Positivity of Weak Solutions of Non-Uniformly Elliptic Equations (*).

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Summary. - *Let A be a symmetric $N \times N$ real-matrix-valued function on a connected region Ω in R^N, with A positive definite a.e. and A, A^{-1} locally integrable. Let b and c be locally integrable, non-negative, real-valued functions on Ω, with c positive, a.e. Put $a(u, v) = \int_\Omega ((A\nabla u, \nabla v) + buv)\,dx$. We consider the boundary value problem $a(u, v) = \int_\Omega fvc\,dx$, for all $v \in C_0^\infty(\Omega)$, and the eigenvalue problem $a(u, v) = \lambda \int_\Omega uvc\,dx$, for all $v \in C_0^\infty(\Omega)$. Positivity of the solution operator for the boundary value problem, as well as positivity of the dominant eigenfunction (if there is one) and simplicity of the corresponding eigenvalue are proved to hold in this context.*

1. - Introduction.

Consider a second-order self-adjoint boundary value problem of the form

$$Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} + b(x)u = f\,, \qquad \text{in } \Omega\,, \tag{1.1}$$

$$Bu = \beta(x)u + \delta \frac{\partial}{\partial \nu} u = 0 \qquad \text{on } \partial\Omega\,, \tag{1.2}$$

Here Ω is a bounded region in R^N having smooth boundary, ν is given by $\nu(x) = A(x)\,n(x)$ where $A(x) = (a_{ij}(x))_{i,j=1,\dots,N}$ and $n(x)$ is the unit outward normal to $\partial\Omega$ at x; $\beta(x)$ is a real-valued function on $\partial\Omega$ and δ is a constant, and either $\beta(x) \equiv 1$ and $\delta = 0$ or $\beta(x) \geqslant 0$ and $\delta = 1$. Suppose that Ω is connected and that L is uniformly elliptic in Ω. If $b(x)$ and $\beta(x)$ do not both vanish identically (on Ω and $\partial\Omega$ respectively) then the differential operator $\mathfrak{L}$ on $L^2(\Omega)$ determined by L and B is positive definite and has a compact inverse. Under these assumptions, together with the classical smoothness conditions on the boundary $\partial\Omega$ and on the coefficients in L and B, it follows from the maximum principle that the Green's function G for $\mathfrak{L}$

(*) Entrata in Redazione il 13 giugno 1973.

(1) Research partially supported by the National Science Foundation under Grant GP-21512.

(2) Research Grant-DA-AROD-31-124-71-G17 Army Research Odce (Durham).

(3) Partially supported by the National Science Foundation under Grant GP-28377.

satisfies

$$G(x, y) > 0 , \quad x \neq y , \quad x, y \in \Omega .$$

Under the same assumptions, the least eigenvalue of the eigenvalue problem

$$Lu = \lambda c(x) u \quad \text{in } \Omega , \qquad Bu = 0 \quad \text{on } \partial\Omega , \tag{1.3}$$

where $c(x) > 0$ on Ω, is positive and simple and the corresponding eigenfunction is of one sign and does not vanish in Ω. Finally, this eigenfunction minimizes the Rayleigh quotient

$$J(u) = \Big[\int_\Omega \Big(\sum_{i,j=1}^N a_{ij}(x) u_{x_i} u_{x_j} + bu^2 \Big) dx + \int_{\partial\Omega} \beta(x) u^2(x) \, d\sigma \Big] \Big/ \Big(\int_\Omega u^2(x) c(x) \, dx \Big)$$

in the class of functions $u \in C^1(\bar{\Omega})$ that satisfy

$$Bu = 0 \quad \text{on } \Gamma = \{x \in \partial\Omega : \beta(x) > 0\} .$$

The purpose of this paper is to establish analogous results for the weak problems corresponding to (1.1), (1.2), and (1.3) when the coefficients are *not* necessarily continuous, when L is *not* necessarily uniformly elliptic, and when Ω is *not* necessarily either bounded or smoothly bounded. For problems of this generality there is available neither a strong maximum principle nor, even when $b(x) \equiv 0$, a Harnack inequality (see however the remark following the proof of Theorem 4.1). In fact we obtain our results not by a local analysis of solutions of (1.1) but rather by analysis of the properties of the Sobolev type function spaces naturally associated with (1.1), (1.2). We are primarily interested in the Dirichlet problem, and the hypotheses which we impose are too weak to permit formulation of general self-adjoint boundary conditions. Thus we do not attempt here to treat boundary conditions of the generality of those discussed above. Our results however do apply to mixed boundary conditions consisting of the Dirichlet condition on a portion of the boundary and natural boundary conditions on the remainder of the boundary. Formally, such boundary conditions can be written

$$u = 0 \quad \text{on } \Gamma_1 \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_2 , \tag{1.4}$$

where ν is as above, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\Gamma_1 \cup \Gamma_2 = \Omega$.

The relation between certain of our methods and the methods used in [3] should be emphasized. This connection is explained further in the remarks following the proofs of Lemmas 3.6 and 4.3.

For the classical existence and uniqueness theory of (1.1), (1.2) see Miranda [18]. Some references for positivity properties of solutions of (not necessarily self-adjoint) second order boundary value problems are [3], [8], [22], [27]. The indicated properties of the first eigenvalue and its corresponding eigenfunction are proved, at least for special cases, in [8], [11], [31]. In general these follow from the theory of positive operators [13], [14], [15], although the references cited generally make over-restrictive hypotheses which, in particular, rule out the Dirichlet boundary conditions; see however the remark on page 923, [13].

Some sources in which elliptic equations are treated under boundedness and ellipticity conditions similar to (but in all cases somewhat stronger than) ours are KRUZKOV, [16], MURTHY and STAMPACCHIA, [21], and TRUDINGER, [29] and [30]. These authors treat non-self-adjoint operators and although they are concerned with problems essentially different from those which are our main concern, there is some overlap of ideas between our work and theirs. In fact we have been guided somewhat in our choice of notation by [30]. We note that under their somewhat stronger assumptions together with some further additional hypotheses, the HARNACK inequalities of KRUZKOV [16] and TRUDINGER [29] can be used to prove a positivity result of the sort we prove here. See the remark following the proof of Theorem 4.1.

The original motivation for proving the results in this paper came from certain problems arising in connection with the work [7] on uniqueness of positive solutions of quasilinear elliptic boundary value problems. Indeed the main result of [7], Theorem 1, can be regarded as a non-linear analogue of Theorem 5.1 below.

2. – Preliminaries.

Let Ω be a connected open set in R^N. Below we shall use the following conventions and notations. First, since such distinctions are not critical for our purpose, we shall not explicitly distinguish between an equivalence class of functions (with respect to equality almost everywhere) and a representative of such an equivalence class. By a subset of Ω we will always understand a measurable subset; set inclusions and set inequalities are to be understood as holding to within a set of measure zero. Finally, an inequality asserted for a function f on a set E is to be understood as holding almost everywhere on E.

We will denote Lebesgue measure by μ; the characteristic function, defined on Ω, of the set $E \subseteq \Omega$ will be denoted by χ_E. For a measurable function f defined on Ω,

$$s(f) = \{x \in \Omega : f(x) \neq 0\} .$$

Following a standard notation we will let $H^{1,1}_{\text{loc}}(\Omega)$ denote the space of real valued functions which are locally of class L^1 in Ω and are locally strongly L^1 differentiable. For $u \in H^{1,1}_{\text{loc}}(\Omega)$, ∇u will have the obvious meaning.

LEMMA 2.1. – *Let* $u \in H^{1,1}_{\text{loc}}(\Omega)$.

a) If $u(x) = \text{const.}$ *a.e. on a measurable set* $G \subseteq \Omega$ *then* $\nabla u = 0$ *a.e. on* G.

b) If Ω' is a connected open subset of Ω and

$$\nabla u = 0 \quad \text{a.e. in } \Omega',$$

then

$$u(x) = c \quad \text{a.e. in } \Omega',$$

for some constant c.

c) $|u| \in H^{1,1}_{\mathrm{loc}}(\Omega)$ *and*

$$\nabla|u| = \operatorname{sgn} u \, \nabla u \quad \text{a.e. in } \Omega.$$

Proof. – Assertion *a*) is Theorem 3.2.2 on page 69 of [20].

Assertion *b*) follows readily from the fact that a distribution on Ω' whose distribution gradient is zero is a function constant almost everywhere, [12], [24].

Finally, assertion *c*) follows from a chain rule given in [17], since the function $g(x, t) = |t|$ satisfies the hypotheses of Theorem 2.1 of that paper and $|u|(x) = g(x, u(x))$.

