Prime-Independent Arithmetical Functions.

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Summary. – *This paper studies arithmetical functions that are prime-independent and multiplicative. Firstly, it establishes necessary and sufficient conditions for such functions to possess simple formulae relating them to the zeta function. Then it investigates asymptotic average-values and moments of such functions. The results apply to functions of ideals in algebraic numbers fields, or of isomorphism classes in certain categories, as well as to functions of positive integers.*

Introduction.

It is well known that a large class of the arithmetical functions $f(n)$ of a positive integer n that occur naturally consists of those that are both *multiplicative* (i.e. $f(1)=1$ and $f(mn)=f(m)f(n)$ for $(m,n)=1$) and *prime-independent* (i.e. for any prime-power p^r , $f(p^r)$ may vary with r but not with the prime p). Further, many such functions f have an associated Dirichlet series $\tilde{f}(z)=\sum_{n=1}^{\infty} f(n)n^{-z}$ that admits a simple multiplicative formula in terms of the Riemann zeta function $\zeta(z)$. For example, the standard *divisor* function d satisfies $\tilde{d}(z)=[\zeta(z)]^2$, while the *unitary-divisor* function d_* such that $d_*(n)=\sum_{d|n, (d,n/d)=1} 1$ satisfies $\tilde{d}_*(z)=[\zeta(z)]^2/\zeta(2z)$, and so on.

The first question that this paper seeks to answer is concerned with finding necessary and sufficient conditions on an arithmetical function f as above, in order that $\tilde{f}(z)$ should admit a simple « ζ -formula» of the above kind. This question is considered in §§ 1-2 below, in the more general setting of functions on an *arithmetical semigroup* – for applications of, and further references on, this concept, see for example the papers [9].

Next, if $f(n)$ is a prime-independent and multiplicative arithmetical function, we investigate the asymptotic behaviour of its «*average value*» $1/x \sum_{n \leq x} f(n)$ as $x \rightarrow \infty$. This is done by extending an asymptotic formula of Titchmarsh [11] and Wilson [12]. The resulting theorem allows one not only to re-derive standard asymptotic results of many authors, including MERTENS, LANDAU and RAMANUJAN in particular, but also to obtain a variety of apparently new propositions.

We shall not attempt to list all such new applications here, but mention only some examples. Thus, it is shown below that the point-wise ratios $d(n)/d_*(n)$ and $d_*(n)/d(n)$ both have finite k -th moments for each $k=1, 2, \dots$, where the k -th mo-

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ment of $f(n)$ is defined to be

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} [f(n)]^k$$

(if the limit exists). Similarly, if $a(n)$ denotes the total number of isomorphism classes of *finite abelian groups* of order n , and $S(n)$ denotes the total number of isomorphism classes of *semisimple finite rings* of cardinal n , we show that *both $a(n)$ and $S(n)$ have finite k -th moments for each $k = 1, 2, \dots$* . Various other results of this nature are given below, together with references to earlier related theorems.

We note that, in obtaining the above-mentioned asymptotic statements, it is again advantageous to work with functions on an *arithmetical semigroup* rather than on the positive integers; however, now we impose a suitable *distribution axiom* (« Axiom A »). By following this procedure, it becomes possible to simultaneously obtain the above results and extend previous work of LANDAU, E. COHEN and one of us, concerning the semigroup G_K of all non-zero integral ideals in a given *algebraic number field* K , the semigroup $\mathcal{A}$ of all isomorphism classes of *finite abelian groups* relative to direct product, or the arithmetical semigroups arising in a similar way from many other *natural categories* of objects; for more details, see [9].

Finally, we should like to thank A. M. SINCLAIR for his helpful suggestions on these topics.

1. - ζ -formulae.

As mentioned in the introduction, the discussion below is formulated in the context of general arithmetical semigroups. By an *arithmetical semigroup* G , we shall understand a free abelian semigroup G with identity that has a finite or countably infinite set P of free generators (whose elements are called the *primes* of G) together with a real-valued *norm* mapping $|\cdot|$ on G such that (i) $|1| = 1$, $|p| > 1$ for $p \in P$, (ii) $|ab| = |a||b|$ for all $a, b \in G$, and (iii) the total number $N_G(x)$ of elements $a \in G$ of norm $|a| \leq x$ is finite, for each $x > 0$.

Any complex-valued function f on G will be called an *arithmetical function* on G . The set $\text{Dir}(G)$ of all such functions on G forms a complex, vector space under the point-wise operations, which is actually a commutative and associative algebra with identity under the *convolution* multiplication $*$ defined by $(f * g)(a) = \sum_{bc=a} f(b)g(c) [a \in G]$.

Following the convention in [9] Part V, we shall call $\text{Dir}(G)$ the *Dirichlet algebra* of G , and represent its elements f in the form of *formal Dirichlet series over G* : $f = \sum_{a \in G} f(a) a^{-z}$, where z is an unusual type of « indeterminate ». In this representation, multiplication will be denoted by juxtaposition, so that $f(x)g(z) = \sum_{a \in G} (f * g)(a) a^{-z}$.

In the following discussion it is necessary to use certain elementary facts about the algebra $\text{Dir}(G)$; we refer to [9] Part V for any such facts that are not treated in detail here.

A function of paramount importance in the study of G , is its *zeta function* $\zeta_G(z) = \sum_{a \in G} a^{-z}$. More generally, in the study of special arithmetical functions of interest, one is also concerned with $\zeta_G(mz) = \sum_{a \in G} (a^m)^{-z} = \sum_{a \in G} a^{-mz}$ where $m = 1, 2, \dots$. A function $f \in \text{Dir}(G)$ will be said to possess a *finite ζ -formula* in the case when $f(z)$ can be written as a finite product of functions of the form $\zeta_G(mz)$ or $[\zeta_G(mz)]^{-1}$. (A function $g \in \text{Dir}(G)$ is a *unit* if and only if $g(1) \neq 0$; if g is a unit then $[g(z)]^{-1}$ can be expressed in terms of a « binomial series » expansion.) As was mentioned in the introduction, many natural arithmetical functions do possess finite ζ -formulae; see Hardy and Wright [6] and [9] Part V, say. Our first aim is to establish some necessary and sufficient conditions for a function $f \in \text{Dir}(G)$ to possess such a formula.

