Multifunctions on Abstract Measurable Spaces and Application to Stochastic Decision Theory (*).

C. J. HIMMELBERG - F. S. VAN VLECK (Lawrence, Kansas) (**)

Summary. – *The main results are some very general theorems about measurable multifunctions on abstract measurable spaces with compact values in a separable metric space. It is shown that measurability is equivalent to the existence of a pointwise dense countable family of measurable selectors, and that the intersection of two compact-valued measurable multifunctions is measurable. These results are used to obtain a Filippov type implicit function theorem, and a general theorem concerning the measurability of* $y(t) = \min f(\{t\} \times \Gamma(t))$ *when* f *is a real valued function and* Γ *a compact valued multifunction. An application to stochastic decision theory is given generalizing a result of Benes.*

Introduction.

There are numerous situations (for example, in game theory, mathematical economics, and decision theory) which give rise to multifunctions defined on an abstract measurable space and having compact values in a separable metric space. The purpose of this note is to prove for such multifunctions some general theorems which arise frequently in applications. We consider when the intersection of two multifunctions is measurable, and as a consequence obtain a Filippov type implicit function theorem. We also show the measurability of certain functions and multifunctions which arise naturally in optimization problems. In particular, we consider the measurability of $f(t, x)$ and of $\min f(\{t\} \times \Gamma(t))$ given that f is measurable in t and continuous in x, and that Γ is measurable with compact values. Our results and methods are natural extensions of those of CASTAING [C] and ROCKAFELLAR [R]. They differ from Castaing's in that the multifunctions involved are defined on an abstract measurable space instead of on a locally compact space with Radon measure, and they differ from Rockafellar's in that values are allowed to be taken in a separable metric space instead of in a Euclidean space. The paper concludes with an application to the stochastic decision problem considered by BENES [B].

We generalize his results by requiring that admissible strategies be selectors, except on a set of probability 0, for a given constraint multifunction.

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1. – Measurable multifunctions.

A multifunction $F\colon T\to X$ is a function whose value for each t in T is a non-empty subset of X. Equivalently, F is a relation in $T\times X$ whose domain is T. Throughout this paper we assume that T is a measurable space with σ-algebra $\mathcal{A}$, and we will refer to members of $\mathcal{A}$ as measurable subsets of T. If X is a topological space then $F\colon T\to X$ is measurable iff $F^{-1}(B)=\{t|F(t)\cap B\neq\emptyset\}$ is measurable for each closed subset B of X. If U is a set of functions from T to X, then $U(t)$ denotes the set $\{u(t)|u\in U\}$. If $u\colon T\to X$ is a function such that $u(t)\in F(t)$ for all t, then u is called a selctor for F.

THEOREM 1. – Let X be a separable metric space and $F\colon T\to X$ be a multifunction with compact values. Then F is measurable iff there exists a countable set U of measurable selectors for F such that $F(t)=\overline{U(t)}$ for all $t\in T$.

PROOF. – This theorem was proved by Castaing [C, Theorem 5.3] in case T is a locally compact space and $\mathcal{A}$ is the σ-algebra of μ-measurable sets for some Radon measure μ on T. His proof of the « only if » part of the theorem does not use the topological structure of T, and in fact works in the present circumstances. On the other hand, let U be a set of measurable selectors such that $F(t)=\overline{U(t)}$ for each t, and let B be a closed subset of X. Define $B_n=\{x\in X|d(x,B)<1/n\}$. Then, by the compactness of $F(t)$, $F^{-1}(B)=\{t|F(t)\cap\bar{B}_n\neq\emptyset \text{ for all } n\}$. Since $\bar{B}_n\subset B_m$, when $n>m$, the right hand side of this equation is $\{t|F(t)\cap B_n\neq\emptyset \text{ for all } n\}$. But $F(t)\cap B_n\neq\emptyset$ iff $t\in u^{-1}(B_n)$ for some $u\in U$. Thus, $F^{-1}(B)=\bigcap_{n=1}^{\infty}\bigcup_{u\in U}u^{-1}(B_n)$, and $F^{-1}(B)$ is measurable since U is countable.

