) > 0$ for $t > T_2$ and t near T_2, contradicting the choice of T_2. It follows that $T_2 - 1 - |x(T_2)| = z_2$ and in particular that $T_2 > z_2 + 1$.

It is clear now that the same arguments show there exists a zero $z_3 > T_2$, $x'(z_3) < 0$, and $x'(t) \leqslant 0$ for $T_2 \leqslant t \leqslant T_3$, where $T_3 > z_3 + 1$. If we define $\psi(s) = x(z_2 + 2 + s)$ for $-2 \leqslant s \leqslant 0$, then we have $\psi \in S$, so the arguments above apply, and the lemma is proved. Q.E.D.

Given $\varphi \in S$ we can define a map $F: S \to S$ by using Lemma 4.2: if $\varphi \in S$, let z_2 denote the second zero of the corresponding solution $x(t)$ of (4.1) and define $F(\varphi) = \psi$, where $\psi(s) = x(z_2 + 2 + s)$ for $-2 \leqslant s \leqslant 0$. Using Lemma 4.1, one can verify that F is continuous in the topology of Y. Obviously, fixed points of F correspond to non-zero periodic solutions of equation (4.1).

LEMMA 4.3. – If A is a bounded subset of S, $F(A)$ is a precompact subset of Y.

PROOF. – Suppose that $\varphi \in A$, $x(t)$ is the corresponding solution of (4.1) and z_2 is the second zero of x on $[-2, \infty)$. By the Ascoli theorem it suffices to show that $x''(t)$, which is defined except at the zeros of x, is bounded on $[z_2, z_2 + 2]$ by a bound independent of φ. Since $|x(t)| < 1$ for all $t \geqslant -2$, we see that $|x'(t)| \leqslant \alpha$ for $t \geqslant 0$; and because A is bounded, there exists a constant β (independent of φ in A) such that $|x'(t)| \leqslant \beta$ for $t \geqslant -2$. Differentiating both sides of equation (4.1) we now see that $x''(t)$ is uniformly bounded on $[z_2, z_2 + 2]$ (where it is defined). Q.E.D.

LEMMA 4.4. – Suppose that $\varphi \in S$, $x(t)$ is the corresponding solution of equation (4.1), and $\alpha > \pi/2$. Then if $z > 0$ is a zero of x and if $\delta = \sup_{t \geqslant z}\big(\max(|x(t)|, |x'(t)|)\big)$, there exists a positive constant a, independent of φ and z, such that $\delta \geqslant a$.

PROOF. – By Lemma 2.5 there exists a solution λ of the equation $\lambda + \alpha \exp[-\lambda] = 0$ such that $\operatorname{Re}(\lambda) > 0$ and $0 < \operatorname{Im}(\lambda) < \pi$. If z_j and T_j are as in Lemma 4.2, integration by parts gives

$$(4.3) \qquad \int_{T_j}^{\infty} x'(t) \exp[-\lambda t]\, dt = \lambda \int_{T_j}^{\infty} x(t) \exp[-\lambda t]\, dt - x(T_j) \exp[-\lambda T_j].$$

On the other hand, if we substitute for $x'(t)$ from equation (4.1) and define $\Delta(t) = x(t-1) - x\big(t-1-|x(t)|\big)$ for notational convenience, one obtains

$$(4.4) \qquad \int_{T_j}^{\infty} x'(t) \exp[-\lambda t]\, dt = -\alpha \int_{T_j}^{\infty} \big(x(t-1) - \Delta(t)\big)\big(1 - x^2(t)\big) \exp[-\lambda t]\, dt$$
$$= -\alpha \int_{T_j}^{\infty} x(t-1) \exp[-\lambda t]\, dt - \alpha \int_{T_j}^{\infty} x(t-1) x^2(t) \exp[-\lambda t]\, dt +$$
$$+ \alpha \int_{T_j}^{\infty} \Delta(t)\big(1 - x^2(t)\big) \exp[-\lambda t]\, dt$$

If we define

$$R(T)=-\alpha\int_T^\infty x(t-1)x^2(t)\exp[-\lambda t]\,dt+\alpha\int_T^\infty \Delta(t)\big(1-x^2(t)\big)\exp[-\lambda t]\,dt\,,$$

equation (4.4) becomes

$$\int_{T_j}^\infty x'(t)\exp[-\lambda t]\,dt=-\alpha\exp[-\lambda]\int_{T_j-1}^\infty x(t)\exp[-\lambda t]\,dt+R(T_j)\,. \tag{4.5}$$

Combining equations (4.4) and (4.6) one finds

$$\alpha\exp[-\lambda]\int_{T_j-1}^{T_j} x(t)\exp[-\lambda t]\,dt-x(T_j)\exp[-\lambda T_j]=R(T_j)\,. \tag{4.6}$$

If one integrates again by parts one obtains

$$\alpha\exp[-\lambda]\int_{T_j-1}^{T_j} x(t)\exp[-\lambda t]\,dt=x(T_j)\exp[-\lambda T_j]-\\ -x(T_j-1)\exp[-\lambda(T_j-1)]-\int_{T_j-1}^{T_j} x'(t)\exp[-\lambda t]\,dt\,. \tag{4.7}$$

Substituting from equation (4.7) in (4.6) and multiplying both sides by $\exp[\lambda(T_j-\frac12)]$ yields

$$-x(T_j-1)\exp[\lambda/2]-\int_{T_j-1}^{T_j} x'(t)\exp[-\lambda(t-T_j+\tfrac12)]\,dt=\exp[\lambda(T_j-\tfrac12)]R(T_j)\,. \tag{4.8}$$

If $\lambda=\mu+i\nu$, the real part of the left hand side of equation (4.8) can be written $-x(T_j-1)\exp[\mu/2]\cos\nu/2-\int_{T_j-1}^{T_j} x'(t)\exp[-\mu(t-T_j+\frac12)]\cos\nu(t-T_j+\frac12)\,dt$. Since $x'(t)$ and $x(T_j-1)$ have the same sign for $T_j-1\leqslant t\leqslant T_j$ and since $\cos\nu(t-T_j+\frac12)\cdot\exp[-\mu(t-T_j+\frac12)]\geqslant\exp[-\mu/2]\cos\nu/2$ for $T_j-1\leqslant t\leqslant T_j$, the absolute value of the above expression is greater than $\big|x(T_j-1)\exp[\mu/2]\cos\nu/2+\exp[-\mu/2]\cos\nu/2\cdot\big(x(T_j)-x(T_j-1)\big)\big|$, which is greater than $\exp[-\mu/2]\cos\nu/2|x(T_j)|$.