For this purpose, consider some preliminary concepts. A function $f \in \text{Dir}(G)$ is called *multiplicative* if $f(1) = 1$ and $f(ab) = f(a)f(b)$ whenever the g.c.d. $(a, b) = 1$; it is called *prime-independent* if and only if for each $r = 1, 2, \dots$, $f(p_1^r) = f(p_2^r)$ for all $p_1, p_2 \in P$. If f is multiplicative, it is easy to see that it can be represented formally as a product

$$f(z) = \prod_{p \in P} \left(\sum_{r=0}^{\infty} f(p^r) p^{-rz} \right),$$

and in particular $\zeta_G(z)$ satisfies the *Euler product formula*

$$\zeta_G(z) = \prod_{p \in P} (1 + p^{-z} + p^{-2z} + \dots) = \prod_{p \in P} (1 - p^{-z})^{-1}.$$

The main reason for introducing the formal Dirichlet series notation for functions $f \in \text{Dir}(G)$, in [9] Part V, was in order to be able to make full use of ordinary operations of analysis in this way, without the immediate necessity of considering delicate convergence questions. Then, in order to analyse such formal operations rigorously, the following concept of *pseudo-convergence* was defined for infinite sums and products in $\text{Dir}(G)$.

Let $f_1, f_2, \dots \in \text{Dir}(G)$ be a sequence of functions such that, for each $a \in G$, $f_n(a) \neq 0$ for at most a finite number of indices n . Then $\sum_{n=1}^{\infty} f_n$ is called a *pseudo-convergent series*, and its *sum* is defined to be

$$\sum_{n=1}^{\infty} f_n = \sum_{a \in G} \left(\sum_{n=1}^{\infty} f_n(a) \right) a^{-z}.$$

It turns out that this «sum» coincides with $\lim_{N \rightarrow \infty} \sum_{n=1}^N f_n$ with respect to the complete ultrametric on $\text{Dir}(G)$ induced by the norm

$$\|f\| = \begin{cases} \max\{|a|^{-1}: f(a) \neq 0\} & \text{if } f \neq 0, \\ 0 & \text{if } f = 0. \end{cases}$$

Similarly, if for example $f_n(z) = 1 + \bar{f}_n(z)$ where $\|\bar{f}_n\| < 1$ and $\sum_{n=1}^{\infty} \bar{f}_n$ is pseudo-convergent, then $\lim_{N \rightarrow \infty} \prod_{n=1}^N f_n$ exists and coincides with

$$1 + \sum_{\substack{a \in G, \\ a \neq 1}} \left\{ \sum_{\substack{b_1, b_2, \dots, b_r = a, \\ r \geq 1, n_i \geq 1}} \prod_{i=1}^r f_{n_i}(b_i) \right\} a^{-z};$$

in this situation, we call $\prod_{n=1}^{\infty} f_n$ a *pseudo-convergent* product and define its *value* by the above formula. The usual algebraic properties of ordinary absolutely convergent sums and products in analysis now turn out to be valid for pseudo-convergent sums and products in $\text{Dir}(G)$; see [9] Part V, for further details.

With the aid of the facts just mentioned (or by a direct but less intuitive inductive argument), it is now easy to verify that the set $\text{PIM}(G) = \text{PIM}$ of all *prime-independent multiplicative functions in $\text{Dir}(G)$* forms a subgroup of the group of units of $\text{Dir}(G)$. Further, if $\mathbf{C}[[y]]$ denotes a formal power series algebra in an indeterminate y over the complex field $\mathbf{C}$, then there is a *multiplicative monomorphism*

$$\hat{}: \text{PIM} \rightarrow \mathbf{C}[[y]]$$

given by

$$\hat{f} = \sum_{r=0}^{\infty} f(p^r) y^r \quad [f \in \text{PIM}, p \in P].$$

The mapping $\hat{}$ is *continuous* relative to $\|\|$ on $\text{Dir}(G)$ and the standard (y)-adic topology on $\mathbf{C}[[y]]$ (c.f. Zariski and Samuel [13]), and in fact $\hat{}$ is a *homeomorphism* of PIM with $\text{Im}(\hat{})$ —the set of all power series with constant term 1.

In order to state the first theorem, consider one more definition. Let $\mathbf{Z}[[y]]$ denote the subring of $\mathbf{C}[[y]]$ consisting of all power series with rational integer coefficients. A series $g \in \mathbf{Z}[[y]]$ will be called a *cyclotomic rational* if it can be expressed as a finite product of cyclotomic polynomials or inverses of these, where a *cyclotomic* polynomial is one of the form $\Phi_m(y) = \prod_i (y - \alpha_i) \in \mathbf{Z}[y]$ (where $\alpha_1, \alpha_2, \dots$ denote the distinct primitive m -th roots of unity), if $m > 1$, or $\Phi_1(y) = 1 - y$. If g is a product of cyclotomic polynomials alone, it will be called a *cyclotomic integer* in $\mathbf{Z}[[y]]$.

(1.1) THEOREM. — A function $f \in \text{Dir}(G)$ possesses a finite ζ -formula if and only if $f \in \text{PIM}$ and its associated power series $\hat{f}$ is a cyclotomic rational.

PROOF. — It is clear that $\zeta_G(mz)$ is a PIM-function. Therefore, since PIM forms a group and any function f that possesses a finite ζ -formula lies in the multiplicative group generated by the functions $\zeta_G(mz)$, every such $f \in \text{PIM}$. Now suppose that $f(z) = \prod_i [\zeta_G(m_i z)]^{k_i}$ where the m_i and k_i are integers and $m_i > 0$. Then the general product decomposition for PIM-functions mentioned above, including the Euler product formula, implies that

$$\hat{f} = \prod_i (1 - y^{m_i})^{-k_i} = \prod_i \prod_{d|m_i} [\Phi_d(y)]^{-k_i}.$$

Thus $\hat{f}$ is a cyclotomic rational.