Recall from [HV] that $F\colon T\to X$ is $(\mathcal{A},\mathrm{C})$-measurable iff $F^{-1}(B)$ is measurable whenever B is compact. Using this concept of measurability, Theorem 1 becomes

THEOREM 1′. – i) Let X be separable metric and $F\colon T\to X$ be a multifunction with complete values. Then F is $(\mathcal{A},\mathrm{C})$-measurable iff there exists a countable set U of measurable selectors for F such that $F(t)=\overline{U(t)}$ for all $t\in T$. (For « if », the values of F need only be closed instead of complete.)

ii) If X is separable metric and σ-compact (i.e., $X=\bigcup_n X_n$ where each X_n is compact), and if $F\colon T\to X$ has closed values, then F is measurable (in the usual sense) iff there exists a countable set U of measurable selectors for F such that $F(t)=\overline{U(t)}$ for all $t\in T$.

PROOF. – (i) Without loss of generality assume X is complete, replacing X by its completion if necessary. The « only if » part follows from the proof of the « only if » part of [C, Theorem 5.4]. To prove « if », it is sufficient, in the proof of the « if » part of Theorem 1, to prove that, if B is compact, then $t\in F^{-1}(B)$ iff $F(t)\cap\bar{B}_n\neq\emptyset$ for all n. Clearly $F(t)\cap\bar{B}_n\neq\emptyset$ if $t\in F^{-1}(B)$. So suppose $F(t)\cap\bar{B}_n\neq\emptyset$ for all n. For each n, choose $x_n\in F(t)\cap\bar{B}_n$ and $y_n\in B$ such that $d(x_n,y_n)<1/n$. If B is compact, there exists a convergent subsequence (y_{n_k}), of (y_n), say $y_{n_k} \to y$. Then also $x_{n_k} \to y$. So $y \in B \cap F(t)$, and hence $t \in F^{-1}(B)$. (Note that, for this « if » argument, it is sufficient for the values of F to be closed and not necessarily complete).

(ii) Let $X = \bigcup_n X_n$, with each X_n compact. Assume F is measurable. For each n, let T_n be the measurable set $F^{-1}(X_n)$ and define $F_n: T_n \to X_n$ by $F_n(t) = F(t) \cap X_n$. Then $F(t) = \bigcup_n F_n(t)$ for each $t \in T$, and each F_n satisfies the conditions of Theorem 1 (with measurability of F_n defined in terms of the restriction of $\mathcal{A}$ to T_n). Hence, for each n, there exists a countable family V_n of measurable selectors for F_n such that $F_n(t) = \overline{V_n(t)}$ for all $t \in T_n$. Let $u_0: T \to X$ be any fixed measurable selector for F (u_0 can easily be constructed by piecing together parts of measurable selectors for the F_n's), and let U_n be the family of selectors for F obtained by extending each member of V_n to agree with u_0 on $T \sim T_n$. Clearly each member of U_n is measurable. Finally let $U = \bigcup_n U_n$. Then U is a countable family of measurable selectors for F and $F(t) \supset \overline{U(t)} \supset \bigcup_n \overline{U_n(t)} \supset \bigcup_n F_n(t) = F(t)$. On the other hand assume U is a countable set of measurable selectors for F such that $F(t) = \overline{U(t)}$ for each t. Since, in the proof of the « if » part of (i), it is sufficient for the values of F to be closed, it follows that F is $(\mathcal{A}, \mathcal{C})$-measurable. So let B be closed in X. Then $F^{-1}(B) = F^{-1}\big(\bigcup_n (B \cap X_n)\big) = \bigcup_n F^{-1}(B \cap X_n) \in \mathcal{A}$.

THEOREM 2. – i) Let E be a separable metric linear space, and let $F_1, F_2: T \to E$ be compact valued measurable multifunctions. Then the multifunction $F: T \to E$ defined by $F(t) = F_1(t) + F_2(t)$ is measurable.

ii) If, in (i), measurability is replaced by $(\mathcal{A}, \mathcal{C})$-measurability, then only one of $F_1(t)$, $F_2(t)$ need be compact for each t, the other complete.

iii) If, in (i), E is also σ-compact, then only one of $F_1(t)$, $F_2(t)$ need be compact, the other closed, for each t.

PROOF. – We prove (i), (ii), and (iii) simultaneously, using Theorems 1 and 1′, and, in (i), the fact that $F(t) = F_1(t) + F_2(t)$ is compact, in (ii), the fact that $F(t)$ is complete, in (iii), the fact that $F(t)$ is closed.