The space $H^{1,1}_{\mathrm{loc}}(\Omega)$, with its natural topology, is a Frechet space; the collection of all sets of the form $\{u \in H^{1,1}_{\mathrm{loc}}(\Omega) : \int_G (|\nabla u| + |u|)\, dx < \varepsilon\}$ where $\varepsilon > 0$ and G is bounded, $\bar{G} \subseteq \Omega$, forms a basis for the neighborhoods of zero in this topology. A family $\{N_n\}$ of semi-norms on $H^{1,1}_{\mathrm{loc}}(\Omega)$ is a *complete family of semi-norms* for $H^{1,1}_{\mathrm{loc}}(\Omega)$ if the set $\{u \in H^{1,1}_{\mathrm{loc}}(\Omega) : N_n(u) < \varepsilon\}$ is open for each n and each $\varepsilon > 0$, and the totality of sets of this form is a subbasis for the neighborhoods of zero in $H^{1,1}_{\mathrm{loc}}(\Omega)$. For example, one complete countable family of semi-norms is given by

$$N_n(u) = \int_{G_n} (|\nabla u| + |u|)\, dx, \quad n = 1, 2, \ldots, \tag{2.1}$$

where $\{G_n\}$ is a countable cover for Ω consisting of bounded open sets G_n with $\bar{G}_n \subseteq \Omega$, $n = 1, 2, \ldots$. If the sequence $\{G_n\}$ is increasing then the semi-norms given by (2.1) satisfy

$$N_n(u) \leqslant N_m(u), \quad u \in H^{1,1}_{\mathrm{loc}}(\Omega), \tag{2.2}$$

for $n, m = 1, 2, \ldots$, and $m \geqslant n$.

Remark 1. – If $\{N_n\}$ is a countable family of semi-norms satisfying (2.2), if $\{N'_n\}$ is any other countable family of semi-norms, and if there exist constants k_n, K_n, $n = 1, 2, \ldots$, such that for $u \in H^{1,1}_{\mathrm{loc}}(\Omega)$ $N'_n(u) \leqslant k_n N_n(u)$ for all n while $N_n(u) \leqslant K_n \cdot N'_n(u)$ for all sufficiently large n, then $\{N'_n\}$ is a complete family of semi-norms for $H^{1,1}_{\mathrm{loc}}(\Omega)$ provided $\{N_n\}$ is.

Remark 2. – If $\{N_n\}$ is a complete family of semi-norms on $H^{1,1}_{\mathrm{loc}}(\Omega)$ and if T is a linear mapping from a normed linear space Z into $H^{1,1}_{\mathrm{loc}}(\Omega)$ then it is easily seen that T is continuous if and only if it is bounded with respect to each N_n, i.e. if and only if for each n there exists λ_n such that $N_n(Tu) \leqslant \lambda_n \|u\|_Z$ for all $u \in Z$. Such a map T is uniformly continuous and therefore has a unique continuous linear extension, $\hat{T}: \hat{Z} \to H^{1,1}_{\text{loc}}(\Omega)$, to the completion $\hat{Z}$ of Z.

Finally, a topological linear space X contained as a linear manifold in $H^{1,1}_{\text{loc}}(\Omega)$ will be called *stronger than* $H^{1,1}_{\text{loc}}(\Omega)$ if the natural embedding $e: X \to H^{1,1}_{\text{loc}}(\Omega)$ is continuous.

LEMMA 2.2. – *Let $\{G_n\}$ be an increasing sequence of bounded, smoothly bounded connected open subsets of Ω, with $\bar{G}_n \subseteq \Omega$, $n = 1, 2, \ldots$, $\Omega = \bigcup\limits_{n=1}^{\infty} G_n$. Let $S \subseteq \Omega$ be a measurable set of positive measure. Then $\{N'_n\}$, where*

$$N'_n(u) = \int\limits_{G_n} |\nabla u|\, dx + \int\limits_{S \cap G_n} |u|\, dx\,, \tag{2.3}$$

is a complete family of semi-norms for $H^{1,1}_{\text{loc}}(\Omega)$.

PROOF. – It is clear that each N'_n is a semi-norm. Thus by Remark 1 above it suffices to prove that for all sufficiently large n, N'_n is equivalent to the semi-norm N_n, given by (2.1). Suppose n is sufficiently large that $\mu(S \cap G_n) > 0$, but that N'_n is not equivalent to N_n. Then there is a sequence $\{u_k\}$ in $H^{1,1}_{\text{loc}}(\Omega)$ such that

$$N_n(u_k) = 1\,, \qquad k = 1, 2, \ldots \tag{2.4}$$

and

$$\lim_{k \to \infty} N'_n(u_k) = 0\,. \tag{2.5}$$

Let u'_k denote the restriction of u_k to G_n, $k = 1, 2, \ldots$. Then $N_n(u_k)$ is just the $H^{1,1}(G_n)$ norm of u'_k, so that by (2.4) and the Sobolev embedding theorem [20, p. 75] we can assume the original sequence to have been chosen so that $\{u'_k\}$ is convergent in $L^1(G_n)$. Then from (2.3) and (2.5) it follows that $\{u'_k\}$ is convergent in $H^{1,1}(G_n)$, say to u'_0. Thus by (2.4)

$$\int\limits_{G_n} |\nabla u'_0|\, dx + \int\limits_{G_n} |u'_0|\, dx = 1\,, \tag{2.6}$$

while by (2.5),

$$\int\limits_{G_n} |\nabla u'_0|\, dx + \int\limits_{S \cap G_n} |u'_0|\, dx = 0\,. \tag{2.7}$$

The latter equation implies that $\nabla u'_0(x) = 0$ a.e. in G_n, and consequently, by Lemma 2.1 *b*) that $u'_0(x) = \text{const.}$ a.e. in G_n. However (2.7) also implies that $u'_0(x) = 0$ a.e. in $S \cap G_n$ and thus that $u'_0(x) = 0$ a.e. in G_n, which contradicts (2.6). Thus the lemma is proved.

Let A be a measurable $N\times N$ real matrix valued function on Ω and let b be a measurable real valued function on Ω and assume that A is a.e. self-adjoint and positive definite while b is a.e. non-negative. Let

$$W^{1,2}(A, b, \Omega) = \left\{u \in H^{1,1}_{\mathrm{loc}}(\Omega): \int_\Omega \big((A\nabla u, \nabla u) + bu^2\big)\,dx < \infty\right\}.$$

In what follows $W^{1,2}(A, b, \Omega)$ and linear manifolds in it are regarded as inner product spaces with the semi-definite inner product

$$\langle u, v\rangle = \int_\Omega \big((A\nabla u, \nabla v) + buv\,dx\big)\,. \tag{2.8}$$

Thus, a linear manifold $X \subseteq W^{1,2}(A, b, \Omega)$ is a pre-Hilbert space if $\langle .,.\rangle$ is positive definite on X; if in addition X is complete with respect to $\langle .,.\rangle$ then X is a Hilbert space.

We shall be interested in Hilbert spaces $X \subseteq W^{1,2}(A, b, \Omega)$ such that $C_0^\infty \subseteq X$, $C^\infty(\Omega) \cap X$ is dense in X, and X is stronger than $H^{1,1}_{\mathrm{loc}}(\Omega)$. (This last condition is necessary and sufficient in order that the Hilbert space $X \subseteq W^{1,2}(A, b, \Omega)$ be a μ-measurable functional Hilbert space in the sense of [3]; see also Lemma 3.4, below and the remark following.) If such spaces exist at all then it is not difficult to see that there exists a (unique) smallest one which is characterized by the property that it contains $C_0^\infty(\Omega)$ as a dense linear manifold.

DEFINITION 2.1. – If there exists a Hilbert space in $W^{1,2}(A, b, \Omega)$ which is stronger than $H^{1,1}_{\mathrm{loc}}(\Omega)$ and contains $C_0^\infty(\Omega)$ as a dense linear manifold then this uniquely determined space will be denoted by $H_0(A, b, \Omega)$. If there exists a Hilbert space in $W^{1,2}(A, b, \Omega)$ which is stronger than $H^{1,1}_{\mathrm{loc}}(\Omega)$ and contains both $C_0^\infty(\Omega)$ and $C^\infty(\Omega) \cap W^{1,2}(A, b, \Omega)$, the latter as a dense linear manifold, then this space, which is also uniquely determined, will be denoted by $H(A, b, \Omega)$.

The uniqueness assertions in the above definition are contained in Lemma 3.1 below. Criteria for $H_0(A, b, \Omega)$ and $H(A, b, \Omega)$ to be defined are given in [30]. Similar criteria, appropriate to the context of this paper, will be developed in section 3.

Another property which we shall generally require of the spaces under consideration here is that they be vector lattices with respect to almost everywhere pointwise order. This is equivalent to invariance of the space under the non-linear mapping $u \to |u|$. Since it is this form of the lattice property which is most important for our purposes we shall invariably refer to it in this version and simply call the reader's attention, at this point, to the fact that this property is equivalent to the lattice property.