If we define $\delta_1=\sup_{t\geqslant z}|x(t)|$ and $\delta_2=\sup_{t\geqslant z}|x'(t)|$, it is clear that $\delta=\max(\delta_1,\delta_2)$. Select $z_j\geqslant z$ and the corresponding T_j such that $|x(T_j)|\geqslant\frac12\delta_1$. It then follows from our previous remarks that $\frac12\exp[-\mu/2](\cos\nu/2)\delta_1\leqslant\big|\mathrm{Re}\big(\exp[\lambda(T_j-\frac12)]R(T_j)\big)\big|$. By using very crude estimates we find

$$(4.9)\qquad |\exp[\lambda(T_j-\tfrac{1}{2})]R(T_j)|\leqslant\alpha\delta_1^3\int_{T_j}^{\infty}\exp[-\mu(t-T_j+\tfrac{1}{2})]dt+$$

$$+\alpha\delta_1\delta_2\int_{T_j}^{\infty}\exp[-\mu(t-T_j+\tfrac{1}{2})]dt=\frac{\alpha\exp[-\frac{1}{2}\mu]}{\mu}[\delta_1^3+\delta_1\delta_2].$$

It follows that for an appropriate constant $c>0$,

$$(4.10)\qquad c\delta_1\leqslant\delta_1^3+\delta_1\delta_2.$$

Since it is easy to see that $\delta_1>0$, this implies that $c\leqslant\delta_1^2+\delta_2$, which certainly implies our result. Q.E.D.

LEMMA 4.5. - If $\alpha>\pi/2$, there exists a positive constant b such that for any $\varphi\in S$ there exists an integer N such that $x(T_{2N})\geqslant b$ (T_j is defined as in Lemma 4.2).

PROOF. - According to Lemma 4.4, there exists a positive constant a such that either $\limsup_{t\to\infty}|x'(t)|\geqslant a$ or $\limsup_{t\to\infty}|x(t)|\geqslant a$. It follows from equation (4.1) that if $b=\limsup_{t\to\infty}|x(t)|$, $\limsup_{t\to\infty}|x'(t)|\leqslant\alpha b$, and this implies that we must have $\limsup_{t\to\infty}|x(t)|\geqslant\geqslant a/\alpha$. It is clear that $\limsup_{t\to\infty}|x(t)|=\limsup_{N\to\infty}|x(T_N)|$, and using Lemma 4.2 one can see that $\limsup_{N\to\infty}|x(T_{2n+1})|\leqslant 2\alpha\limsup_{N\to\infty}|x(T_{2N})|$. Thus we see that $\limsup_{N\to\infty}|x(T_{2N})|\geqslant\geqslant a/2\alpha^2$. Q.E.D.

LEMMA 4.6. - Suppose that $\alpha>1$, $\varphi\in S$ and $x(t)$ is the corresponding solution of equation 4.1. Then if $a=\sup\{|x(t)|: z_n\leqslant t\leqslant z_{n+1}\}$, $|x'(z_{n+1})|\geqslant c$, where

$$c=\min\{a,\sqrt{\alpha(\alpha-1)},\ \alpha/2\ a(1-a^2),\ \alpha^2/4\ a(1-a^2)(1+\alpha a)^{-1}\}.$$

PROOF. - We can assume for convenience that $x(t)\geqslant 0$ for $z_n\leqslant t\leqslant z_{n+1}$, so that $a=x(T_n)$. According to Lemma 4.2, $t-1-|x(t)|\geqslant z_n$ for $T_n\leqslant t\leqslant z_{n+1}$, and in fact $(d/dt)(t-1-|x(t)|)=1-x'(t)\geqslant 1$ for $T_n\leqslant t<z_{n+1}$. It follows that $x'(t)$ is decreasing for $T_n\leqslant t\leqslant\min(z_{n+1},T_n+1)$; and thus if $z_{n+1}\leqslant T_n+1$ we find that

$$a=-\int_{T_n}^{z_{n+1}}x'(t)\,dt\leqslant -x'(z_{n+1}).$$

If $z_{n+1}>T_n+1$, we define s_n to be the unique point on (T_n, z_{n+1}) such that $s_n-1-|x(s_n)|=T_n$. There are two subcases to consider. First, suppose that

$s_n + 1 \leqslant z_{n+1}$. We claim that $\alpha(1 - x^2(z_{n+1} - 1)) \leqslant 1$. If not we have $x(t - 1 - |x(t)|) \geqslant x(z_{n+1} - 1)$ for $z_{n+1} - 1 \leqslant t \leqslant z_{n+1}$ and $x'(t) > x(z_{n+1} - 1)$ for $z_{n+1} - 1 \leqslant t \leqslant z_{n+1}$. But this would imply that $x(z) = 0$ for some z, $z_{n+1} - 1 < z < z_{n+1}$, a contradiction. Therefore we have that $\alpha(1 - x^2(z_{n+1} - 1)) \leqslant 1$, so that $x(z_{n+1} - 1) \geqslant \sqrt{(\alpha - 1)/\alpha}$. It follows that $x'(z_{n+1}) \leqslant -\sqrt{\alpha(\alpha - 1)}$.