Conversely, if $f \in \text{PIM}$ and f is a cyclotomic rational, with $\hat{f} = \prod [\Phi_{n_i}(y)]^{r_i}$, then the standard inversion formula for cyclotomic polynomials implies that $\hat{f} = \prod_i \prod_{d|n_i} (1 - y^d)^{\mu(n_i/d)r_i}$ where μ denotes the Möbius function. Hence

$$f(z) = \prod_i \prod_{d|n_i} [\zeta_G(dz)]^{-\mu(n_i/d)r_i},$$

so that f has a finite ζ -formula.

(1.2) COROLLARY. — If $f \in \text{Dir}(G)$ possesses a finite ζ -formula then f takes only rational integer values.

This is immediate, since one may apply binomial series expansions to each function $[\zeta_G(mz)]^k$.

(1.3) COROLLARY. — If $f \in \text{PIM}$ and $\hat{f}$ is a polynomial, then f has a finite ζ -formula if and only if $\hat{f}$ is a cyclotomic integer.

PROOF. — Let $g \in \mathbf{Z}[y]$ be a cyclotomic rational. Then recall that the cyclotomic polynomials are irreducible, and note that $g = h_1/h_2$ where h_1, h_2 are coprime cyclotomic integers. Then $h_1 = gh_2$ and since $\mathbf{Z}[y]$ is a unique factorization domain this implies that g is a cyclotomic integer. The corollary is now immediate.

The above criterion can be used to show the non-existence of finite ζ -formulae for various special arithmetical functions, and even to show that the existence of such formulae for many of the most immediate natural functions is in a sense merely a coincidence. (See the appendix-§ 5 below.)

2. — Infinite ζ -formulae.

In the theory of enumeration of isomorphism classes of objects in various specific types of categories, certain natural arithmetical functions arise that do not have finite ζ -formulae, but instead have the following property: A function $f \in \text{Dir}(G)$ will be

possess an *infinite* ζ -formula if $f(z)$ can be expressed as a pseudo-convergent infinite product of functions of the form $[\zeta_G(mz)]^k$ where m, k are integers and $m > 0$. (Strictly speaking, this admits finite products also.)

For example, if $a(n)$ and $S(n)$ are the functions defined in the introduction, then

$$a(z) = \prod_{r=1}^{\infty} \zeta(rz) \quad \text{and} \quad S(z) = \prod_{r=1}^{\infty} \prod_{m=1}^{\infty} \zeta(rm^2z) ;$$

cf. KENDALL and RANKIN [8], and [9] Part I—where these functions are also interpreted as the zeta functions of the relevant categories and arithmetical semigroups (in a slightly different sense from above).

Next, let D denote the ring of all algebraic integers in some algebraic number field K , and let $\mathcal{F}$ denote the category of all D -modules of finite cardinal. The set F of all isomorphism classes of objects in $\mathcal{F}$ has the structure of an arithmetical semigroup, and it is pointed out in [9] Part IV that there is a natural epimorphism $\Phi: F \rightarrow G_K$ (where G_K is the arithmetical semigroup of all non-zero ideals in D), given by $\Phi([D/P']) = P'$ for each prime ideal P in D . Then, for each $I \in G_K$, the total number $a_D(I)$ of classes $[M] \in F$ such that $\Phi([M]) = I$ is finite and defines a function on G_K satisfying

$$a_D(z) = \prod_{r=1}^{\infty} \zeta_K(rz)$$

where $\zeta_K(z) = \zeta_{G_K}(z)$ is essentially the *Dedekind* zeta function of K ; as with the special case $a(z)$, $a_D(z)$ may be interpreted as a special type of zeta function for $\mathcal{F}$ or F .

A similar function on G_K , to be considered again later, arises from the category $\mathcal{S}$ of all semisimple finite D -algebras. This category also has an associated arithmetical semigroup (S) , and by [9] Part IV there is an epimorphism $\Phi: S \rightarrow G_K$ given by $\Phi([M_n(D/P)_r]) = P^{rn}$ for each prime ideal P in D , where $(D/P)_r$ denotes an r -degree extension of the field D/P . If one lets $S_D(I)$ denote the total number of classes $[A] \in S$ such that $\Phi([A]) = I \in G_K$, then one now obtains a function on G_K satisfying

$$S_D(z) = \prod_{r=1}^{\infty} \prod_{m=1}^{\infty} \zeta_K(rm^2z) .$$

Examples like these suggest the question as to what types of functions possess infinite ζ -formulae; the answer is simple:

(2.1) THEOREM. — Let G denote an arbitrary arithmetical semigroup, and let $f \in \text{Dir}(G)$. Then f possesses an infinite ζ -formula if and only if $f \in \text{PIM}$ and f takes only rational integer values.

PROOF. — The necessity of the two stated conditions is easy to see, if one bears in mind the remarks about pseudo-convergent products in $\text{Dir}(G)$ made in § 1. Conversely, suppose that $f \in \text{PIM}$ and f takes only rational integer values. Then

$\hat{f} = 1 + k_1 y^{m_1} + \sum_{r > m_1} c_r y^r$, say, where k_1, m_1 are integers. Therefore

$$(1 - y^{m_1})^{k_1} \hat{f} = 1 + k_2 y^{m_2} + \sum_{r > m_2} c'_r y^r = \hat{f}_2,$$

say, where k_2, m_2 are integers, $m_2 > m_1$ and $\hat{f}_2$ is the image under $\hat{}$ of a PIM-function f_2 such that f_2 takes only rational integer values and

$$f(z) = [\zeta_G(m_1 z)]^{k_1} f_2(z).$$

Now apply the same procedure to f_2 in order to obtain a similar function f_3 such that

$$f_2(z) = [\zeta_G(m_2 z)]^{k_2} f_3(z).$$

By carrying on in this way, one obtains a sequence of functions $f, f_2, f_3, \dots$ such that $(\hat{f}_n - 1) \rightarrow 0$ as $n \rightarrow \infty$, in the (y) -adic topology for $\mathbf{C}[[y]]$. The topological property of $\hat{}$ mentioned in § 1 then implies that $f_n \rightarrow 1$ as $n \rightarrow \infty$, relative to $\|\|$ in $\text{Dir}(G)$. If

$$F_N = [\zeta_G(m_1 z)]^{k_1} \dots [\zeta_G(m_N z)]^{k_N},$$

it now follows with the aid of the definition of $\|\|$ that $\|F_N\| \rightarrow \|f\| = 1$ as $N \rightarrow \infty$, and hence the previous conclusion implies that $F_N \rightarrow f$ in $\text{Dir}(G)$. Thus f has an infinite ζ -formula.