3. – The space $W^{1,2}(A, b, \Omega)$.

Suppose now that A and b are as in § 2. We first give a criterion for a pre-Hilbert space in $W^{1,2}(A, b, \Omega)$ to have a concrete completion in $W^{1,2}(A, b, \Omega)$ which is stronger than $H^{1,1}_{\mathrm{loc}}(\Omega)$.

LEMMA 3.1. – *Let Z be a pre-Hilbert space in $W^{1,2}(A, b, \Omega)$ which is stronger than $H^{1,1}_{loc}(\Omega)$. Then there exists a unique Hilbert space $X \subseteq W^{1,2}(A, b, \Omega)$ such that X is stronger than $H^{1,1}_{loc}(\Omega)$ and Z is a dense linear manifold in X.*

REMARK 3. – If b is positive on a set of positive measure then X is unique even without its being required to be stronger than $H^{1,1}_{loc}(\Omega)$; in general, however, this is not true.

PROOF. – Let e denote the natural embedding $Z \to H^{1,1}_{loc}(\Omega)$; by hypothesis e is continuous. Let $\sigma: Z \to L^2(\Omega, R^{N+1})$ be defined by

$$\sigma(u) = (A^{\frac{1}{2}} \nabla u, b^{\frac{1}{2}} u) , \tag{3.1}$$

then σ is an isometry and thus $\overline{\sigma(Z)}$ can be identified with the abstract completion of Z. Note that if

$$(g_1, \ldots, g_{N+1}) \in \overline{\sigma(Z)}$$

then

$$g^{N+1}(x) = 0 \qquad \text{a.e. on } \Omega \setminus S_0 \tag{3.2}$$

where $S_0 = \{x \in \Omega : b(x) > 0\}$. We now define $\tau: \sigma(Z) \to H^{1,1}_{loc}(\Omega)$ as follows

$$\tau = e\sigma^{-1} . \tag{3.3}$$

We clearly have, for $g \in \sigma(Z)$ and $\omega = \tau g$,

$$\nabla\omega = A^{-\frac{1}{2}} \tilde{g} \quad \text{a.e. on } \Omega, \qquad \omega = b^{-\frac{1}{2}} g_{N+1} \quad \text{a.e. on } S_0, \tag{3.4}$$

where $\tilde{g} = (g_1, \ldots, g_N)$. Since τ, defined by (3.3), is a continuous linear map it has, by Remark 2, a unique continuous extension $\bar{\tau}: \overline{\sigma(Z)} \to H^{1,1}_{loc}(\Omega)$. Now suppose that $g \in \overline{\sigma(Z)}$ and let $\{g^n\}$ be a sequence in $\sigma(Z)$ converging to g, with $\omega_n = \tau(g^n)$, $n = 1, 2, \ldots$, so that $\{\omega_n\}$ converges to $\omega = \bar{\tau}(g)$ in $H^{1,1}_{loc}(\Omega)$. We can assume, moreover, that $\{g^n\}$ was selected so that $g^n(x) \to g(x)$, $\nabla\omega_n(x) \to \nabla\omega(x)$ and $\omega_n \to \omega(x)$ for almost all $x \in \Omega$. It then follows that (3.4) holds for $g \in \overline{\sigma(Z)}$, $\omega = \bar{\tau}(g)$. We now show that $\bar{\tau}$ is one-to-one. Indeed if $\omega = \bar{\tau} g$ and $\omega' = \bar{\tau} g'$ and $\omega = \omega'$ then clearly $g_i = g'_i$ a.e. on Ω for $i = 1, 2, \ldots, N$ and $g_{N+1} = g'_{N+1}$ on S_0, so by (3.2), $g = g'$. Using again the relation (3.4) for $g \in \overline{\sigma(Z)}$, $\omega = \bar{\tau} g$, we conclude that $\bar{\tau}(\overline{\sigma(Z)}) \subseteq W^{1,2}(A, b, \Omega)$, i.e. that $\langle \omega, \omega \rangle$ as defined by (2.8), is finite for $\omega \in \bar{\tau}(\overline{\sigma(Z)})$. Let X be the subspace of $W^{1,2}(A, b, \Omega)$ whose elements are just the elements of $\bar{\tau}(\overline{\sigma(Z)})$. It is easily seen from this construction that the isometry σ extends to a surjective isometry $\tilde{\sigma}: X \to \overline{\sigma(Z)}$, with

$$\tilde{\sigma}(u) = (A^{\frac{1}{2}} \nabla u, b^{\frac{1}{2}} u) , \qquad u \in X.$$

Thus X is a Hilbert space with Z dense in X. The natural embedding $\tilde{e}: X \to H^{1,1}_{loc}(\Omega)$ satisfies $\tilde{e} = \bar{\tau}\tilde{\sigma}$ so that $\tilde{e}$ is continuous and thus X is stronger than $H^{1,1}_{loc}(\Omega)$. On the other hand, if X is a Hilbert space in $W^{1,2}(A, b, \Omega)$ which is stronger than $H^{1,1}_{loc}(\Omega)$

and if Z is dense in X then the elements of X are $H^{1,1}_{\mathrm{loc}}(\Omega)$-limits of Cauchy sequences in the pre-Hilbert space Z. It follows that X is uniquely determined.

LEMMA 3.2. – (*a*) *In order that* $C_0^\infty(\Omega) \subseteq W^{1,2}(A, b, \Omega)$ *it is necessary and sufficient that* $\|A\|$, $b \in L^1_{\mathrm{loc}}(\Omega)$.

(*b*) *A linear manifold* $Z \subseteq W^{1,2}(A, b, \Omega)$ *satisfies the hypotheses of Lemma* 3.1 *provided that either of the following holds*:

(i) $Z = C_0^\infty(\Omega)$, *Ω is bounded, and* $\|A^{-1}\| \in L^1(\Omega)$.

(ii) *Z is arbitrary,* $\|A^{-1}\| \in L^1_{\mathrm{loc}}(\Omega)$ *and b is positive on a set of positive measure.*

PROOF. – The sufficiency of the condition in assertion (*a*) is obvious. Conversely, suppose that $C_0^\infty(\Omega) \subseteq W^{1,2}(A, b, \Omega)$. Then necessarily $(A\nabla u, \nabla u) + bu^2 \in L^1(\Omega)$ for each $u \in C_0^\infty(\Omega)$. First we show that this implies $b \in L^1_{\mathrm{loc}}(\Omega)$. To this end let G be an arbitrary open set in Ω with $\bar{G}$ compact, $\bar{G} \subseteq \Omega$. Let $u_0 \in C_0^\infty(\Omega)$ with $u_0 \equiv 1$ on G. Then $u_0 \in W^{1,2}(A, b, \Omega)$ implies that b is integrable over G. Since G was arbitrary it follows that $b \in L^1_{\mathrm{loc}}(\Omega)$.

Next we show that the diagonal elements of A belong to $L^1_{\mathrm{loc}}(\Omega)$. Let G and u_0 be as above and let $u(x) = x_i u_0(x)$ so that $\partial u/\partial x_i = 1$ on G, $\partial u/\partial x_j = 0$ on G where i is a fixed index $1 \leqslant i \leqslant N$, $j \neq i$. Then on G $(A\nabla u, \nabla u) + bu^2 = a_{ii} + bu^2$. Since we already know that $b \in L^1_{\mathrm{loc}}(\Omega)$, it follows that a_{ii} must be integrable over G and hence, since G was arbitrary, $a_{ii} \in L^1_{\mathrm{loc}}(\Omega)$. Finally let G and u_0 be as before and let $u(x) = (x_i + x_j)u_0(x)$, so that on G $\partial u/\partial x_i = \partial u/\partial x_j = 1$ and $\partial u/\partial x_k = 0$ for $k \neq i, j$, where i and j are distinct indices $1 \leqslant i, j \leqslant N$. Then on G, $(A\nabla u, \nabla u) + bu^2 = a_{ii} + a_{jj} + 2a_{ij} + bu^2$, and thus we conclude that $a_{ij} \in L^1_{\mathrm{loc}}(\Omega)$.