Finally, we have to consider the possibility that $s_n < z_{n+1} < s_n + 1$. Since $x(t)$ is convex downward and positive on $[T_n, T_{n+1}]$, it follows that $x(u) \geqslant \frac{1}{2}x(T_n)$ if $0 \leqslant u \leqslant T_n + \frac{1}{2}$. Because $t - 1 - |x(t)| > T_n = s_n - 1 - |x(s_n)|$ for $s_n < t < z_{n+1}$ and because $|(t - 1 - |x(t)|) - (s_n - 1 - |x(s_n)|)| \leqslant (t - s_n) + \alpha x(T_n)(t - s_n)$, we see that $T_n \leqslant t - 1 - |x(t)| \leqslant T_n + \frac{1}{2}$ if $s_n \leqslant t \leqslant \frac{1}{2}(1 + \alpha a)^{-1} + s_n$ and $t < z_{n+1}$. If we define $\beta = \min\left(\frac{1}{2}(1 + \alpha a)^{-1}, z_{n+1} - s_n\right)$ all of this implies that $x(t - 1 - |x(t)|) \geqslant \frac{1}{2}x(T_n) = \frac{1}{2}a$ for $s_n \leqslant t \leqslant \beta + s_n$ and consequently that $x'(t) \leqslant -(\alpha/2)a(1 - x^2(s_n))$ for $s_n \leqslant t \leqslant s_n + \beta$. There are two possibilities: If $x(s_n) \leqslant (\alpha/4)a(1 - x^2(s_n))(1 + \alpha a)^{-1}$, then $z_{n+1} - s_n \leqslant \frac{1}{2}(1 + \alpha a)^{-1}$ and consequently $x'(z_{n+1}) \leqslant -(\alpha/2)a(1 - x^2(s_n)) \leqslant -(\alpha/2)a(1 - a^2)$. If we assume $x(s_n) > (\alpha/4)a(1 - x^2(s_n))(1 + \alpha a)^{-1}$, we must have $x(s_n) > (\alpha/4)a(1 - a^2) \cdot (1 + \alpha a)^{-1}$. This implies that $x'(z_{n+1}) = -\alpha x(z_{n+1} - 1) \leqslant -\alpha x(s_n) \leqslant -(\alpha^2/4)a(1 - a^2) \cdot (1 + \alpha a)^{-1}$. Q.E.D.

According to Lemma 4.5 there exists a constant a_0 such that if $\varphi \in S$ and $x(t)$ is the corresponding solution of (4.1), $\limsup_{N\to\infty} x(T_{2N}) > a_0$. Since trivial estimates imply that $x(T_{2N}) \leqslant 2\alpha|x(T_{2N-1})|$, it follows that if $x(T_{2N}) > a_0$, $|x(T_{2N-1})| \geqslant a_0/2\alpha$. Therefore Lemma 4.6 implies that there exists a constant b_0 such that if $x(T_{2N}) > a_0$, $x'(z_{2N}) \geqslant b_0$ and $x'(z_{2N+1}) \leqslant -b_0$. If we take c_0 to be a positive constant such that $c_0 < \min(a_0, b_0)$, then for any $\varphi \in S$, if $x(t)$ is the corresponding solution of (4.1), there exists a positive integer N such that $x(T_{2N}) > c_0$, $x'(z_{2N}) > c_0$ and $x'(z_{2N+1}) < -c_0$.

For a fixed constant $A > \alpha$, define $U = \{\varphi \in S : \varphi'(z) \neq 0$ if $\varphi(z) = 0$ and $\sup_{-2 \leqslant t \leqslant 0} |\varphi'(t)| < A\}$. It is easy to check that U is a bounded open subset of the closed, convex set $\{\psi \in Y : \psi(-2) = 0\}$ and that $F(S) \subset U$. If $\psi \in F(U)$, it is clear from equation (4.1) that $\sup_{-2 \leqslant t \leqslant 0} |\psi'(t)| \leqslant \alpha$ and (by taking second derivatives that) $\sup_{-2 \leqslant t \leqslant s \leqslant 1} |\psi'(t) - \psi'(s) \leqslant B|t - s|$, where $B = \alpha A(3 + A)$. Finally, if $x(t)$ satisfies equation (4.1), then we obtain

$$(4.11) \qquad \frac{1}{2}\log\left(\frac{1 + x(T_2)}{1 - x(T_2)}\right) = \int_{z_2}^{T_2} x'(t)/1 - x^2(t)\,dt = -\alpha\int_{z_2}^{T_2} x(t - 1 - |x(t)|)\,dt \leqslant 2\alpha.$$

This implies that $x(T_2) \leqslant k < 1$, where k is the largest solution of

$$\frac{1}{2}\log\left(\frac{1 + k}{1 - k}\right) = 2\alpha$$

such that $0 < k < 1$.

For each c, $0 < c \leqslant c_0$, we define a subset M_c of S as follows: If $\psi \in S$, then $\psi \in M_c$ if 1) there exists a time T, $-1 \leqslant T \leqslant 0$, ($T$ dependent on ψ) such that ψ is monotonic increasing on $[-2, T]$ and monotonic decreasing on $[T, 0]$; 2) $x'(t)$ is monotonic decreasing for $T \leqslant t \leqslant 0$; 3) $\sup_{-2 \leqslant t \leqslant 0} |x(t)| \leqslant k$, $\sup_{-2 \leqslant t \leqslant 0} |x'(t)| \leqslant \alpha$ and $\sup_{-2 \leqslant t < s \leqslant 0} |x'(t) - x'(s)| \leqslant B|t-s|$, where k and B are as above; 4) $x'(-2) \geqslant c$ and $x(T) \geqslant c$. It is not hard to check that M_c is a compact subset of Y and that $F(\psi)$ satisfies conditions $1-3$ above for any $\psi \in U$. Furthermore, condition 2) implies (by an argument used in Lemma 4.6) that if $\psi \in M_c$ and $\psi(z) = 0$ (necessarily $z \geqslant T$), then we have $\psi'(z) \leqslant -c$. It follows that $M_c \subset U$. Observe that if $\varphi_0 \in U$ our previous lemmas imply that there exists an integer N such that $\psi_0 = F^N \varphi_0$ satisfies $\psi_0'(-2) > c_0$ and $\sup_{-2 \leqslant t \leqslant 0} \psi_0(t) > c_0$: It follows that if $\varphi \in U$ is close enough to φ_0 then $\psi = F^N \varphi$ also satisfies the condition $\psi'(-2) > c_0$ and $\sup_{-2 \leqslant t \leqslant 0} \psi(t) > c_0$. Since $F^N \varphi$ automatically satisfies conditions $1-3$ for any $\varphi \in U$, $F^N \varphi \in M_c$.

Our next lemma is an easy consequence of Lemma 1.5 and is an abstraction of our concrete situation.

LEMMA 4.7. – Suppose that G is a closed, convex subset of a Banach space Y, U is a bounded open subset of G and $F\colon U \to U$ is a continuous, compact map. Assume that there exists a compact set $M \subset U$ such that $F(M) \subset M$ and such that for any $x \in U$ there exists an open neighborhood U_x of x and an integer n_x such that $F^{n_x}(y) \in M$ for $y \in U_x$. Finally, suppose that there exists a compact set $K \supset M$ such that K is contractible in itself to a point. Then we have $i_G(F, U) = 1$ and F has a fixed point in U.