The next result exhibits the relationship between ζ -formulae and certain very similar formulae considered in [9] Part V. For this result, first define a function $\chi \in \text{Dir}(G)$ to be *completely multiplicative* if $\chi(1) = 1$ and $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in G$. If χ is completely multiplicative and f is any element of $\text{Dir}(G)$, let

$$f(z, \chi) = \sum_{a \in G} \chi(a) f(a) a^{-z} = f_\chi(z)$$

where $f_\chi(a) = \chi(a) f(a)$; also let $\zeta_G(z, \chi) = \sum_{a \in G} \chi(a) a^{-z} = \chi(z)$.

(2.2) PROPOSITION. — The function $f \in \text{Dir}(G)$ possesses the finite or infinite ζ -formula $f(z) = \prod_i [\zeta_G(m_i z)]^{k_i}$ if and only if

$$f(z, \chi) = \prod_i [\zeta_G(m_i z, \chi^{m_i})]^{k_i}$$

for every completely multiplicative function $\chi \in \text{Dir}(G)$.

PROOF. — Here χ^m denotes the point-wise m -th power of χ , and in general $g(mz) = \sum_{a \in G} g(a) (a^m)^{-z} = \sum_{a \in G} g(a) a^{-mz}$ for any $g \in \text{Dir}(G)$. Now note that, for any completely

multiplicative function $\chi \in \text{Dir}(G)$, the rule $f \rightarrow f_\chi$ defines an algebra endomorphism of $\text{Dir}(G)$ and this endomorphism is continuous relative to $\|\cdot\|$. (If $\chi(a) \neq 0$ for all $a \in G$ then this endomorphism is both an automorphism and a homeomorphism.) Also note that this endomorphism sends $\zeta_G(mz)$ to $\zeta_G(mz, \chi^m)$. The proposition is now an easy consequence of these statements and the definition of an infinite ζ -formula.

It may be worth remarking that the above examples of special infinite ζ -formulae can also be given analytic significance in the ordinary sense. We note also that ESTERMANN [4] has studied the analytic properties of $\tilde{g}(z)$ for z a complex variable, when g is a special type of integer-valued PIM-function on the semigroup G_z of all positive integers.

3. - Average values of arithmetical functions.

For the rest of this paper we shall assume that G denotes an arithmetical semigroup satisfying the following distribution axiom. (For a discussion of natural examples obeying this axiom, see [9].)

AXIOM A. - There exist positive constants A and δ depending on G and a constant η with $0 \leq \eta < \delta$ such that

$$N_G(x) = Ax^\delta + O(x^\eta) \quad \text{as } x \rightarrow \infty.$$

If confusion seems unlikely, we shall now also refer to the series $\sum_{a \in G} |a|^{-z}$ as the *zeta function* $\zeta_G(z)$ of G ; by [9] Part II, $\zeta_G(z)$ is convergent for complex z with $\text{Re } z > \delta$, and we shall need to refer to [9] Part II for various other analytic or asymptotic statements below. For any $f \in \text{Dir}(G)$, use will now be made of the associated Dirichlet series $\tilde{f}(z) = \sum_{a \in G} f(a)|a|^{-z}$, and for convenience in such a case z may be regarded as a complex variable.

In studying asymptotic average values of various special arithmetical functions, use will be made of the following theorem which extends a result of TITCHMARSH [11] and WILSON [12].

(3.1) THEOREM. - Let $f, g \in \text{Dir}(G)$ be functions such that formally

$$\tilde{f}(z) = [\zeta_G(mz)]^k \tilde{g}(z)$$

for certain positive integers m, k . Suppose that $\tilde{g}(z)$ is absolutely convergent for $\text{Re } z > \nu$ where $\nu < \delta/m$. Then as $x \rightarrow \infty$

$$\sum_{|a| \leq x} f(a) = \frac{A}{(k-1)!} \left(\frac{\delta A}{m}\right)^{k-1} \left[\tilde{g}\left(\frac{\delta}{m}\right) + o(1) \right] x^{\delta/m} (\log x)^{k-1}.$$

PROOF. — Let d_k ($k \geq 2$) denote the generalized divisor function on G such that $d_k(a)$ is the total number of ordered k -tuples $(b_1, \dots, b_k)$ with $b_1 \dots b_k = a$. Then $d_k(z) = [\zeta_G(z)]^k$ in $\text{Dir}(G)$, and by Theorem I.4.1. of [9] Part II there exist constants c_r such that

$$\sum_{|a| \leq x} d_k(a) = Ax^\delta \sum_{r=1}^k c_r (\log x)^{k-r} + O(x^{\delta - (\delta - \eta)/k} (\log x)^{k-2})$$

and $c_1 = (\delta A)^{k-1} / (k-1)!$. Therefore, if $\tilde{F}(z) = [\zeta_G(mz)]^k = \tilde{d}_k(mz)$, one obtains

$$\sum_{|a| \leq x} F(a) = \sum_{|c^m| \leq x} d_k(c) = \sum_{|c| \leq x^{1/m}} d_k(c) = Ax^{\delta/m} \sum_{r=1}^k c_r \left(\frac{1}{m} \log x \right)^{k-r} + O(x^{(\delta - (\delta - \eta)/k)/m} (\log x)^{k-2}).$$

In view of the fact that $\tilde{f}(z) = \tilde{F}(z)\tilde{g}(z)$, the required theorem is therefore a consequence of