Suppose now that condition (i) of (*b*) is satisfied. From Schwarz's inequality and (2.8) each $u \in C_0^\infty(\Omega)$ satisfies

$$\int_\Omega |\nabla u|\, dx \leqslant \Big(\int_\Omega \|A^{-1}\|\, dx\Big)^{\frac12} \langle u, u\rangle^{\frac12}. \tag{3.5}$$

Since Ω is bounded there exists $\varrho > 0$ such that

$$\varrho \int_\Omega (|\nabla u| + |u|)\, dx < \int_\Omega |\nabla u|\, dx\,, \qquad u \in H_0^{1,1}(\Omega), \tag{3.6}$$

see [20, p. 69]. Thus, combining (3.5) and (3.6) we see that (i) implies that $C_0^\infty(\Omega) \cap W^{1,2}(A, b, \Omega)$ is a pre-Hilbert space stronger than $H_0^{1,1}(\Omega)$, hence also stronger than $H^{1,1}_{\mathrm{loc}}(\Omega)$.

Now suppose that (ii) is satisfied and let S be a measurable set of positive measure such that

$$b(x) \geqslant m > 0\,, \qquad \text{for } x \in S,$$

for some positive constant m. We can assume that S is contained in a compact subset of Ω. Let G be a bounded, open subset of Ω with

$$S \subseteq G \subseteq \bar{G} \subseteq \Omega ,$$

then for $u \in W^{1,2}(A, b, \Omega)$,

$$\int_G |\nabla u|\, dx + \int_S |u|\, dx \leqslant \left[\left(\int_G \|A^{-1}\|\, dx\right)^{\frac{1}{2}} + m^{-\frac{1}{2}}(\mu(S))^{\frac{1}{2}}\right] \langle u, u\rangle^{\frac{1}{2}} .$$

It readily follows from Lemma 2.2 that $W^{1,2}(A, b, \Omega)$ is itself a Hilbert space stronger than $H^{1,1}_{\text{loc}}(\Omega)$, and thus any linear manifold $Z \subseteq W^{1,2}(A, b, \Omega)$ is a pre-Hilbert space stronger than $H^{1,1}_{\text{loc}}(\Omega)$.

As a consequence of Lemmas 3.1 and 3.2 we have the following.

LEMMA 3.3. – *Let* $\|A\|$, $b \in L^1_{\text{loc}}(\Omega)$. *If* $\|A^{-1}\| \in L^1_{\text{loc}}(\Omega)$ *and* b *is positive on a set of positive measure then both* $H_0(A, b, \Omega)$ *and* $H(A, b, \Omega)$ *are defined. If* Ω *is bounded and* $\|A^{-1}\| \in L^1(\Omega)$ *then* $H_0(A, b, \Omega)$ *is defined.*

PROOF. – It is immediate from Lemmas 3.1 and 3.2 that under the general hypothesis and either of the two alternative conditions of the above assertion, $H_0(A, b, \Omega)$ is defined. Under the first of the alternative conditions $H(A, b, \Omega)$ is defined as the completion of $C^\infty(\Omega) \cap W^{1,2}(A, b, \Omega)$ in $W^{1,2}(A, b, \Omega)$.

LEMMA 3.4. – *Let* X *be a Hilbert space in* $W^{1,2}(A, b, \Omega)$ *which is stronger than* $H^{1,1}_{\text{loc}}(\Omega)$.

(*a*) *If* $\{u_n\}$ *is a convergent sequence in* X *with limit* u *then there exists a subsequence* $\{u_{n_k}\}$ *such that*

$$\lim_{k\to\infty} u_{n_k}(x) = u(x) \qquad a.e. \text{ in } \Omega \tag{3.7}$$

and

$$\lim_{k\to\infty} \nabla u_{n_k}(x) = \nabla u(x) \qquad a.e. \text{ in } \Omega . \tag{3.8}$$

(*b*) *If* $\{u_n\}$ *is a weakly convergent sequence in* X *and if* (3.7) *holds for some subsequence* $\{u_{n_k}\}$ *then* u *is the weak limit of* $\{u_n\}$.

REMARK 4. – The first assertion of Lemma 3.4 has the following converse. If X is any Hilbert space in $W^{1,2}(A, b, \Omega)$ and if every convergent sequence $\{u_n\}$ in X with limit u has a subsequence $\{u_{n_k}\}$ satisfying (3.7), then X is stronger than $H^{1,1}_{\text{loc}}(\Omega)$. Indeed, if X has this property then one can verify immediately that the graph of the natural imbedding $X \to H^{1,1}_{\text{loc}}(\Omega)$ is closed and therefore that this imbedding is continuous.

PROOF. – Convergence of $\{u_n\}$ to u in X implies convergence of $\{u_n\}$ to u in $H^{1,1}_{\text{loc}}(\Omega)$, and from this assertion (*a*) readily follows.

To prove assertion (b) we can just as well assume that the full sequence is a.e. convergent. By Mazur's theorem there is a sequence $\{w_n\}$ whose terms are convex combinations of the u_n,

$$w_n = \sum a_{n,l} u_l\,, \qquad a_{n,l} \geqslant 0,\ \sum a_{n,l} = 1,$$

such that $\{w_n\}$ converges strongly in X to the weak limit of the sequence $\{u_n\}$. Since u belongs to the closed convex hull of the set $\{u_n, u_{n+1} \ldots\}$, for any value of n, one can construct the sequence $\{w_n\}$ in such a way that

$$a_{n,l} = 0 \qquad \text{for } l < n,$$

and then $\{w_n\}$ will converge almost everywhere to u. As in (a), the sequence $\{w_n\}$ is convergent in $H^{1,1}_{\text{loc}}(\Omega)$ and its limit in this space clearly must coincide with its a.e. limit. The X- and the $H^{1,1}_{\text{loc}}(\Omega)$-limits of the sequence $\{w_n\}$ coincide and this completes the proof.

Lemma 3.5. – *Whenever* $u \in W^{1,2}(A, b, \Omega)$ *then so is* $|u|$; *hence* $W^{1,2}(A, b, \Omega)$ *is a vector lattice. Moreover,* $\langle u, u\rangle = \langle |u|, |u|\rangle$. *If* $H\,(A, b, \Omega)$ $\big(H_0(A, b, \Omega)\big)$ *is defined and* $u \in H(A, b, \Omega)$ $\big(u \in H_0(A, b, \Omega)\big)$ *then* $|u| \in H(A, b, \Omega)$ $\big(|u| \in H_0(A, b, \Omega)\big)$. *Furthermore whenever* X *is as in Lemma* 3.4 *and is invariant under the mapping* $u \to |u|$, (*so that* X *is a vector lattice*) *then that mapping is continuous and so are the mappings* $u \to u_+$, $u \to u_-$.

Remark 5. – An argument similar to that in the proof to follow shows that $u \in \in H(A, b, \Omega)$ $\big(u \in H_0(A, b, \Omega)\big)$ implies $f(u) \in H(A, b, \Omega)$ $\big(f(u) \in H_0(A, b, \Omega)\big)$ whenever f is uniformly Lipschitz continuous and $f(0) = 0$. However the proof of continuity of $u \to f(u)$ is more involved and will be given elsewhere.

An immediate consequence of the last assertion of Lemma 3.5 is the following.

Corollary. – *Let* X *be as in Lemma* 3.4, *and let* X *be invariant under* $u \to |u|$. *If* X_1 *is a subspace of* X *and* V *is a dense linear manifold in* X_1 *which is invariant under* $u \to |u|$, *then* X_1 *itself is invariant under* $u \to |u|$.

Proof of Lemma 3.5. – For the first statement, note that $|u|$ has the same norm as u as follows from Lemma 2.1 (c) and (2.8). To prove the second assertion suppose first that $u \in C^\infty(\Omega)$ $\big(u \in C_0^\infty(\Omega)\big)$. Then $|u|$ can be approximated uniformly by a sequence $\{w_n\}$ in $C^\infty(\Omega)$ $\big(C_0^\infty(\Omega)\big)$ with

$$|\text{grad}\, w_n(x)| \leqslant |\text{grad}\, u(x)|\,, \qquad x \in \Omega. \tag{3.9}$$

This can be done for example by taking $w_n(x) = f_n\big(u(x)\big)$, where $f_n \in C^\infty(R)$, $f_n(0) = 0$, $|f_n'| \leqslant 1$ and the sequence $\{f_n\}$ converges uniformly to $f = |\ |$. The sequence $\{w_n\}$ is clearly bounded in $H(A, b, \Omega)$ because of (3.9) and the fact that $|f(t)| \leqslant |t|$, and thus can be assumed to converge weakly in $H(A, b, \Omega)$. In view of Lemma 3.4 (*b*), this shows that $|u| \in H(A, b, \Omega)$ $\big(|u| \in H_0(A, b, \Omega)\big)$.