PROOF. – By a simple compactness argument there exists an open neighborhood V_0 of K in G and an integer N such that $\mathrm{cl}\, F(V_0) \subset U$ and $F^n(V_0) \subset U$ for $n \geqslant N$. If we define $C_1 = \mathrm{cl}\big(F(V)\big)$ and $C_n = F^{n-1}(C_1)$ for $n \geqslant 2$, then C_n is a compact subset of U and $C_n \subset M$ for $n \geqslant N$. There exists an open neighborhood V_{N-1} of C_{N-1} such that $F(V_{N-1}) \subset V_0$, and continuing inductively there exists an open neighborhood V_j of C_j for $1 \leqslant j \leqslant N-1$ such that $F(V_j) \subset V_{j+1}$, where V_N is defined to be V_0. If we define $V = \bigcup_{j=0}^{N-1} V_j$, it is clear that V is an open neighborhood of K, $F(V) \subset V$, and $F^n(V) \subset M$ for $n \geqslant 2N$. It follows from Lemma 1.5 that $i_G(F, V) = 1$, and since all of the fixed points of F in U lie in M, $i_G(F, V) = i_G(F, U)$. Q.E.D.

THEOREM 4.1. – If $\alpha > \pi/2$, equation (4.1) has a nonzero periodic solution $x(t)$ of period greater than 2.

PROOF. – It suffices to show that $F\colon U \to U$ has a fixed point, and to show this it suffices to reduce to the situation of Lemma 4.7. We take F and U as above and define $G = \{\varphi \in Y\colon \varphi(-2) = 0\}$. If M_c, $0 < c \leqslant c_0$, is as defined before, we have seen that given $\varphi \in U$, there exists an open neighborhood U_φ of φ in G and a positive integer $n = n(\varphi)$ such that $F^n(\psi) \in M_c$ for $\psi \in U_\varphi$. Since M_c is compact it has a finite covering by open sets $U_j = U_{\varphi_j}$ with corresponding integers n_j. We define $N = \max\{n_j\}$ and consider $M = \bigcup_{j=0}^{N} F^j(M_c)$, a compact subset of U. We claim that $F(M) \subset M$. To see this it suffices to show that if $\psi = F^N(\varphi)$ for some $\varphi \in M_c$, then $F(\psi) \in M$. However, by construction we have $\psi \in U_j$ for some j, so $F^{n_j}(\varphi) \in M_c$ and $F^{N+1}(\varphi) = F^{N+1-n_j}(F^{n_j}(\varphi)) \in M$.

By Lemma 4.7 it only remains to construct a compact set $K \supset M$, $K \subset U$, such that K is contractible in itself to a point. Since M is a compact set, it follows that $\inf\{\sup_{-2\leqslant t\leqslant 0} \psi(t) : \psi \in M\} = \delta_1$ is positive, and this easily implies (using Lemma 4.6) that $\inf\{\psi'(-2) : \psi \in M\} = \delta_2$ is positive. It follows that if $\delta = \min(\delta_1, \delta_2)$, $M \subset M_\delta$. We define $M_\delta = K$, and it remains to show M_δ is contractible. Let ψ_0 be a C^1 function such that $\psi_0'(-2) \geqslant \delta$, ψ_0 is monotonic increasing on $[-2, -1]$ and $\psi_0(t) = \delta$ for $-1 \leqslant t \leqslant 0$. It is known that such a ψ_0 exists, and by decreasing δ one can also guarantee that ψ_0 satisfies condition (3) in the definition of M_δ. It is clear that if $\psi \in M_\delta$ and $0 \leqslant \mu \leqslant 1$, then $(1-\mu)\psi_0 + \mu\psi \in M_\delta$, so M_δ is contractible to the point ψ_0. Q.E.D.

It is clear that the same arguments also apply to the equation $x'(t) = -\alpha x(t-1-\varepsilon|x(t)|)(1-x^2(t))$, $\alpha > \pi/2$ and $0 \leqslant \varepsilon \leqslant 1$, and imply existence of periodic solutions. If $\varepsilon = 0$, it is known that the period of the periodic solution is 4 for every $\alpha > \pi/2$. For $\varepsilon > 0$, numerical studies suggest that the period is not constant with α. However, just as for $\varepsilon = 0$, as α increases the periodic solution looks more and more like a step-function alternating between values $+1$ and -1.

For $\varepsilon > 1$, results can be obtained, but technical difficulties are increased.

5. – In this section we wish to consider the following neutral FDE:

$$x'(t) = \left[-\alpha x(t-1) + \frac{k}{m+1}\frac{d}{dt}\left(x(t-1)\right)^{m+1}\right][1 - x^2(t)] \tag{5.1}$$
$$x(t) = \varphi(t) \quad \text{for} \quad -1 \leqslant t \leqslant 0 .$$

If $k = 0$ this reduces to one of the best understood nonlinear FDE's. Work of JONES [18] shows that for each $\alpha > \pi/2$ there exists a periodic solution of period 4, and a remark of A. J. MACINTYRE (see [18]) actually gives an explicit formula in terms of the elliptic function $\mathrm{sn}(u)$. However, if $k \neq 0$, previous methods give no results on (5.1). We shall prove below that if $\alpha > \pi/2$, $m \geqslant 1$, and

$$|k| \leqslant \left(\frac{m+1}{4}\right)\left(1 + \frac{2}{m-1}\right)^{(m-1)/2},$$

then equation (5.1) has a nonzero periodic solution. Even in this simple case this result is far from best possible and leaves a large number of open questions, which we shall discuss later.

LEMMA 5.1. – Let φ be a continuous, real-valued function on $[-1, 0]$ such that $\varphi(-1) = 0$, $\varphi^{m+1}(t)$ is continuously differentiable, $(d/dt)\varphi^{m+1}(0) = 0 = (d/dt)\varphi^{m+1}(-1)$ and $|\varphi(0)| < 1$. Then equation (5.1) has a unique solution $x(t) = x(t; \varphi)$ defined for all $t \geqslant 0$ such that $x(t)$ is continuously differentiable for $t \geqslant 0$ and $|x(t)| < 1$ for all $t \geqslant 0$.