Let U_1, U_2 be sets of measurable selectors for F_1, F_2, respectively as in Theorems 1 and 1′. Define $U = \{u_1 + u_2 | u_1 \in U_1, u_2 \in U_2\}$. Then for each t, we have $F(t) = F_1(t) + F_2(t) \supset \overline{U_1(t) + U_2(t)} \supset \overline{U_1(t)} + \overline{U_2(t)} = F_1(t) + F_2(t)$. So $F(t) = \overline{U(t)}$ for all t. It follows from Theorems 1 and 1′ that F is measurable ($(\mathcal{A}, \mathcal{C})$-measurable in (ii)).

THEOREM 3. – i) Let X be a separable metric space, and let $F_1, F_2: T \to X$ be compact valued measurable multifunctions such that $F_1(t) \cap F_2(t) \neq \emptyset$ for all $t \in T$. Then $F: T \to X$ defined by $F(t) = F_1(t) \cap F_2(t)$, is measurable.

ii) If measurability is replaced by $(\mathcal{A}, \mathcal{C})$-measurability in (i), then the values of F_1 and F_2 need only be complete.

iii) If, in (i), X is also σ-compact, then the values of F_1 and F_2 need only be complete.

PROOF. – (i) By an embedding theorem of Kuratowski and Wojdyslawski (see [H, page 81]), X can be uniformly embedded in a separable normed linear space E (where, if necessary, we replace the metric on X by a uniformly equivalent bounded one). We regard F_1, F_2 as multifunctions into E. Now let B be a closed subset of E. Define $F_1'(t) = F_1(t) \cap B$, and $T' = F_1'^{-1}(E) = \{t \in T | F_1'(t) \neq \emptyset\} = F_1^{-1}(B)$. Then $T' \in \mathcal{A}$ and F_1' is a measurable multifunction from T' to E. By Theorem 2(i), the multifunction $L: T' \to E$ defined by $L(t) = F_1'(t) - F_2(t)$ is measurable. Hence

$$\begin{aligned} F^{-1}(B) &= \{t | F_1(t) \cap F_2(t) \cap B \neq \emptyset\} \\ &= \{t | F_1'(t) \cap F_2(t) \neq \emptyset\} \\ &= \{t | 0 \in F_1'(t) - F_2(t)\} \\ &= L^{-1}(\{0\}) . \end{aligned}$$

Thus F is measurable.

(ii) Use the same proof as for (i), except now assume B is compact. Then F_1' has compact values and we can apply Theorem 2 (ii).

(iii) By (ii), F is $(\mathcal{A}, \mathcal{C})$-measurable. Suppose $X = \bigcup_n X_n$, where each X_n is compact, and let B be a closed subset of X. Then $F^{-1}(B \cap X_n)$ is measurable, and hence so is $F^{-1}(B) = \bigcup_n F^{-1}(B \cap X_n)$.

We are now able to prove the following implicit function theorem.

THEOREM 4. – (i) Let X be a compact metric space, and Y a separable metric space. Let $\Gamma: T \to X$ be a measurable multifunction with closed values, $f: T \times X \to Y$ a function which is measurable in t and continuous in x, and $g: T \to Y$ a measurable function such that $g(t) \in f(\{t\}) \times \Gamma(t))$ for all $t \in T$. Then there exists a measurable function $\gamma: T \to X$ such that $\gamma(t) \in \Gamma(t)$ and $g(t) = f(t, \gamma(t))$ for all $t \in T$.

(ii) In (i), X need only be a complete separable metric space, if we assume that Y is σ-compact metric, and that measurability is everywhere replaced by $(\mathcal{A}, \mathcal{C})$-measurability.

PROOF. – i) Define $K: T \to X$ by $K(t) = \{x | f(t, x) = g(t)\}$. Each value of K is closed and non-empty. If we show that K is measurable, it will follow from Theorem 3 (i) that $K \cap \Gamma$ is a measurable multifunction. Any measurable selector γ of $K \cap \Gamma$ is then the desired function. (One exists by Theorem 1).