For arbitrary $u \in H(A, b, \Omega)$, $\big(u \in H_0(A, b, \Omega)\big)$ we first approximate u in $H(A, b, \Omega)$ by a sequence $\{w_n\}$ in $C^\infty(\Omega)$ $\big($in $C_0^\infty(\Omega)\big)$. By Lemma 3.4, (*a*) $\{w_n\}$ can be assumed to converge almost everywhere in Ω, and thus, by what we have already shown, $\{|w_n|\}$ is a bounded sequence in $H(A, b, \Omega)$ $\big($in $H_0(A, b, \Omega)\big)$ converging almost everywhere in Ω to $|u|$. Using Lemma 3.4 (*b*) as before, we conclude that $|u| \in H(A, b, \Omega)$ $\big(|u| \in H_0(A, b, \Omega)\big)$. That u and $|u|$ have the same norm follows as before.

The continuity of the mapping $u \to |u|$ is proved as follows. Let $\{w_n\}$ be a sequence converging to u in X. Then by Lemma 3.4 (*a*) every subsequence of $\{w_n\}$ has a subsequence converging to u a.e. on Ω. Thus every subsequence of $\{|w_n|\}$ has a subsequence converging to $|u|$ a.e. However, since $\{|w_n|\}$ is a bounded sequence in X, it follows from Lemma 3.4 (*b*) that $\{|w_n|\}$ converges weakly to $|u|$. Finally the facts

$$|w_n| \to |u| \quad \text{weakly in } X,$$
$$\big\||u|\big\| = \|u\| = \lim \big\||w_n|\big\|,$$

together imply that actually the convergence of $\{|w_n|\}$ to $|u|$ is strong convergence in X.

LEMMA 3.6. – *Let X be as in Lemma* 3.4 *and suppose in addition that X is invariant under the mapping $u \to |u|$. If $u \in X$ is non-negative as a linear functional,* i.e. *if*

$$\langle v, u\rangle \geqslant 0 \tag{3.10}$$

for all $v \in X$ with $v(x) \geqslant 0$, a.e. *on Ω, then*

$$u(x) \geqslant 0 \quad \text{a.e. on } \Omega. \tag{3.11}$$

PROOF. – Since $u(x) \leqslant |u(x)|$ a.e. on Ω, the positivity of u as a linear functional implies

$$\langle u, u\rangle \leqslant \langle |u|, u\rangle\,. \tag{3.12}$$

However since u and $|u|$ have the same X-norm, the Schwarz inequality implies

$$\langle |u|, u\rangle \leqslant \langle u, u\rangle$$

with equality only if u and $|u|$ are proportional, i.e. only if u is of fixed sign. The conclusion of the lemma then follows from (3.12).

REMARK 6. – If K denotes the positive cone in X, i.e. the set of functions in X which are a.e. positive on Ω, and if K^* denotes the dual cone

$$K^* = \{u \in X : \langle v, u\rangle \geqslant 0 \ \text{ for all } \ v \in K\}\,,$$

then Lemma 3.6 asserts that

$$K^* \subseteq K .$$

If X is any Hilbert space, K a proper closed convex cone in X, $\geqslant$ the partial order induced in X by K, and K^* the dual cone then the following are equivalent:

(i) *For every* $u \in X$ *there exists* $\tilde{u} \in K$ *such that* $\tilde{u} \geqslant \pm u$, $\|\tilde{u}\| \leqslant \|u\|$, *where* $\|\ \|$ *denotes the* X*-norm.*

(ii) *For every* $u \in K$ *there exists* $u' \in K$ *such that* $u' \geqslant u$, $\|u'\| \leqslant \|u\|$.

(iii) $K^* \subseteq K$.

For a proof of the non-trivial implication, (iii) implies (i), see the proof of Theorem 1, [3].

Note that if for some set E $\chi_E u \in X$ for every $u \in X$ then the mapping $u \to \chi_E u$ would necessarily be an orthogonal projection, since by Lemma 2.1 (*a*) and (2.8),

$$\langle \chi_E u, v \rangle = \int_E \big((A \nabla u, \nabla v) + buv \big)\, dx .$$

Concerning such sets E we have the following result.

LEMMA 3.7. – *Let* X *be a Hilbert space in* $W^{1,2}(A, b, \Omega)$ *with* $C_0^\infty(\Omega) \subseteq X$. *If* E *is a measurable subset of* Ω *with* $\mu(E) > 0$, *and if the mapping* $u \to \chi_E u$, *where* χ_E *denotes the characteristic function of* E, *is an orthogonal projection on* X, *then* $\mu(\Omega \setminus E) = 0$.

PROOF. – Assume $u \to \chi_E u$ is an orthogonal projection and $\mu(\Omega \setminus E) > 0$. Let Ω' be a connected, open subset of Ω having compact closure in Ω and such that $\mu(E \cap \Omega') > 0$ and $\mu(\Omega' \setminus E) > 0$. Let $\varphi \in C_0^\infty(\Omega)$ with $\varphi(x) \equiv 1$ on Ω', so that $\varphi \in X$. Then if $\chi_E \varphi \in X$, we have by Lemma 2.1 (*a*) that $\nabla(\chi_E \varphi) = 0$ a.e. on Ω', and thus by Lemma 2.1 (*c*), $\chi_E \varphi$ is constant a.e. on Ω', which contradicts $\mu(\Omega' \setminus E) > 0$. We must therefore have $\mu(\Omega \setminus E) = 0$ and the result is proved.

4. – The Green's operator.

Let c be a real-valued measurable function on Ω with

$$c(x) > 0 \qquad \text{a.e. on } \Omega. \tag{4.1}$$

For brevity we shall denote by Y the weighted real L^2 space with weight c

$$Y = L^2(\Omega, c(x)\, dx) .$$

The inner product in Y will be denoted $(.,.)$

$$(f, g) = \int_\Omega f(x)\, g(x)\, c(x)\, dx\,.$$

In what follows X will always denote a Hilbert space in $W^{1,2}(A, b, \Omega)$. We shall say that such a space is *admissible* if it satisfies the following three conditions

I) X is a Hilbert space in $W^{1,2}(A, b, \Omega)$ which is stronger than $H^{1,1}_{\mathrm{loc}}(\Omega)$.

II) X is invariant under the mapping $u \to |u|$, i.e. X is a vector lattice.

III) If E is a measurable subset of Ω with $\mu(E) > 0$ and if $\chi_E u \in X$ whenever $u \in X$ then $\mu(\Omega \setminus E) = 0$.

We will say that the pair (X, Y) is *admissible* if X is admissible, Y is as above, the functions in X have finite Y-norm, i.e.

$$\int_\Omega u^2(x)\, c(x)\, dx < \infty\,, \qquad \text{for all } u \in X, \tag{4.2}$$

and X, regarded as a linear manifold in Y, is dense in Y.

It follows from Lemmas 3.5 and 3.7 that the spaces $H(A, b, \Omega)$ or $H_0(A, b, \Omega)$, whenever they are defined, are admissible. We will not discuss in detail the various conditions which imply (4.2) but only record the following trivial criterion for the pair (X, Y) to be admissible.

LEMMA 4.1. – *If X is admissible, if $C_0^\infty(\Omega) \subseteq X$ and if there exists a constant M such that*

$$c(x) \leqslant M b(x)\,, \qquad \text{a.e. } on\ \Omega,$$

then the pair (X, Y) is admissible.

In the remainder of this section we will always assume that the pair (X, Y) is admissible. We will denote by i the natural injection of X into Y.

LEMMA 4.2. – *The operator i is continuous and has dense range in Y. The adjoint operator i^*. $Y \to X$ is continuous, injective and has dense range in X.*

PROOF. – Since by condition I above X is stronger than $H^{1,1}_{\mathrm{loc}}(\Omega)$, the elements of X are (equivalence classes of) measurable functions (a fact which we have already implicitly assumed in the definition of an admissible pair). Therefore, in view of (4.2), i is well-defined with domain X; by (4.1), i is indeed an injection. Further, because of Condition I it follows from Lemma 3.4 (*a*) that the graph of i is closed, and therefore that i is continuous. That i has dense range in Y follows immediately from the admissibility of (X, Y). The assertions concerning i^* follow immediately, by duality, from the properties of i.

We note that, for $f \in Y$, $u = i^* f$ is the solution of the weak problem $\langle u, v\rangle = (f, iv)$ or

$$\int_\Omega ((A\nabla u, \nabla v) + buv)\,dx = \int_\Omega vfc(x)\,dx\,, \qquad \text{all } v \in X.$$

LEMMA 4.3. – *The operator i^* is non-negative, i.e. $f \in Y$ and $f(x) \geq 0$ a.e. on Ω imply $u(x) \geq 0$ a.e. on Ω, where $u = i^* f$.*

PROOF. – Let $f \in Y$, $f(x) \geq 0$ a.e. on Ω. Then if $u = i^* f$ we have

$$\langle u, v\rangle = (f, iv) \tag{4.3}$$

for all $v \in X$. Since the term on the right in (4.3) is non-negative when $v(x) \geq 0$ a.e. on Ω it follows that the solution u of (4.3) is non-negative as a linear functional on X. It follows immediately from Lemma 3.6 that $u(x) \geq 0$ a.e. on Ω.