PROOF. For $0 \leqslant t \leqslant 1$ define $x(t)$ by the equation

$$(5.2)\qquad \frac{1}{2}\log\left(\frac{1+x(t)}{1-x(t)}\right) - \frac{1}{2}\log\left(\frac{1+\varphi(0)}{1-\varphi(0)}\right) = -\alpha\int_{-1}^{t-1}\varphi(s)\,ds + \frac{k}{m+1}\varphi^{m+1}(t-1)\,.$$

This equation is obtained from (5.1) by dividing by $1 - x^2(t)$ and integrating. One can solve equation (5.2) for $x(t)$ and check directly that $|x(t)| < 1$, $x(t)$ is C^1 in $[0, 1]$ and $x'_+(0)$ (the right hand derivative) is zero. Repeating this procedure on $[1, 2]$ we extend $x(t)$ in such a way that $|x(t)| < 1$ on $[1, 2]$, $x|[1, 2]$ is continuously differentiable, and $x(t)$ satisfies (5.1) on $[1, 2]$. Using the condition that $x'_+(0) = 0 = (d/dt)\varphi^{m+1}(0)$, it is easy to check that $x'_+(1) = x'_-(1)$, so $x'(t)$ is continuous also at 1. Continuing in this way, we obtain the result of the lemma Q.E.D.

LEMMA 5.2. – Assume that m is a positive integer $\alpha > \pi/2$, and φ is a continuous monotonic increasing function such that $\varphi(-1) = 0$, $\varphi^m(t)$ is continuously differentiable and $(d/dt)\varphi^m(0) = 0$. Then if $\varphi(0) > 0$, there exists a positive constant a (independent of φ) such that either $\sup\{|(d/dt)x^m(t)|: t \geqslant -1\} \geqslant (\alpha - 1)m/|k|$ or $\limsup_{t\to\infty} \max(|x(t)|, |x'(t)|) \geqslant a$.

PROOF. – The assumptions imply the $\varphi^{m+1} = (\varphi^m)^{m+1/m}$ satisfy the hypotheses of Lemma 5.1, so $x(t)$ is defined. If we have $\sup|(d/dt)x^m(t)| \geqslant (\alpha - 1)m/|k|$, we are done, so we assume for the rest of the proof that $\sup_{t\geqslant -1}|(d/dt)x^m(t)| < (\alpha - 1)m/|k|$. If $t_0 = \sup_{t\geqslant -1}\{t \geqslant 0: \varphi(s) = 0 \text{ on } [0, t-1]\}$, then it is clear that $x'(t) = 0$ for $0 \leqslant t \leqslant t_0$; and for $t > t_0$, we have $x'(t) = x(t-1)[-\alpha + (k/m)(d/dt)x^m(t-1)][1 - x^2(t)]$. Our assumptions imply that $-\alpha + (k/m)(d/dt)x^m(t-1) \leqslant -\beta < -1$ for $t \geqslant 0$. Just as in Section 2 it follows that $x(t)$ has infinitely many isolated zeros z_n, $n \geqslant 1$, that $z_{n+1} - z_n > 1$, $|x(t)| > 0$ for $t \neq z_n$, $x(t)$ is monotonic increasing on $[z_{2n-1} + 1, z_{2n} + 1]$ and $x(t)$ is monotonic decreasing on $[z_{2n} + 1, z_{2n+1} + 1]$.

Let $z > 0$ be a zero of x and define $\delta = \sup_{t\geqslant z+1}|x(t)|$ and $\delta_1 = \sup_{t\geqslant z}|x'(t)|$. Our assumptions imply that $\sup_{t\geqslant z+1}|x'(t)| \leqslant A\delta$, where $A = 2\alpha - 1$, so $\delta_1 < \infty$. If we define $u \geqslant z$ to be a zero of x such that $|x(u+1)| \geqslant \frac{1}{2}\delta$, define $T = u + 1$ and λ to be as in Lemma 2.5, then integration by parts gives

$$\int_T^{\infty} x'(t) \exp[-\lambda t]\,dt = -x(T)\exp[-\lambda T] + \lambda \int_T^{\infty} x(t)\exp[-\lambda t]\,dt. \tag{5.3}$$

If we substitute from equation (5.1) for $x'(t)$ we obtain

$$\int_T^{\infty} x'(t)\exp[-\lambda t]\,dt = -\alpha \exp[-\lambda]\int_{T-1}^{\infty} x(t)\exp[-\lambda t]\,dt + \\ + k\int_T^{\infty} x^m(t-1)x'(t-1)\big(1-x^2(t)\big)\exp[-\lambda t]\,dt - \alpha\int_T^{\infty} x(t-1)x^2(t)\exp[-\lambda t]\,dt. \tag{5.4}$$

If we set equation (5.3) equal to (5.4) and simplify (recalling that

$$\alpha \exp[-\lambda]\int_{T-1}^{T} x(t)\cdot\exp[-\lambda t]\,dt = x(T)\exp[-\lambda T] - \int_{T-1}^{T} x'(t)\exp[-\lambda t]\,dt\Big),$$

we find that if we define

$$R(T) = k\int_T^{\infty} x^m(t-1)x'(t-1)\big(1-x^2(t)\big)\exp[-\lambda(t-T+\tfrac{1}{2})]\,dt - \\ -\alpha\int_T^{\infty} x(t-1)x^2(t)\exp[-\lambda(t-T+\tfrac{1}{2})], \qquad -\int_T^{\infty} x'(t)\exp[-\lambda(t-T+\tfrac{1}{2})]\,dt = R(T). \tag{5.5}$$

The usual estimates imply that if $\lambda = \mu + i\nu$, $|R(T)| \leqslant \big(|k|\delta^m\delta_1 + \alpha\delta^3\big)\exp[\tfrac{1}{2}\mu]\mu^{-1}$. As before, we find the absolute value of the real part of the left hand side of (5.5) is greater than $\tfrac{1}{2}\delta\exp[-\tfrac{1}{2}\mu]\cos t\,\nu/2$. Dividing both expressions by δ we obtain

$$\tfrac{1}{2}\exp[-\mu]\mu\cos\nu/2 \leqslant |k|\delta^{m-1}\delta_1 + \alpha\delta^2. \tag{5.6}$$

Equation (5.6) implies that there exists a positive constant a (independent of φ) such that $\max(\delta, \delta_1) \geqslant a$. Q.E.D.

Observe that to prove $\limsup_{t\to\infty} \max\big(|x(t)|, |x'(t)|\big) \geqslant a$ it sufficed to know that x had the qualitative behaviour (existence of zeros, etc.) used in the second half of the proof.

Our next lemma is a trivial calculus exercise which we leave to the reader.