(3.2) LEMMA. — Let $f, F, g \in \text{Dir}(G)$ be arbitrary functions such that formally $\tilde{f}(z) = \tilde{F}(z)\tilde{g}(z)$. Suppose that

$$\sum_{|a| \leq x} F(a) = Bx^\alpha (\log x)^r + O(x^\beta (\log x)^s)$$

where $\alpha > 0$, $0 \leq \beta < \alpha$, and $r \geq 0$, $s \geq 0$ are integers with the property that $\beta < \alpha$ if $r = 0$, while $s < r$ if $\beta = \alpha$. Suppose also that $\tilde{g}(z)$ is absolutely convergent for $\text{Re } z > \nu$ where $\nu < \alpha$. Then as $x \rightarrow \infty$

$$\sum_{|a| \leq x} f(a) = [B\tilde{g}(\alpha) + o(1)]x^\alpha (\log x)^r.$$

PROOF. — We have

$$\begin{aligned} \sum_{|a| \leq x} f(a) &= \sum_{|b| \leq x} F(b)g(c) = \sum_{|c| \leq x} g(c) \sum_{|b| \leq x/|c|} F(b) = \\ &= \sum_{|c| \leq x} g(c) \left\{ B \left(\frac{x}{|c|} \right)^\alpha \left(\log \frac{x}{|c|} \right)^r + O \left(\left(\frac{x}{|c|} \right)^\beta \left(\log \frac{x}{|c|} \right)^s \right) \right\}. \end{aligned}$$

Now suppose that $r = 0$. Then $\beta < \alpha$ and $x^\beta (\log x)^s = O(x^{\beta'})$ for suitable β' with $\nu < \beta' < \alpha$. Hence

$$\begin{aligned} \sum_{|a| \leq x} f(a) &= Bx^\alpha \sum_{|c| \leq x} g(c) |c|^{-\alpha} + O(x^{\beta'} \sum_{|c| \leq x} |g(c)| |c|^{-\beta'}) = \\ &= Bx^\alpha [\tilde{g}(\alpha) + o(1)] + O(x^{\beta'}) = [B\tilde{g}(\alpha) + o(1)]x^\alpha. \end{aligned}$$

On the other hand, if $r > 0$ then $x^\beta(\log x)^s = O(x^\alpha(\log x)^{s'})$ for a non-negative $s' < r$ if $\beta < \alpha$, while if $\beta = \alpha$ then this is so with $s' = s$, by assumption. Therefore now

$$\begin{aligned} \sum_{|a| \leq x} f(a) &= Bx^\alpha \sum_{|c| \leq x} g(c) |c|^{-\alpha} (\log x - \log |c|)^r + O\left(x^\alpha \sum_{|c| \leq x} |g(c)| |c|^{-\alpha} \left(\log \frac{x}{|c|}\right)^{s'}\right) \\ &= Bx^\alpha \left\{ (\log x)^r [\tilde{g}(\alpha) + o(1)] + O\left(\sum_{t=0}^{r-1} (\log x)^t \sum_{|c| \leq x} |g(c)| |c|^{\varepsilon-\alpha}\right) \right\} + O(x^\alpha (\log x)^s O(1)), \end{aligned}$$

where $\varepsilon > 0$ is arbitrary and may be chosen so that $\alpha - \varepsilon > \nu$. Then

$$\begin{aligned} \sum_{|a| \leq x} f(a) &= Bx^\alpha \{ (\log x)^r [\tilde{g}(\alpha) + o(1)] + O((\log x)^{r-1} O(1)) \} + O(x^\alpha (\log x)^s) = \\ &= x^\alpha (\log x)^r [B\tilde{g}(\alpha) + o(1)]. \end{aligned}$$

In applying Theorem 3.1 to special arithmetical functions, the following lemma is useful.

(3.3) LEMMA. — Let $f \in \text{PIM}$ and suppose that $\hat{f} = 1 + ky^m + \sum_{r>m} a_r y^r$ where $a_r = O(c^r)$ as $r \rightarrow \infty$, for some constant c such that $1 < c < q_0^{\delta/m}$ ($q_0 = \min\{|p| : p \in P\}$). Then

$$f(z) = [\zeta_G(mz)]^k g(z)$$

and $\tilde{g}(z)$ is absolutely convergent whenever $\text{Re } z > \nu$, for some $\nu < \delta/m$.

PROOF. — The function $[\zeta_G(mz)]^k$ can be defined by a pseudo-convergent binomial series expansion, for any complex number k . Then, just as in the proof of Theorem 2.1, one sees that $f(z) = [\zeta_G(mz)]^k g(z)$ where $g \in \text{PIM}$ and $\hat{g} = 1 + k' y^{m'} + \sum_{r>m'} a'_r y^r$ for some $m' > m$. (The operation of taking binomial series expansions of suitable functions is continuous relative to $\|\cdot\|$ in $\text{Dir}(G)$). Thus

$$\hat{g} = \left(1 + \sum_{i=1}^{\infty} \binom{k}{i} (-y^m)^i\right) \left(1 + \sum_{r=1}^{\infty} a_r y^r\right), \quad \text{say,}$$

and so for r sufficiently large

$$\begin{aligned} |a'_r| &= \left| \sum_{i \leq r/m} (-1)^i \binom{k}{i} a_{r-mi} \right| \leq C \sum_{i \leq r/m} \left| \binom{k}{i} a_{r-mi} \right| c^{r-mi} = \\ &= C c^r \sum_{i \leq r/m} \binom{-|k|}{i} (-c^{-m})^i \leq C c^r (1 - c^{-m})^{-|k|} = O(c^r), \end{aligned}$$

since $0 < c^{-m} < 1$, and $|a_r| \leq C c^r$ for some constant $C > 0$.