REMARK 7. – Let $\mathcal{E}$ denote the collection of measurable subsets E of Ω with

$$\int_E c(x)\,dx < \infty\,.$$

Each $u \in X$ determines a function $\tilde{u}$ on $\mathcal{E}$ by

$$\tilde{u}(E) = \int_E u(x)\,c(x)\,dx\,. \tag{4.4}$$

Let $\tilde{X}$ denote the set $\{\tilde{u}: u \in X\}$ furnished with the inner product

$$[\tilde{u}, \tilde{v}] = \langle u, v\rangle\,, \qquad u, v \in X. \tag{4.5}$$

Then $\tilde{X}$ is a proper functional Hilbert space in the sense of [3]; (X is a μ-measurable functional Hilbert space in the sense of [3]) As a proper functional Hilbert space, $\tilde{X}$ has a reproducing kernel $\tilde{K}$ defined on $\mathcal{E} \times \mathcal{E}$, and it is easily seen that

$$\tilde{K}(E, E') = \int_E (i^* \chi_{E'})\,c(x)\,dx = \int_{E'} (i^* \chi_E)\,c(x)\,dx\,. \tag{4.6}$$

If $u \in X$ and $w = |u|$, then clearly $\tilde{w}(E) \geq \tilde{u}(E)$ for all $E \in \mathcal{E}$. In view of this, since $\tilde{X}$ is a space of real functions, it follows from Theorem 1 of [3] that $\tilde{K}$ is non-negative on $\mathcal{E} \times \mathcal{E}$, and this implies Lemma 4.3.

We now consider the operator $k = k_{X,Y}\colon Y \to Y$ defined by

$$k = ii^*\,. \tag{4.7}$$

LEMMA 4.4. – *The operator k is self-adjoint, positive definite and preserves non-negativity.*

PROOF. – The self adjointness of k is clear from (4.7). Positive definiteness follows from the injectivity of i^* (Lemma 4.2) and the identity

$$(kf, f) = \langle i^* f, i^* f\rangle .$$

Finally, the non-negativity follows from the non-negativity of i^*.

Recall that $s(f)$ denotes the set of points in Ω where f is not equal to zero.

LEMMA 4.5. – *Let $f \in Y$, f not identically zero, and*

$$f(x) \geqslant 0 \quad \text{on } \Omega.$$

Then the sequence $\{s(k^n f)\}$ is an increasing sequence with

$$\bigcup_{n=1}^{\infty} s(k^n f) = \Omega .$$

PROOF. – Let f be as above and suppose $G = s(f) \setminus s(kf)$. Put $f_1 = \chi_G f$ where χ_G is the characteristic function of G. From the non-negativity of k, since $0 < f_1 \leqslant f$ on Ω,

$$0 \leqslant kf_1 \leqslant kf \quad \text{on } \Omega,$$

and thus $kf_1 = 0$ on G. But by definiteness of k

$$(f_1, kf_1) > 0$$

unless $f_1 = 0$; thus we must have $f_1 = 0$, i.e. G of measure zero. The increasing character of the sequence $\{s(k^n f)\}$ obviously follows. Let now

$$F = \bigcup_{n=1}^{\infty} s(k^n f) , \qquad E = \Omega \setminus F.$$

Let $g \in Y$ be any non-negative function with $s(g) \subseteq E$, and suppose that $s(kg) \cap F$ has positive measure. Then for suitably large n, $s(kg) \cap s(k^n f)$ will also have positive measure; but this leads via

$$0 < (k^n f, kg) = (k^{n+1} f, g) = 0$$

to a contradiction. Thus $s(g) \subseteq E$ implies $s(kg) \subseteq E$, at least for non-negative g; but then the same immediately follows for arbitrary $g \in Y$ Let now P denote the orthogonal projection on Y defined by

$$Pg = \chi_E g .$$

We have shown that

$$kP = PkP\,,$$

and from this it follows that kP is self-adjoint and hence P and k commute:

$$kP = Pk\,.$$

Thus Y can be represented as the direct sum

$$Y = M \oplus N$$

where $g \in N$ if and only if

$$s(g) \subseteq E\,, \tag{4.8}$$

and $h \in M$ if and only if

$$s(h) \subseteq F\,, \tag{4.9}$$

and M and N are invariant manifolds for k. Since $s(ii^*g) = s(i^*g)$, (4.8) implies

$$s(i^*g) \subseteq E$$

and (4.9) implies

$$s(i^*h) \subseteq F\,,$$

so that, by (2.8) and Lemma 2.1 (*a*), i^*M and i^*N are orthogonal in X. Thus, since i^*Y is dense in X,

$$X = U \oplus V$$

where U is the closure of $i^*(M)$ and V is the closure of $i^*(N)$. By Lemma 3.4 the functions in U vanish on E and those in V vanish on F. This means, however, that the orthogonal projection of X onto U is given by $u \to \chi_F u$, but then, by condition III, we must have $\mu(E) = 0$ and the result is proved.

We now prove that k is *strictly* positive in the sense that $f \in Y$, $f \geqslant 0$ on Ω and $f \not\equiv 0$ implies $(kf)(x) > 0$ a.e. on Ω. For this we use Lemma 4.5 and the following device: we introduce an operator k_1 with the same properties as k and related to k by

$$k^{-1} = k_1^{-1} - I_Y \tag{4.10}$$

so that k may be expressed

$$k = k_1 + k_1^2 + \ldots \tag{4.11}$$

To this end we introduce the Hilbert space X_1 which is simply X furnished with the equivalent inner product

$$\langle u, v\rangle_1 = \langle u, v\rangle + (iu, iv)\,. \tag{4.12}$$

We denote by j the identification

$$j\colon X \to X_1\,,$$

and by i_1 the immersion $X_1 \to Y$. Then (4.12), written more formally, becomes

$$\langle u, v\rangle = \langle ju, jv\rangle_1 - (iu, iv)$$

or

$$\langle u, v\rangle = \langle j^* ju, v\rangle - \langle i^* iu, v\rangle\,,$$

so that

$$I_X = j^* j - i^* i\,, \tag{4.13}$$

where I_X is the identity on X. From (4.13), since $i_1 = ij^{-1}$, we deduce that

$$k_1^{-1} = i_1^{*-1} i_1^{-1} = i^{*-1} j^* j i^{-1} = i^{*-1} i^{-1} + I_Y = k^{-1} + I_Y\,.$$

Finally to justify (4.11) we note that, since k is self-adjoint

$$\|k\|^{-1} = \{\inf(k^{-1}f, f)\colon f \in \text{domain of } k^{-1},\ \|f\|_Y = 1\}\,,$$

and a similar formula holds for $\|k_1\|^{-1}$. Thus by (4.10)

$$\|k\|^{-1} = \|k_1\|^{-1} - 1$$

and hence

$$\|k_1\| = \|k\|/(1 + \|k\|)$$

so that $k_1 + \mu k_1^2 + \mu^2 k_1^3 + \ldots$ converges for $\mu\|k\| < 1 + \|k\|$, in particular for $\mu = 1$.

Since Lemmas 4.4 and 4.5 clearly apply to k_1, if $f \in Y$, f not identically zero, and $f \geqslant 0$ on Ω then from (4.11)

$$s(kf) = \bigcup_{n=1}^{\infty} s(k_1^n f) = \Omega$$

(again we emphasize that the equality is only to within sets of measure zero). We thus have proved the following.

THEOREM 4.1. – *The operator k is positive in the sense that if $f \in Y$, $f \geqslant 0$ and f is not zero almost everywhere then kf is positive almost everywhere on Ω.*

COROLLARY. – *Let f be a non-negative eigenfunction of k. Then f is positive almost everywhere. If f is any eigenfunction of k with*

$$kf = \|k\| f \tag{4.14}$$

then f or $-f$ is positive almost everywhere on Ω and $\|k\|$ is a simple eigenvalue of k.

PROOF. – That a non-negative eigenfunction of k must be positive almost everywhere follows immediately from Theorem 4.1.

Suppose now that (4.14) holds for some $f \neq 0$. Then since $|kf| \leq k|f|$,

$$\|k\| \|f\|^2 = (kf, f) \leq (k|f|, |f|) \leq \|k\| \|f\|^2 ,$$

so that by Schwarz's inequality $|f|$ is an eigenfunction of k. We must therefore have $f = \pm |f|$ since otherwise f_+ would be a non-negative eigenfunction of k vanishing on a set of positive measure in contradiction to the first assertion of the lemma.