LEMMA 5.3. – If $\beta \geqslant 0$ and $c_\beta = \sup_{0\leqslant u\leqslant 1} u^\beta(1-u^2)$, then $c_0 = 1$ and

$$c_\beta = \left(\frac{\beta}{\beta+2}\right)^{\beta/2}\left(\frac{2}{\beta+2}\right) \quad \text{for} \quad \beta > 0.$$

LEMMA 5.4 – Assume that φ is as in Lemma 5.1 and

$$\left|\frac{k}{m+1}\frac{d}{dt}\varphi^{m+1}(t)\right| \leqslant B \quad \text{for} \quad -1 \leqslant t \leqslant 0 .$$

Then if

$$|k|c_m\left(\frac{\alpha+B}{B}\right) \leqslant 1 , |k|\,|x^m(t)x'(t)| \leqslant B \quad \text{for all} \quad t \geqslant 0 .$$

If $\varphi^m(t)$ is continuously differentiable and

$$k/m\,\frac{d}{dt}\varphi^m(t) \leqslant B \quad \text{for} \quad -1 \leqslant t \leqslant 0 ,$$

then if

$$|k|c_{m-1}\left(\frac{\alpha+B}{B}\right) \leqslant 1 , |k|\,|x^{m-1}(t)x'(t)| \leqslant B \quad \text{for all} \quad t \geqslant 0 .$$

If in this second case, $\alpha > 1$, φ is monotonic increasing and $\varphi(0) > 0$, $B = \alpha$, and $2|k|c_{m-1} \leqslant 1$, then $x(t)$ has a first zero z_1, and x is monotonic decreasing on $[0, z_1 + 1]$.

PROOF. – First suppose that $|k/(m+1)(d/dt)\varphi^{m+1}(t)| \leqslant B$. Then for $0 \leqslant t \leqslant 1$ we have

$$|k|\,|x^m(t)x'(t)| \leqslant |k|\,|x(t)|^m\big(1-x^2(t)\big)\big(\alpha|x(t-1)| + B\big) . \tag{5.6}$$

Since $0 \leqslant |x(t)| < 1$, $|x(t)|^m\big(1-x^2(t)\big) \leqslant c_m$, and we find that $|k|\,|x^m(t)x'(t)| \leqslant B$ for $0 \leqslant t \leqslant 1$. Generally, if we assume that $|k|\,|x^m(t)x'(t)| \leqslant B$ for $j \leqslant t \leqslant j+1$, the same argument proves $|k|\,|x^m(t)x'(t)| \leqslant B$ for $j+1 \leqslant t \leqslant j+2$.

Now assume $\varphi^m(t)$ is continuously differentiable and

$$\left|\frac{k}{m}\frac{d}{dt}\varphi^m(t)\right| \leqslant B \quad \text{for} \quad -1 \leqslant t \leqslant 0 .$$

Then for $0 \leqslant t \leqslant 1$ the equation (5.1) reduces to

$$x'(t) = \varphi(t-1)\left[-\alpha + \frac{k}{m}\frac{d}{dt}\varphi^m(t-1)\right][1 - x^2(t)] .$$

Multiplying both sides by $kx^{m-1}(t)$ and taking absolute values gives $|kx^{m-1}(t)x'(t)| < c_{m-1}|k|(\alpha+B) \leqslant B$. Generally, if $|kx^{m-1}(t)x'(t)| \leqslant B$ for $j \leqslant t \leqslant j+1$, the same argument shows that $|kx^{m-1}(t)x'(t)| < B$ for $j+1 \leqslant t \leqslant j+2$.

Finally, suppose that in addition $\alpha > 1$, $B = \alpha$, φ is monotonic increasing and $\varphi(0) > 0$ and $2|k|c_{m-1} \leqslant 1$. Then by the remarks above $|kx^{m-1}(t)x'(t)| < \alpha$ for

$0 \leqslant t < \infty$: Since equation (5.1) can be written as

$$x'(t) = \varphi(t-1)\left[-\alpha + \frac{k}{m}\frac{d}{dt}\varphi^m(t-1)\right][1-x^2(t)] \quad \text{for} \quad 0 \leqslant t \leqslant 1$$

and

$$x'(t) = x(t-1)[-\alpha + kx^{m-1}(t-1)x'(t-1)][1-x^2(t)] \quad \text{for} \quad t \geqslant 1\,,$$

it follows that $x(t)$ is monotonic decreasing on $[0, z_1+1]$, where z_1 denotes the first zero of x (possibly $z_1 = +\infty$).

We wish to show $z_1 < +\infty$, and to do this we proceed by contradiction. If $x(t) > 0$ for all t, let $b = \lim_{t\to\infty} x(t)$ (since $x(t)$ is monotonic decreasing) and let $d = x(0) < 1$. Then the argument used above actually shows that $|kx^{m-1}(t)x'(t)| \leqslant d\alpha$ for $0 \leqslant t \leqslant 1$ and generally $|kx^{m-1}(t)x'(t)| \leqslant d^j\alpha$ for $j-1 \leqslant t \leqslant j$. It follows that $\lim_{t\to\infty} kx^{m-1}(t)x'(t) = 0$, and therefore we have $\lim_{t\to\infty} x'(t) = -\alpha b(1-b^2)$. This implies that $b = 0$, but if we select t_0 so large that $[-\alpha + kx^{m-1}(t-1)x'(t-1)][1-x^2(t)] < -1$ for $t \geqslant t_0$, the usual argument implies that x has a zero on $[t_0, t_0+1)$. Q.E.D.