Now choose $\varepsilon > 0$ such that $c < q_0^{\delta/m-\varepsilon}$ and $\delta/m - \varepsilon > \delta/(m+1)$, and let $\tau = (m+1)(\delta/m - \varepsilon) > \delta$. Then for $p \in P$ and $\operatorname{Re} z = \sigma > \delta/m - \varepsilon$

$$\left| \sum_{r=1}^{\infty} g(p^r) |p|^{-zr} \right| \leq \sum_{r=1}^{\infty} |g(p^r)| |p|^{-\sigma r} = O\left(\sum_{r=m+1}^{\infty} c^r |p|^{-\sigma r} \right),$$

since $g(p^r) = 0$ for $r \leq m$ and $g(p^r) = a'_r = O(c^r)$ for r large enough. Therefore

$$\left| \sum_{r=1}^{\infty} g(p^r) |p|^{-zr} \right| = O\left\{ (c|p|^{-\sigma})^{m+1} \sum_{r=0}^{\infty} (c|p|^{-\sigma})^r \right\} = O(|p|^{-\sigma(m+1)}) = O(|p|^{-\tau}),$$

since

$$c|p|^{-\sigma} < q_0^{\delta/m-\varepsilon} |p|^{-\sigma} \leq q_0^{\delta/m-\varepsilon-\sigma} < 1.$$

Now, by Chapter I, § 2, of [9] Part II, the series $\sum |p|^{-z}$ converges for $\operatorname{Re} z > \delta$. Hence $\sum_{p \in P} |p|^{-\tau}$ converges, and therefore the product $\prod_{p \in P} \left(1 + \sum_{r=1}^{\infty} g(p^r) |p|^{-zr} \right)$ is absolutely convergent for $\operatorname{Re} z = \sigma > \delta/m - \varepsilon$. By a standard argument (of the type used for Lemma 2A of BATEMAN and DIAMOND [1], for example), it follows now that $\tilde{g}(z)$ is also absolutely convergent for $\operatorname{Re} z > \delta/m - \varepsilon$ (and that it converges to the same value as the infinite product). This proves the lemma.

It may be interesting to note that the initial argument above together with the method of proof of Theorem 2.1 yields the following corollary, in which Axiom A is *not* needed.

(3.4) COROLLARY. — Let $f \in \operatorname{Dir}(G)$. Then $f \in \operatorname{PIM}$ if and only if f can be expressed as a pseudo-convergent product

$$f(z) = \prod_i [\zeta_G(m_i z)]^{k_i},$$

where the m_i are positive integers.

3. — Moments of special arithmetical functions.

By means of Theorem 3.1 and Lemma 3.3, it is possible to re-derive the dominant terms of a large number of asymptotic estimates for the average values of arithmetical functions; c.f. [6] and [9] Part II, say. However, this section will be confined to noting a number of applications of the above results that, apart from Theorem 4.1 below, appear to be new even for the semigroup $G_{\mathbf{Z}}$ of *positive integers*; all the results below appear to be new for the semigroup G_K of all non-zero integral ideals in a given *algebraic number field*, and also for the associated arithmetical semigroups of all the *natural categories* discussed in [9] Part II (Chapter I) and Parts III-IV.

First consider the *divisor* function d on G such that $d(a) = \sum_{d|a} 1$.

(4.1) THEOREM. - For any $k = 1, 2, \dots$, there is a constant B_k such that

$$\sum_{|a| \leq x} [d(a)]^k \sim B_k x^\delta (\log x)^{2^k - 1} \quad \text{as } x \rightarrow \infty.$$

PROOF. - For any positive integer k , the point-wise k -th power $\hat{d}^k$ of d lies in PIM and $\hat{d}^k = 1 + 2^k y + 3^k y^2 + \dots + (r+1)^k y^r + \dots$. Therefore the coefficient a_r of y^r is $O(r^c)$ as $r \rightarrow \infty$, for every $c > 1$. Hence Lemma 3.3 gives

$$\hat{d}^k(z) = \sum_{a \in G} \hat{d}^k(a) a^{-z} = [\zeta_G(z)]^{2^k} g(z)$$

where $\tilde{g}(z)$ is absolutely convergent whenever $\operatorname{Re} z > \nu$, for some $\nu < \delta$. Therefore the theorem follows from Theorem 3.1, with

$$B_k = \frac{A}{(2^k - 1)!} (\delta A)^{2^k - 1} \tilde{g}(\delta).$$

In the case when G is the semigroup $G_{\mathbb{Z}}$ of positive integers, Theorem 4.1 is due to Ramanujan (see WILSON [12]). For general G , when $k = 1, 2$, see [9] Part II.

Now consider the *unitary-divisor* function d_* defined by $d_*(a) = \sum_{d|a, (d, a/d)=1} 1$.

(4.2) THEOREM. - For any $k = 1, 2, \dots$, there is a constant B_k^* such that

$$\sum_{|a| \leq x} [d_*(a)]^k \sim B_k^* x^\delta (\log x)^{2^k - 1} \quad \text{as } x \rightarrow \infty.$$

PROOF. - By Proposition 1.6.1 of [9] Part II, $d_*(a) = 2^{\omega(a)}$ where $\omega(a)$ denotes the total number of distinct prime divisors of a . Hence the point-wise k -th power $\hat{d}_*^k \in \text{PIM}$ and $\hat{d}_*^k = 1 + 2^k(y + y^2 + \dots)$. Here the coefficient of y^r is constant, and so the result follows in the same way as for $\hat{d}^k$ above. (When $k = 1$, compare Theorem 1.6.2 of [9] Part II.)

(4.3) THEOREM. - For any $k = 1, 2, \dots$, the point-wise ratios $\hat{d}/\hat{d}_*$ and $\hat{d}_*/\hat{d}$ both possess finite k -th moments:

$$\begin{aligned} \text{(i)} \quad \lim_{x \rightarrow \infty} \frac{1}{N_G(x)} \sum_{|a| \leq x} \left[\frac{d(a)}{d_*(a)} \right]^k &= \prod_{p \in P} \left\{ 1 + 2^{-k} \sum_{r=2}^{\infty} [(r+1)^k - r^k] |p|^{-r\delta} \right\}, \\ \text{(ii)} \quad \lim_{x \rightarrow \infty} \frac{1}{N_G(x)} \sum_{|a| \leq x} \left[\frac{d_*(a)}{d(a)} \right]^k &= \prod_{p \in P} \left\{ 1 + 2^k \sum_{r=2}^{\infty} [(r+1)^{-k} - r^{-k}] |p|^{-r\delta} \right\}. \end{aligned}$$