Suppose finally that $\|k\|$ were not a simple eigenvalue of k, then there would be a second eigenfunction g of k, corresponding to the eigenvalue $\|k\|$ and orthogonal to f. This g would not be essentially of one sign, and thus this supposition leads to the same contradiction.

REMARK 8. – Suppose that Ω is bounded, $b = 0$ and

$$\|A^{-1}\| \in L^t(\Omega) , \qquad \|A^{-1}\| \|A\|^2 \in L^s(\Omega),$$

where $t, s \geq 1$ and

$$\frac{1}{t} + \frac{1}{s} < \frac{2}{N} .$$

Then in particular $\|A\|$, $\|A^{-1}\| \in L^1(\Omega)$ and thus, by Lemma 3.3, $H_0(A, 0, \Omega)$ is defined. If c is such that $(H_0(A, 0, \Omega), Y)$ is admissible (e.g. if $c \in L^{s_0}(\Omega)$, $1/s_0 + 1/t = 2/N$) then in this case the conclusion of Theorem 4.1 follows from Lemma 4.4 and a Harnack inequality proved by TRUDINGER [29, Theorem 4.1]. (Indeed in this case, when $f \geq 0$ on Ω, $f \neq 0$, then $u = i^* f$ has a positive lower bound on compact subsets of Ω.)

We next prove that if the operator k has a non-negative eigenfunction f, say

$$kf = \mu f$$

then necessarily

$$\|k\| = \mu .$$

We shall actually prove something more general—we note first however that in view of the Corollary to Theorem 4.1 and Lemma 4.4 we can assume that the eigenfunction f is positive a.e. on Ω. The following result is trivial if k is compact but the general result is more subtle and does not appear to be contained in the extensive literature concerning positive operators.

THEOREM 4.2. – *Let k be a self-adjoint, bounded operator on Y which preserves non-negativity. If $f \in Y$ and*

$$f(x) > 0\,, \qquad \text{a.e. } on\ \Omega,$$

and

$$(kf)(x) \leqslant \mu f(x)\,, \qquad \text{a.e. } on\ \Omega, \tag{4.15}$$

then

$$\|k\| \leqslant \mu\,. \tag{4.16}$$

If equality holds in (4.15), i.e. *if f is an eigenfunction of k, then $\|k\| = \mu$.*

PROOF. – Suppose first that k is compact, then by the theory of compact self-adjoint operators k has an eigenfunction g with

$$kg = \|k\|\, g\,,$$

and by the same argument as that used in the proof of the Corollary to Theorem 4.1 g can be taken to be non-negative.

But from (4.15),

$$0 \geqslant (kf - \mu f, g) = (f, kg - \|k\|\, g) + (\|k\| - \mu)(f, g)$$

so

$$0 \geqslant (\|k\| - \mu)(f, g)\,,$$

and the result follows since $(f, g) > 0$.

Now consider the general case and let

$$\Omega = E_1 \cup E_2 \cup \ldots \cup E_n \tag{4.17}$$

be a partitioning of Ω into measurable sets of positive measure. Put

$$f_i = \sigma_i^{-1} \chi_{E_i} f\,, \qquad i = 1, \ldots, n \tag{4.18}$$

where χ_{E_i} is the characteristic function of E_i and where

$$\sigma_i = \Big(\int_{E_i} f^2 c\, dx\Big)^{\frac{1}{2}}.$$

Thus

$$(f_i, f_j) = \delta_{ij}\,,$$

where δ_{ij} is the Kronecker delta. We have

$$f = \sigma_1 f_1 + \ldots + \sigma_n f_n \,,$$

and

$$\begin{aligned}\sum_{j=1}^{n} (f_i, kf_j)\sigma_j &= (f_i, kf)\\ &\leqslant \mu(f_i, f)\\ &\leqslant \mu \sum_{j=1}^{n} (f_i, f_j)\sigma_j \,,\end{aligned}$$

or

$$K\sigma \leqslant \mu\sigma \tag{4.19}$$

where K is the non-negative symmetric matrix defined by

$$K = \big((f_i, kf_j)\big) \,,$$

and the inequality in (4.19) has the obvious meaning. As in the case of compact k, inequality (4.19) for σ having all positive components implies that the largest eigenvalue of K does not exceed μ.

If P denotes the orthogonal projection of Y onto the subspace spanned by $f_1, \ldots, f_n$, then K is the matrix of PkP relative to the basis $f_1, \ldots, f_n$. By choosing a sequence of finer and finer partitions (4.17) we obtain a corresponding sequence of projections $\{P_m\}$ such that, because of (4.17) and the fact that $c(x) > 0$ a.e. on Ω, P_m tends strongly to I. Thus also $P_m k P_m$ tends strongly to k. Since $\|P_m k P_m\| \leqslant \mu$ for each m it follows that $\|k\| \leqslant \mu$. The opposite inequality when f is actually an eigenfunction is obvious.

REMARK 9. – Note that when k is defined by (4.7) then (4.15) even for a *non-negative* f, $f \not\equiv 0$, implies (4.16).

5. – Applications.

Let (X, Y) be admissible and let $u \in X$ be in the range of i^*

$$u = i^* g \,, \qquad g \in Y \,.$$

Put

$$f = iu = kg \,.$$

Then since k is positive definite

$$(f, f) = (kg, kg) \leqslant \|k\|(g, kg) = \|k\|\langle u, u\rangle \,,$$

which implies

$$\langle u, u\rangle \geqslant \|k\|^{-1}(iu, iu) \tag{5.1}$$

for u in the range of i^*. Since the range of i^* is dense in X the inequality is valid for all $x \in X$. We are now in a position to prove our main result.

THEOREM 5.1. – *Let A, b be as in* § 2, $c > 0$, *and let* (X, Y) *be an admissible pair. If the weak eigenvalue problem*

$$\int_\Omega \Big(\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} v_{x_j} + b(x) uv \Big) dx = \lambda \int_\Omega uvc(x)\, dx\,, \qquad v \in X\,, \tag{5.2}$$

has a non-negative eigenfunction u_1 corresponding to the eigenvalue λ_1 then $u_1(x) > 0$ a.e. *on Ω, and for all $u \in X$,*

$$\int_\Omega \Big(\sum_{i,j=1}^{N} a_{ij}(x) u_{x_i} u_{x_j} + b(x) u^2 \Big) dx \geqslant \lambda_1 \int_\Omega u^2 c(x)\, dx\,. \tag{5.3}$$

Moreover, λ_1 is a simple eigenvalue and consequently (5.3) *is strict unless u is proportional to u_1.*

PROOF. – The function u_1 is an eigenfunction of (5.2), corresponding to the eigenvalue λ_1 if and only if

$$ku_1 = \lambda_1^{-1} u_1\,,$$

thus the almost everywhere positivity of u_1 follows from the Corollary to Theorem 4.1. From Theorem 4.2, and in view of Lemma 4.4, it then follows that $\|k\| = \lambda_1^{-1}$, and thus the simplicity of λ_1 follows from the Corollary of Theorem 4.1. The inequality (5.3) then follows from (5.1).

For applications it is desirable to relax the requirements on b and c. We do this in the following.

THEOREM 5.2. – *Let A be as before with*

$$\|A\|,\ \|A^{-1}\| \in L^1_{\mathrm{loc}}(\Omega) \tag{5.4}$$

and let b_0, c_0 be real valued measurable functions on Ω with

$$b_0,\ c_0 \in L^1_{\mathrm{loc}}(\Omega)\,. \tag{5.5}$$

Finally, let there exist a linear manifold V and a non-negative function g such that

(*a*) $C_0^\infty(\Omega) \subseteq V \subseteq H^{1,1}_{\mathrm{loc}}(\Omega)$,

(*b*) $v \in V$ *implies* $|v| \in V$,

(*c*) $g \in L^1_{\mathrm{loc}}(\Omega)$, $g(x) > 0$ a.e. *on* Ω,

(*d*) *for all* $v \in V$,

$$\int_\Omega \Big((A\nabla v, \nabla v) + \big(|b_0| + |c_0| + g\big)\, v^2\Big)\, dx < \infty\,. \tag{5.6}$$

If $u_1 \in V$, $u_1 \geqslant 0$, $u_1 \not\equiv 0$, *and for some* $\lambda_1 > 0$ u_1 *satisfies*

$$\int_\Omega \big((A\nabla u_1, \nabla v) + b_0 u_1 v\big)\, dx \geqslant \lambda_1 \int u_1 v c_0\, dx\,, \qquad v \in V, \tag{5.7}$$

then $u_1(x) > 0$ a.e. *on* Ω *and*

$$\int_\Omega (A\nabla u, \nabla u) + b_0 u^2)\, dx \geqslant \lambda_1 \int_\Omega u^2 c_0\, dx \tag{5.8}$$

for all $u \in V$, *with equality only if* u *is proportional to* u_1.