If $z_1 \geqslant 1$, then it is clear from the above remarks that $x'(z_1) < 0$ and z_1 is isolated. However, if $z_1 < 1$ and if $(d/dt)\varphi^m(z_1) = \alpha m/k$, it is possible that z_1 is not isolated. In this case, it is not hard to see that there exists z_1^*, $z_1^* < 1$, such that $x(t) = 0$ for $z_1 \leqslant t \leqslant z_1^*$, $x(t) < 0$ for $z_1^* < t \leqslant z_1^* + 1$, and x is monotonic decreasing on $[z_1^*, z_1^* + 1]$. In any event, the same arguments show that x has a first zero $z_2 > z_1^* + 1$. Since $|kx^{m-1}(t-1)x'(t)| < \alpha$ for $t \geqslant 1$, it is easy to see that $x'(z_2) > 0$ and z_2 is isolated. Now applying Lemma 5.4 to $x|[z_2, z_2+1]$, x has a first zero $z_3 > z_2 + 1$, and $z'(z_3) < 0$. Continuing in this way we find zeros z_j, $j \geqslant 2$, which are isolated for $j \geqslant 2$, and x is monotonic decreasing on $[0, z_1^* + 1]$, monotonic increasing on $[z_1^* + 1, z_2 + 1]$, etc. If $\alpha > \pi/2$ and $|k| \leqslant (2c_{m-1})^{-1}$, it follows by Lemma 5.2 and the remark following it that $\limsup_{t\to\infty} |x(t)| \geqslant a$ or $\limsup_{t\to\infty} |x'(t)| \geqslant a$ (a as in Lemma 5.2). However under the assumptions on x, $|kx^{m-1}(t-1)x'(t-1)| < \alpha$ for all $t > 1$, so by equation (5.1) we must have $\limsup_{t\to\infty} |x'(t)| \leqslant 2\alpha \limsup_{t\to\infty} |x(t)|$ and $\limsup_{t\to\infty} |x(t)| \geqslant a/2\alpha$.

We now proceed as usual. Let S denote the set of continuous monotonic increasing functions φ on $[-1, 0]$ such that φ^m is continuously differentiable,

$$\sup_{-1\leqslant t\leqslant 0}\left|\frac{k}{m}\frac{d}{dt}\varphi^m(t)\right| \leqslant \alpha\,, \quad \frac{d}{dt}\varphi^m(0) = 0\,, \quad \varphi(-1) = 0 \quad \text{and} \quad 0 < \varphi(0) < 1\,.$$

If $\varphi \in S$, $\alpha > 1$ and $|k| \leqslant (2c_{m-1})^{-1}$, let $x(t)$ denote the corresponding solution of (5.1) and define $F(\varphi) \in S$ by $F(\varphi)(s) = x(z_2 + 1 + s)$, $-1 \leqslant s \leqslant 0$. It is not hard to show that F is continuous on S in the sense that given $\varphi_0 \in S$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $\varphi \in S$ and

$$\sup_{-1\leqslant t\leqslant 0}\left|\frac{d}{dt}\varphi^m(t) - \frac{d}{dt}\varphi_0^m(t)\right| < \delta\,,$$

then $\sup_{-1\leqslant t\leqslant 0} |y'(t)-y_0'(t)|<\varepsilon$, where $y=F(\varphi)$ and $y_0=F(\varphi_0)$. It is clear that fixed points of F correspond to nonzero periodic solutions of (5.1), but for technical reasons F is not quite the right map to consider and S is not quite the right set of initial functions. Our next lemma suggests another condition for functions in S.

LEMMA 5.5. – Assume that $\varphi\in S$, $\alpha>1$ and $|k|\leqslant(2c_{m-1})^{-1}$. Then if $x(t)$ is the corresponding solution of (5.1), $|x(t)|\leqslant A$ for $t\geqslant z_1$, where $A<1$ and A is independent of φ.

PROOF. – According to Lemma 5.4 and the remarks following it, $\sup_{t\geqslant z_1}|x(t)|= =\sup\{|x(z_j+1)|: j\geqslant 1\}$. However, if $x(z)=0$, we then obtain

$$(5.7)\qquad \frac{1}{2}\left|\log\left(\frac{1+x(z+1)}{1-x(z+1)}\right)\right| = \left|\int_z^{z+1} -\alpha x(t-1)+\frac{k}{m+1}\frac{d}{dt}x^{m+1}(t-1)\,dt\right| = \left|-\alpha\int_z^{z+1} x(t-1)\,dt+\frac{k}{m+1}x^{m+1}(z-1)\right| \leqslant \alpha+\left|\frac{k}{m+1}\right|.$$

It follows that $|x(z+1)|\leqslant A$, where $A<1$ is the largest solution of

$$\frac{1}{2}\log\left(\frac{1+A}{1-A}\right)=\alpha+\left|\frac{k}{m+1}\right|. \qquad \text{Q.E.D.}$$

We are now almost done. Let G denote the set of continuously differentiable functions ψ on $[-1,0]$ such that $\psi(-1)=0$, $\psi(0)\leqslant A^m$, $\psi'(t)\geqslant 0$ for $-1\leqslant t\leqslant 0$, $\psi'(0)=0$ and $|(k/m)(d/dt)\psi(t)|\leqslant\alpha$ for $-1\leqslant t\leqslant 0$. If we view G as a subset of $C^1([-1,0])$, then it is clear that G is closed, bounded and convex. Now assume that $\alpha>\pi/2$ and $|k|\leqslant(2c_{m-1})^{-1}=\big((m+1)/4\big)\big(1+2/(m-1)\big)^{(m-1)/2}$ and define $\Phi(\psi)= =\big(F(\psi^{1/m})\big)^m$ for $\psi\in G-\{0\}$. It is not hard to see (using Lemmas 5.4 and 5.5) that $\Phi\colon G-\{0\}\to G-\{0\}$ and Φ is continuous. Furthermore, by Lemma 5.2 and the remarks following Lemma 5.4, if $\psi\in G-\{0\}$ and $x(t)$ is the solution of (5.1) corresponding to $\psi^{1/m}$, then $\limsup_{t\to\infty}|x(z_n+1)|\geqslant a/2\alpha$. Using this result, it is easy to see that 0 is an ejective point of Φ.

If for $A\subset G$ we define $\mu(A)=\limsup_{\delta\to\infty}\{|\psi'(t)-\psi'(s)|:\psi\in A,\ |t-s|\leqslant\delta\}$, to find a fixed point of Φ, it suffices by the remarks in Section 1 to show that $\mu\big(\Phi(A)\big)\leqslant \leqslant c\mu(A)$ for some constant $c<1$, c independent of A.

LEMMA 5.6. – If $\alpha>\pi/2$ and $|k|\leqslant(2c_{m-1})^{-1}$, there exists a constant $c<1$ such that $\mu\big(\Phi(A)\big)\leqslant c\mu(A)$ for every subset A of G.