PROOF. - For any $k = 1, 2, \dots$, consider the point-wise k -th powers $f_1 = (\hat{d}/\hat{d}_*)^k$ and $f_2 = (\hat{d}_*/\hat{d})^k$. Then $f_1, f_2 \in \text{PIM}$ and

$$\hat{f}_1 = 1 + \sum_{r=1}^{\infty} 2^{-k} (r+1)^k y^r, \quad \hat{f}_2 = 1 + \sum_{r=2}^{\infty} 2^k (r+1)^{-k} y^r.$$

The coefficients of these power series satisfy the requirements of Lemma 3.3 and therefore

$$f_1(z) = \zeta_G(z)g_1(z), \quad f_2(z) = \zeta_G(z)g_2(z)$$

where both $\tilde{g}_1(z)$ and $\tilde{g}_2(z)$ are absolutely convergent whenever $\operatorname{Re} z > \nu$, for some $\nu < \delta$. Hence Theorem 3.1 implies that f_1 and f_2 have the finite mean-values

$$\lim_{x \leftarrow \infty} \frac{1}{N_G(x)} \sum_{|a| \leq x} f_1(a) = \tilde{g}_1(\delta), \quad \lim_{x \rightarrow \infty} \frac{1}{N_G(x)} \sum_{|a| \leq x} f_2(a) = \tilde{g}_2(\delta).$$

In other words, d/d_* and d_*/d have the finite k -th moments $\tilde{g}_1(\delta)$ and $\tilde{g}_2(\delta)$, respectively. Now, $\hat{g}_i = (1-y)\hat{f}_i$, and the theorem therefore follows from the fact that

$$\tilde{g}_i(\delta) = \prod_{p \in P} \left(1 + \sum_{r=1}^{\infty} g_i(p^r) |p|^{-r\delta}\right).$$

The fact that d/d_* has a finite mean-value when G is the semigroup $G_{\mathbf{Z}}$ of positive integers is proved in a different way, by KAC [7]. The following theorem may be proved by the same method as for Theorem 4.3; for the case $k = 1$, see also [9] Part II.

(4.4) THEOREM. — Let β denote the prime-divisor function on G such that $\beta(1) = 1$ and $\beta(p_1^{\alpha_1} \dots p_r^{\alpha_r}) = \alpha_1 \alpha_2 \dots \alpha_r$ for distinct $p_i \in P$. Then, for any $k = 1, 2, \dots$, β has the finite k -th moment

$$\lim_{x \rightarrow \infty} \frac{1}{N_G(x)} \sum_{|a| \leq x} [\beta(a)]^k = \prod_{p \in P} \left\{ 1 + \sum_{r=2}^{\infty} [r^k - (r-1)^k] |p|^{-r\delta} \right\}.$$

We note that, when G is the semigroup $G_{\mathbf{Z}}$ of positive integers, the formulae of Theorems 4.3 and 4.4 agree with those obtainable by substitution in the formula (3.2) of ERDÖS and RÉNYI [2], which concerns non-negative multiplicative functions on $G_{\mathbf{Z}}$. However, even for PIM-functions on $G_{\mathbf{Z}}$, the sufficiency criteria of § 3 above appear to be more readily verifiable than those of [2]. Similar remarks apply to Corollaries 4.6 and 4.8 below.

Now consider the enumerative functions $a(n)$, $S(n)$, $a_D(I)$ and $S_D(I)$ discussed in § 2 above.

(4.5) THEOREM. — For any $k = 1, 2, \dots$, the function $a_D(I)$ has a finite k -th moment $\lim_{x \rightarrow \infty} 1/N_K(x) \sum_{\|I\| \leq x} [a(I)]^k$ over G_K , where $N_K(x)$ denotes the total number of ideals of norm at most x in G_K and $\|I\|$ denotes the norm of $I \in G_K$.

PROOF. — By LANDAU [10], G_K is an arithmetical semigroup satisfying Axiom A with $\delta = 1$ and $\eta = 1 - 2/(1 + [K:\mathbf{Q}])$. Next, a discussion of the structure of objects in

the category $\mathcal{F}$ given in [9] Part III, shows that $a_D(I)$ is a PIM-function on G_K such that $a_D(P^r) = p(r)$ for any prime ideal P in D , where $p(r)$ is the ordinary arithmetical *partition* function. Therefore, if f denotes the point-wise k -th power of $a_D(I)$ on G_K then $\hat{f} = 1 + \sum_{r=1}^{\infty} [p(r)]^k y^r$.

Now, a well result of HARDY and RAMANUJAN [5] implies that there is a positive constant B such that $p(r) \leq \exp[B\sqrt{r}]$. This implies that $[p(r)]^k = O(c^r)$ as $r \rightarrow \infty$ for any $c > 1$. Therefore the above remark about G_K together with Lemma 3.3 implies that $f(z) = \zeta_K(z)g(z)$ where $\tilde{g}(z)$ is absolutely convergent whenever $\operatorname{Re} z > \nu$, for some $\nu < 1$. Hence the theorem now follows from Theorem 3.1.

(4.6) COROLLARY. - For any $k = 1, 2, \dots$, the function $a(n)$ has a finite k -th moment $\lim_{x \rightarrow \infty} 1/x \sum_{n \leq x} [a(n)]^k$.

The case $k = 1$ of Theorem 4.5 is a consequence of Corollary 2.3 of [9] Part III; the case $k = 1$ of the corollary was first proved by ERDÖS and SZEKERES [3].

(4.7) THEOREM. - For any $k = 1, 2, \dots$, the function $S_D(I)$ has a finite k -th moment $\lim_{x \rightarrow \infty} 1/N_K(x) \sum_{|I| \leq x} [S_D(I)]^k$ over G_K .