So let us prove K is measurable. Let B be a closed subset of X and let D be a countable dense subset of B. Then

$$\begin{aligned} K^{-1}(B) &= \{t | K(t) \cap B \neq \emptyset\} \\ &= \{t | f(t, x) = g(t) \text{ for some } x \in B\} \\ &= \bigcup_{x \in B} \{t | f(t, x) = g(t)\} \\ &= \bigcap_n \bigcup_{x \in D} \{t | d(f(t, x), g(t)) < 1/n\} \end{aligned}$$

To prove the last equality above, note on the one hand that for each t in the left hand side there exists $x \in B$ and a sequence (x_n) in D such that $x_n \to x$ and $d(f(t, x_n), g(t)) < 1/n$ for each n. On the other hand, suppose for each n that there exists $x_n \in D$ such that $d(f(t, x_n), g(t)) < 1/n$. By the compactness of B there is a subsequence (x_{n_k}) of (x_n) which converges, say, to $x \in B$. Then $f(t, x) = g(t)$.

To show $K^{-1}(B) \in \mathcal{A}$, it is thus sufficient to show $C = \{t | d(f(t, x), g(t)) < 1/n\} \in \mathcal{A}$ for all n and all $x \in D$. So fix n and $x \in D$. Define $h: T \to Y \times Y$ by $h(t) = (f(t, x), g(t))$. Then $C = h^{-1}(\{(y, z) \in Y \times Y | d(y, z) < 1/n\})$. The set $\{(y, z) \in Y \times Y | d(y, z) < 1/n\}$ is the union of countably many closed rectangles $A_m \times B_m$, and for each m, $h^{-1}(A_m \times B_m) = \{t | f(t, x) \in A_m\} \cap g^{-1}(B_m)$ is measurable. Hence $C = \bigcup_m h^{-1}(A_m \times B_m) \in \mathcal{A}$.

ii) Proceed as in the proof of (i), except now use Theorem 3 (ii) to show that $K \cap \Gamma$ is measurable, and Theorem 1′ to obtain a selector for $K \cap \Gamma$. The proof of the measurability of K still works, since now we assume that B is compact, and we may assume, by the σ-compactness of Y, that each of the rectangles $A_m \times B_m$ is compact.

We will conclude this section with a sufficient condition for the measurability of $\min f(\{t\} \times \Gamma(t))$. We need the following theorem on the joint measurability of $f(t, x)$. The proof is from KURATOWSKI [K, p. 378], where the same argument is used to obtain a more precise result. The σ-algebra on $T \times X$ is the product σ-algebra of $\mathcal{A}$ and the σ-algebra of Borel subsets of X.

THEOREM 5. – Let X be a separable metric space, Y a metric space, and let $f: T \times X \to Y$ be a function measurable in t and continuous in x. Then f is measurable. In fact, for each closed subset B of Y, $f^{-1}(B)$ is the countable intersection of countable unions of rectangles $A \times F$ with $A \in \mathcal{A}$ and F closed in X.

PROOF. – Let D be a countable dense subset of X, let B be a closed subset of Y, and let $B_n = \{y \in Y | d(y, B) \leqslant 1/n\}$. Then $f(t, x) \in B$ iff for each integer n there exists $a \in D$ such that $d(x, a) \leqslant 1/n$ and $f(t, a) \in B_n$. (The «only if» part of this statement is trivial. To see the «if» part, choose $a_n \in D$ for each n such that $d(x, a_n) \leqslant 1/n$ and $f(t, a_n) \in B_n$. Then $a_n \to x$, so $f(t, a_n) \to f(t, x)$ and

$$d(f(t, x), B) = \lim_{n \to \infty} d(f(t, a_n), B) = 0 .$$

Thus $f(t, x) \in B$.) It follows that

$$f^{-1}(B) = \bigcap_n \bigcup_{a \in D} \{t | f(t, a) \in B_n\} \times \{x | d(x, a) \leqslant 1/n\} ,$$

so that $f^{-1}(B)$ is measurable.

THEOREM 6. – Let X be a separable metric space, let $f: T \times X \to R$ be measurable in t and continuous in x, and let $\Gamma: T \to X$ be a measurable multifunction with compact values. Then the function $y: T \to R$, defined by $y(t) = \min f(\{t\} \times \Gamma(t))$, is measurable.