PROOF. – Suppose that $A \subset G - \{0\}$ and $\mu(A) < d$. Given $\psi \in G$, let $x(t)$ denote the solution of (5.1) corresponding to $\psi^{1/m}(s)$. According to Lemma 5.4, $|kx^{m-1}(t)x'(t)| \leqslant \alpha$ for all $t \geqslant 0$. It follows that the set of functions $x^m(t)$ $0 \leqslant t < \infty$, corresponding to $\psi^{1/m}$, $\psi \in G$, is equicontinuous; and this easily implies that the set of functions $x(t)$, $0 \leqslant t < \infty$, corresponding to $\psi^{1/m}$, $\psi \in G$, is equicontinuous. Of course $\{\psi^{1/m} : \psi \in G\}$ is equicontinuous.

By applying the above remarks, it is easy to see that given $\varepsilon > 0$ there exists a $\delta > 0$, δ independent of $\psi \in G$, such that $|x(t-1)x^{m-1}(t)(1-x^2(t)) - x(s-1)x^{m-1}(s) \cdot (1-x^2(s))| < \varepsilon$ if $t, s \geqslant 0$ and $|t-s| < \delta$. Since $\mu(A) < d$, by taking δ smaller we can also assume that $\sup\{|\psi'(t) - \psi'(s)| : \psi \in A,\ |t-s| < \delta\} < d$. If $x(t)$ is a solution of (5.1) corresponding to ψ with $\psi \in A$, then for $0 \leqslant t$, $s \leqslant 1$ and $|t-s| < \delta$ we obtain the following equations:

$$(5.8) \qquad \left|\frac{d}{dt}x^m(t) - \frac{d}{dt}x^m(s)\right| \leqslant \alpha m |x(t-1)\,x^{m-1}(t)\,(1-x^2(t)) - x(s-1)\,x^{m-1}(s)\,(1-x^2(s))| + |k|\,|x(t-1)\psi'(t-1)x^{m-1}(t)(1-x^2(t)) - x(s-1)\psi'(s-1)x^{m-1}(s)(1-x^2(s))|\,.$$

The first term on the right is bounded by $\alpha m\varepsilon$, by assumption. It is easy to check that the second term on the right in equation (5.8) is less than the following expression:

$$(5.9) \qquad |k|\,|\psi'(t-1) - \psi'(s-1)|\,|x(t-1)|(1-x^2(t)) + |k|\,|\psi'(s-1)|\,|x(t-1)x^{m-1}(t)(1-x^2(t)) - x(s-1)x^{m-1}(s)(1-x^2(s))|\,.$$

Since $|k|\,|x^{m-1}(t)(1-x^2(t))| \leqslant \frac{1}{2}$ and $|\psi'(s-1)| \leqslant \alpha m/k$, the expression (5.9) is dominated by $\frac{1}{2}d + \alpha m\varepsilon$. Therefore, for δ so small that $4\alpha m\varepsilon < \frac{1}{4}d$, we have

$$\left|\frac{d}{dt}x^m(t) - \frac{d}{dt}x^m(s)\right| \leqslant \frac{3}{4}d < d \quad \text{if} \quad 0 \leqslant s,\ t \leqslant 1 \text{ and } |t-s| < \delta\,.$$

Since $x(t)$ is monotonic decreasing on $[0, 1]$ and monotonic increasing on $[-1, 0]$ it is easy to see that

$$\left|\frac{d}{dt}x^m(t) - \frac{d}{dt}x^m(s)\right| < d \quad \text{if} \quad -1 \leqslant t,\ s \leqslant 1 \text{ and } |t-s| < \delta\,.$$

Now assume that $\psi \in A$, $x(t)$ is the solution of (5.1) corresponding to $\psi^{1/m}$ and $z = z_2$ is the second zero of x, $z > 1$. If for $r \geqslant -1$ we define

$$C_r = \sup\left\{\left|\frac{d}{dt}x^m(t) - \frac{d}{dt}x^m(s)\right| : r \leqslant t,\ s \leqslant r+1,\ |t-s| < \delta\right\},$$

then exactly the same proof used before shows

$$C_z \leqslant \tfrac{1}{2} C_{z-1} + 2\alpha m \varepsilon . \tag{5.10}$$

Iterating this estimate we find that

$$C_z \leqslant \frac{1}{2^j} C_{z-j} + 2\alpha m \varepsilon \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{j-1}}\right). \tag{5.11}$$

If we select j such that $-1 \leqslant z - j \leqslant 0$, we already have proved that $C_{z-j} < d$, so we have $C_z \leqslant \frac{1}{2} d + 4\alpha m \varepsilon \leqslant \frac{3}{4} d$.

The estimate above shows that $\mu(\Phi(A)) \leqslant \frac{3}{4} d$, and since d was any number greater than $\mu(A)$, $\mu(\Phi(A)) \leqslant \frac{3}{4} \mu(A)$. Q.E.D.

Lemma 5.6 implies (together with our previous remarks) that Φ has a fixed point. This in turn gives a fixed point of F and a nontrivial, C^1 solution of (5.1). Thus we have proved the following theorem:

THEOREM 5.1. – If $\alpha > \pi/2$, $m \geqslant 1$, and $|k| \leqslant ((m+1)/4)(1 + 2/(m-1))^{(m-1)/2}$ ($|k| \leqslant \frac{1}{2}$ if $m = 1$) then equation (5.1) has a nontrivial continuously differentiable solution x such that $x(-1) = 0$, x is monotonic increasing on $[-1, 0]$ and $0 < x(0) < 1$. Furthermore, if Φ and G are as above and U is an open neighborhood of the origin such that $\Phi(x) \neq x$ for $x \in \overline{U} - \{0\}$, $i_G(\Phi, G - \overline{U}) = 1$.

Theorem 5.1 is not best possible. Computer numerical studies suggest that nontrivial periodic solutions occur for a larger range of k. However, these periodic solutions may not be as nice as those guaranteed by Theorem 5.1. For example, they may not be monotonic increasing from their minima to their maxima and vice versa, or their maxima may occur before $z + 1$, z a zero of the periodic solution. Furthermore, numerical studies for the case $m = 1$ suggest that if k becomes too large, nondamped, oscillatory, nonperiodic behaviour occurs. If m is even, however, it appears that « nice » periodic behaviour occurs for a much larger range of k than that given by Theorem 5.1. If this is true, it is completely mysterious from the standpoint of our techniques.

Acknowledgements. – I would like to thank my colleague, DON ORTH, who very kindly ran computer numerical simulations of a number of the equations discussed here. These simulations proved invaluable in gaining insight into the equations and in fact encouraged substantial improvements in early versions of Sections 3-5.

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