PROOF. - A discussion of the structure of algebras in the category $\mathcal{S}$ given in [9] Part III implies that $S_D(I)$ is a PIM-function on G_K such that $S_D(P^r) = s(r)$ for any prime ideal P in D , where $s(r)$ denotes a certain explicit *generalized partition function*. By Theorem A of [9] Part I, it follows that there exists a positive constant B' such $s(r) \leq \exp[B'\sqrt{r}]$. Therefore the point-wise k -th power h of $S_D(I)$ on G_K is a PIM-function satisfying $\hat{h} = 1 + \sum_{r=1}^{\infty} [s(r)]^k y^r$ where $[s(r)]^k = O(c^r)$ as $r \rightarrow \infty$, for any $c > 1$. Since $s(1) = 1$, the theorem now follows in the same way as Theorem 4.5.

(4.8) COROLLARY. - For any $k = 1, 2, \dots$, the function $S(n)$ has a finite k -th moment $\lim_{x \rightarrow \infty} 1/x \sum_{n \leq x} [S(n)]^k$.

For the case $k = 1$ of Theorem 4.7, see also Theorem 4.2 of [9] Part III.

5. - Appendix: Non-existence of finite ζ -formulae in various cases.

Let G denote an arbitrary arithmetical semigroup, *not* necessarily satisfying Axiom A. Suppose that $f \in \text{PIM}$ and that $\hat{f} = 1 + \sum_{r=1}^{\infty} a_r y^r$ where $a_r = a(r)$ is a polynomial of degree t in r . Consider

$$g_k(z) = [\zeta_G(z)]^{-(tk+1)} f^k(z)$$

where f^k denotes the *point-wise* k -th power of f . Then

$$\hat{g}_k = (1 - y)^{tk+1} \hat{f}^k,$$

and so the coefficient of y^r in $\hat{g}_k$ may be regarded as the $(tk + 1)$ -th repeated « difference » of a_r^k . Since $a_r^k = [a(r)]^k$ is a polynomial of degree tk in r , it follows that this « difference » coefficient must be zero for $r > tk + 1$, i.e. that $\hat{g}_k$ is a polynomial of degree at most $tk + 1$ in y . By Corollary 1.3 above, it therefore follows that f^k has a finite ζ -formula if and only if $\hat{g}_k$ is a cyclotomic integer.

(5.1) EXAMPLES. - (i) Let $f = d_*$, the unitary-divisor function. Then $a_r = d_*(p^r) = 2[p \in P]$. Hence $t = 0$ and the above process leads to the polynomial $\hat{g}_k = 1 + (2^k - 1)y$, which is a cyclotomic integer if and only if $k = 1$. Thus d_*^k has a finite ζ -formula if and only if $k = 1$. (For this and other formulae referred to below, see [9] Part V.)

(ii) Now let f denote the ordinary divisor function d on G . In this case, $a(r) = r + 1$ and so $t = 1$. Now $a(0) = 1$, and, whenever this happens in the general case considered above, the resulting polynomial $\hat{g}_k$ must be of degree tk or less in y . Thus here $\hat{g}_k$ is of degree at most k in y , and calculation shows that $\hat{g}_k = 1 + (2^k - k - 1)y + \dots$.

In order to determine whether or not this is a cyclotomic integer, we observe that, if a complex polynomial $1 + c_1 y + \dots + c_n y^n$ has all its zeros on the unit circle, then $|c_r| \leq \binom{n}{r}$. (This follows easily by considering the equation $c_r = (-1)^{n-r} c_n \sigma_{n-r}$ where σ_i denotes the i -th elementary symmetric polynomial in the roots.) In particular, this remark applies to cyclotomic integers. But the above polynomial $\hat{g}_k$ has first coefficient $2^k - k - 1 > k = \binom{k}{1}$ if $k > 2$. Therefore $\hat{g}_k$ is not a cyclotomic integer, and so d^k does not have a finite ζ -formula, if $k > 2$. For $k = 1, 2$, there do exist finite ζ -formulae, the second case going back to Ramanujan when $G = G_Z$. (In a sense, the present discussion explains why Ramanujan produced a formula for d^2 but not for any higher power d^k .)

(iii) Lastly, let f denote the function β considered in Theorem 4.4 above. Here the coefficient $a_r = r$; thus $t = 1$ and the corresponding polynomial $\hat{g}_k$ is of degree at most $k + 1$ in y . Calculation now shows that

$$\hat{g}_k = 1 - ky + [2^k + \frac{1}{2}(k+1)(k-2)]y^2 + \dots$$

The second coefficient is greater than $\binom{k+1}{2}$ when $k > 1$, and so the principle used in (ii) now shows that β^k does not have a finite ζ -formula if $k > 1$. (For $k = 1$, there is such a formula.)

In addition to having some intrinsic interest, the above examples also provide concrete illustrations of the the following proposition.

(5.2) PROPOSITION. — Let $f \in \text{PIM}$ and suppose that $\hat{f} = 1 + \sum_{r=1}^{\infty} a_r y^r$ where $a_r = a(r)$ is a polynomial of degree t in r , which has rational integer coefficients and is not identically 1, 0 or -1 . Then at most a finite number of the point-wise powers f^k possess finite ζ -formulae.

PROOF. — Let n denote the least positive integer r such that $|a(r)| > 1$, and let $w = |a(n)|$. In terms of the earlier notation, if $tk + 1 \geq n$ then the coefficient of y^n in $\hat{g}_k$ is

$$\pm w^k + \sum_{r=1}^{n-1} (-1)^r \binom{tk+1}{r} [a(n-r)]^k + (-1)^n \binom{tk+1}{n}.$$

Since $|a(r)| \leq 1$ for $r < n$, the coefficient of y^n therefore has modulus not less than $w^k - \sum_{r=1}^n \binom{tk+1}{r}$.

The last quantity is greater than $\binom{tk+1}{n}$ for k sufficiently large, because $w^k \geq 2^k$ and $\binom{tk+1}{n} + \sum_{r=1}^n \binom{tk+1}{r}$ is a polynomial in k . Therefore the principle used in Example 5.1 ii) above shows that $\hat{g}_k$ is not a cyclotomic integer for k sufficiently large. This proves the proposition.

It is not difficult to verify that this proposition is false for those functions f such that $a(r)$ is identically 1 or -1 ; also $f(z) = 1$ on G if $a(r) \equiv 0$.

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