PROOF. – Using Theorem 1, let U be a countable collection of measurable functions from T to X such that $\Gamma(t)=\overline{U(t)}$ for each $t\in T$. Then for any real number r, we have

$$\begin{aligned} y^{-1}((-\infty, r)) &= \{t|\min f(\{t\}\times\Gamma(t))<r\} \\ &= \{t|\min \overline{f(\{t\}\times U(t))}<r\} \\ &= \{t|\inf f(\{t\}\times U(t))<r\} \\ &= \{t|f(t, u(t))<r \text{ for some } u\in U\} \\ &= \bigcup_{u\in U}\{t|f(t, u(t))<r\}. \end{aligned}$$

Hence, it is sufficient to prove that $\{t|f(t, u(t))<r\}$ is measurable. But this follows since the function $t\to f(t, u(t))$ is the composition of $t\to(t, u(t))$ and $(t, x)\to f(t,x)$ The first is measurable in the sense that $t^{-1}(S)$ is measurable for each measurable subset S of $T\times X$. The second is measurable by Theorem 5.

2. – An application.

We can now apply the results of the previous section to generalize the stochastic decision problem solved by Benes in [B]. Indeed, much of section 1 parallels a similar development in [B]. For a more complete description of the background to this problem, we refer the reader to Benes' article.

Let $(\Omega, P, \mathcal{B})$ be a probability space, and let $\{x(t, \omega)|0\leqslant t\leqslant 1\}$ be a measurable separable stochastic process with values in R^n and having continuous sample paths with probability one. As does Benes, we use the process only to restrict the pattern of information available in the formation of admissible strategies. In particular, for each $t\in[0, 1]$, let $\mathcal{K}_t$ be the σ-algebra on Ω generated by all sets of the form

$$\{\omega\in\Omega|x(s, \omega)\in A\}, \qquad \text{with } 0\leqslant s\leqslant t,\ A \text{ Borel in } R^n.$$

$\mathcal{K}_t$ is the σ-algebra representing knowledge of the past up to time t. Let $\mathcal{F}_t$ be a sub-σ-algebra of $\mathcal{K}_t$.

We assume given a compact metric space X of control points; a function $c: I\times\Omega\times X\to R^+$ representing cost per unit time as a function of the time t, the event ω and applied control x; and a *constraint multifunction* $\Gamma: I\times\Omega\to X$. The function $c(t, \omega, x)$ is continuous in x and measurable in the variables t, ω together relative to the product σ-algebra $\mathcal{L}\times\mathcal{B}$ on $I\times\Omega$ (where $\mathcal{L}$ is the family of Lebesgue measurable subsets of I). Moreover, we assume that $c(t, \omega, x)$ is uniformly bounded by a measurable function $r: \Omega\to R^+$ with finite expectation. $\Gamma(t, \omega)$ is assumed to be measurable relative to $\mathcal{L}\times\mathcal{B}$ and in addition $\mathcal{F}_t$-measurable in ω for each t.

Let λ denote Lebesque measure on $\mathfrak{L}$ and let $\lambda \times P$ denote the product probability measure on $\mathfrak{L} \times \mathfrak{B}$. An *admissible strategy* is a function $\gamma: I \times \Omega \to X$ which is: (i) a selector for $\Gamma: I \times \Omega \to X$ with probability 1 (i.e., $\gamma(t, \omega) \in \Gamma(t, \omega)$ for all (t, ω) in some set of measure 1), (ii) measurable relative to $\mathfrak{L} \times \mathfrak{B}$, and (iii) $\mathcal{F}_t$-measurable in ω for each $t \in [0, 1]$. An *optimal strategy* is an admissible strategy which minimizes the integral

$$E\int_0^1 c\big(t, \omega, \gamma(t, \omega)\big)\, dt .$$

The problem is to show that such optimal strategies exist.

Our description of this problem differs from Benes' in two ways. He allows admissible strategies to take values anywhere in X at all times, i.e., the constraint multifunction Γ has the constant value X. There is also a technical difference. Benes describes full knowledge of the past in terms of σ-algebras on the space $C(I)$ of continuous functions from I to R^n. The cost function has $I \times C(I) \times X$ for its domain, and admissible strategies are defined on $I \times C(I)$. The problem then is to find $\gamma: I \times \times C(I) \to R^+$ minimizing

$$E\int_0^1 c\Big(t, x(\cdot, \omega), \gamma\big(t, x(\cdot, \omega)\big)\Big)\, dt .$$

In his solution, Benes uses a measure theoretic lemma to recast the problem in terms of Ω and $x(t, \omega)$. We prefer, for the sake of simplicity, to describe the problem in terms of Ω and $x(t, \omega)$ to begin with.

THEOREM 7. – There exists an optimal strategy.

PROOF. – Let $\mathcal{F}$ be the sub-σ-algebra of $\mathfrak{L} \times \mathfrak{B}$ formed of all measurable sets E such that every t-section of E is in $\mathcal{F}_t$. Then the process $x = x(t, \omega)$ is measurable with respect to $\mathcal{F}$, and the criteria for admissibility of $\gamma: I \times \Omega \to X$ are equivalent to:

i) $\gamma(t, \omega) \in \Gamma(t, \omega)$ for all (t, ω) in a member of $\mathcal{F}$ of $\lambda \times P$-measure 1.

ii) γ is measurable with respect to $\mathcal{F}$.

Let $\bar{\mathcal{F}}$ be the completion of $\mathcal{F}$ with respect to $\lambda \times P$, and define $K: I \times \Omega \times X \to R^+$ by

$$K(t, \omega, x) = E\{c(t, \omega, x) | \mathcal{F}\} .$$

K is, by definition, an $\bar{\mathcal{F}}$-measurable function of (t, ω) for each x. Moreover, $K(t, \omega, x)$ is a continuous function of x for each (t, ω) in a member M of $\mathcal{F}$ having $\lambda \times P$-measure 1. (This latter fact is not entirely trivial. But using the fact that $c(t, \omega, x)$ (regarded as a stochastic process on $I \times \Omega$ with parameter set X) can be replaced by a separable process, it can be shown, for each (t, ω) in some member of $\bar{\mathcal{F}}$ with

$\lambda\times P$-measure 1, that

$$\sup_{d(x_1,x_2)<h} |E\{c(t,\omega,x_1)|\mathcal{F}\} - E\{c(t,\omega,x_2)|\mathcal{F}\}| \leqslant E\Big\{\sup_{d(x_1,x_2)\leqslant h} |c(t,\omega,x_1) - c(t,\omega,x_2)|\,|\mathcal{F}\Big\}.$$

Since $c(t,\omega,x)$ is bounded by $r(\omega)$ with $E(r(\omega))<\infty$, it follows that for (t,ω) in some member of $\mathcal{F}$ with $\lambda\times P$-measure 1 that the right hand side of the inequality above tends to 0 as $h\to 0$. Hence $K(t,\omega,x)$ is a continuous function of x for (t,ω) in a member M of $\mathcal{F}$ of measure 1.)

By Theorem 6, the function $y\colon M\to R$ defined by

$$y(t,\omega) = \min K(\{t,\omega\}\times\Gamma(t,\omega))\,, \qquad \text{if } (t,\omega)\in M\,,$$

is $\overline{\mathcal{F}}$-measurable.

By Theorem 4, there exists on $\overline{\mathcal{F}}$-measurable function $\gamma_1\colon M\to X$ such that

$$\gamma_1(t,\omega)\in\Gamma(t,\omega), \qquad \text{and} \qquad y(t,\omega) = K(t,\omega,\gamma_1(t,\omega))\,,$$

for all $(t,\omega)\in M$. Let γ_2 be any extension of γ_1 over $I\times\Omega$ to an $\overline{\mathcal{F}}$-measurable selector for Γ, and let γ_3 be an $\mathcal{F}$-measurable function obtained by altering γ_2 only on a set of measure 0. Then γ_3 is an admissible strategy. It is also optimal. For let γ be any admissible strategy. Then

$$K(t,\omega,\gamma_3(t,\omega)) \leqslant K(t,\omega,\gamma(t,\omega)) \qquad \text{a.e.}$$

By applying [B, Lemma 3], this inequality is equivalent to

$$E\{c(t,\omega,\gamma_3(t,\omega))|\mathcal{F}\} \leqslant E\{c(t,\omega,\gamma(t,\omega))|\mathcal{F}\} \qquad \text{a.e.}$$

Then integrating both sides and reversing the order of integration yields the desired optimality of γ_3.

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