PROOF. – We put

$$c = c_0 + g_1$$
$$b = b_0 + \lambda_1 g_1$$

where

$$g_1 = |c_0| + 2\lambda_1^{-1}|b_0| + g\,.$$

With b defined as above, it follows from Lemma 3.2, in view of (5.4), (5.5), and assumption (*c*) that $W^{1,2}(A, b, \Omega)$ is a Hilbert space stronger than $H^{1,1}_{\mathrm{loc}}(\Omega)$ and containing $C_0^\infty(\Omega)$. By assumptions (*a*) and (*d*), $V \subseteq W^{1,2}(A, b, \Omega)$; we define X to be the closure of V in $W^{1,2}(A, b, \Omega)$. By Lemma 3.5 $W^{1,2}(A, b, \Omega)$ is invariant under $u \to |u|$ and therefore by assumption (*b*) and the corollary to Lemma 3.5 so is X, consequently X satisfies condition II of § 4. By (*a*), $C_0^\infty(\Omega) \subseteq X$ and therefore, by Lemma 3.7, X satisfies condition III of § 4. Finally, from the definitions of b, c and g_1 we have

$$0 < c \leqslant 2\lambda_1^{-1} b$$

and therefore, since we have already seen that $C_0^\infty(\Omega) \subseteq X$, it follows from Lemma 4.1 that (X, Y) is admissible.

By adding $\lambda_1 \int_\Omega u_1 v g_1\, dx$ to both sides of (5.7) and taking into account the definitions of b and c we see that u_1 is an eigenfunction of (5.2), thus the positivity assertion concerning u follows from Theorem 5.1, as does the inequality (5.3) for $u \in X$. Upon subtracting $\lambda_1 \int_\Omega g_1 u^2\, dx$ from both sides of (5.3) we obtain (5.8). By Theorem 5.1, equality holds in (5.3) only if u and u_1 are proportional, hence the same is true of (5.8).

COROLLARY. – *Let* $2 \leqslant p \leqslant \infty$, *let* A *be as in Theorem* 5.2 *and in addition suppose that*

$$\|A\| \in L^{p/(p-2)}(\Omega)\,, \tag{5.9}$$

and

$$b_0,\, c_0 \in L^r(\Omega) \tag{5.10}$$

where

$$r = 1, \quad r > 1 \qquad or \qquad r = Np/(Np - 2(N-p)) \tag{5.11}$$

according as $p > N$, $p = N$ *or* $p < N$.

If $u_1 \in W_0^{1,p}(\Omega)$, $u_1 \geqslant 0$, $u_1 \not\equiv 0$, *and* u_1 *satisfies* (5.7) *for all* $v \in W_0^{1,p}(\Omega)$ *then* u_1 *is positive almost everywhere and* (5.8) *holds for all* $u \in W_0^{1,p}(\Omega)$, *with equality only if* u *and* u_1 *are proportional.*

PROOF. – By (5.9)

$$\int_\Omega (A\nabla u,\, \nabla u)\, dx < \infty$$

for $u \in W^{1,p}(\Omega)$ while by Sobolev's theorem (5.10) and (5.11) imply

$$\int_\Omega u^2(|b_0| + |c_0|)\, dx < \infty$$

for $u \in W^{1,p}(\Omega)$. Finally, one can choose $g > 0$ with $g \in L^r(\Omega)$. The last condition implies

$$\int_\Omega u^2 g\, dx < \infty$$

for $u \in W_0^{1,p}(\Omega)$. Thus, with this g and with $V = W_0^{1,p}(\Omega)$ the hypotheses of Theorem 5.2 are satisfied and the result follows.

6. – Maximum principle.

In this section we discuss the dependence of the operator k on boundary conditions and prove a maximum principle and an eigenvalue estimate. The maximum principle which we prove can be regarded as an analogue, for the boundary value problems which we treat, of a result of AMANN [2] for classical subsolutions of non-self-adjoint boundary value problems, see also SERRIN [25]. A similar result for weak subsolutions of equations with discontinuous coefficients was proved by CHICCO [6]. We also prove a partial converse—an eigenvalue estimate—to this maximum principle. This eigenvalue estimate is the analogue of a theorem of BARTA [4] for the Laplace operator with Dirichlet data. For generalizations of Barta's result see DUFFIN [9], PROTTER and WEINBERGER [23], and CHICCO [6]; the analogue of these latter results for ordinary differential operators is given by a theorem of WINTNER [33].

Let (X, Y) be an admissible pair and let $\lambda_1(X, Y) = \|k_{X,Y}\|^{-1}$. Recall that by Lemma 3.5 whenever $u \in W^{1,2}(A, b, \Omega)$, so are $|u|$, u_+ and u_-.

LEMMA 6.1. – *Let $\lambda < \lambda_1(X, Y)$ and let $u \in W^{1,2}(A, b, \Omega)$ be such that*

$$u_- \in X\,,$$

$$\langle u, v\rangle \geq \lambda \int_\Omega uvc(x)\, dx$$

for all $v \in X$ with $v \geq 0$ on Ω. Then

$$u \geq 0 \quad \text{on } \Omega.$$

PROOF. – Since $u_- \in X$ we have, by Lemma 2.1

$$\begin{aligned}\langle u_-, u_-\rangle = -\langle u, u_-\rangle &\leq -\lambda \int_\Omega uu_- c(x)\, dx \\ &\leq \lambda \int_\Omega u_-^2 c(x)\, dx\,.\end{aligned}$$

In view of (5.1), since $\lambda < \lambda_1$ this is only possible if $u_- = 0$. Thus the lemma is proved.

DEFINITION 6.1. – Let X be admissible and let X' be a subspace of X which is also admissible hence a vector sublattice of X. We shall say that X' is *solid relative to* X if whenever $u \in X'$, $w \in X$ and $|u| \geq |w|$ on Ω then $w \in X'$.

DEFINITION 6.2. – Let X be admissible and let Γ be a closed subset of $\bar{\Omega}$. Then, X_Γ will denote the closure in X of the linear manifold

$$\{u \in X : u = 0 \text{ on a neighborhood of } \Gamma\}\,.$$

REMARK 10. – Suppose that $C_0^\infty(\Omega) \subseteq X$ and that $X \cap C^\infty(\Omega)$ is dense in X. It can be shown that if $\|A\|\,|b^{-1}| \in L^\infty_{\text{loc}}(\Omega/K)$ for some compact set $K \subseteq \Omega$ then for $\Gamma = \partial\Omega$,

$$X_\Gamma = H_0(A, b, \Omega)\,.$$

Actually the result holds even without this assumption as will be shown elsewhere.

LEMMA 6.2. – *Let Γ be a closed subset of $\partial\Omega$. Then X_Γ is admissible and is solid relative to X. If $C_0^\infty(\Omega) \subseteq X$ and $X \cap C^\infty(\Omega)$ is dense in X then $H_0(A, b, \Omega)$ is solid relative to X.*

PROOF. – That X_Γ is admissible follows immediately from the Corollary to Lemma 3.5 and the proof of Lemma 3.7.

We now show that X_Γ is solid relative to X. Let $u \in X_\Gamma$, $w \in X$ and suppose that $|u| \geqslant |w|$ on Ω. There is no loss of generality in assuming, as we shall, that $u \geqslant w \geqslant 0$ on Ω. Let

$$u = \lim_{n\to\infty} u_n \quad \text{in } X$$

where for each $n = 1, 2, \ldots, u_n \in X$, and u_n vanishes on a neighborhood of Γ; by the continuity assertion of Lemma 3.5 we can assume $u_n \geqslant 0$ on Ω for each n. Moreover, for each value of n,

$$v_n = u_n - (u_n - w)_+$$

belongs to X and vanishes on a neighborhood of Γ. Since

$$w = u - (u - w)_+ \,, \tag{6.1}$$

it follows from the last assertion of Lemma 3.5 that $w \in X_\Gamma$.

Suppose now that $C_0^\infty(\Omega) \subseteq X$ and that $C^\infty(\Omega) \cap X$ is dense in X. Clearly then $H_0(A, b, \Omega)$ is defined and contained in X. Let $u \in H_0(A, b, \Omega)$, $w \in X$ with $u \geqslant w \geqslant 0$ on Ω, and let

$$u = \lim_{n\to\infty} u_n \,, \qquad w = \lim_{n\to\infty} w_n \,, \quad \text{in } X$$

where the sequences $\{u_n\}$, $\{w_n\}$ are in $C_0^\infty(\Omega)$ and $C^\infty(\Omega)$ respectively. We can assume that these sequences converge a.e. in Ω. Consider the sequence $\{v_n\}$ where

$$v_n = u_n - (u_n - w_n)_+ \,.$$

As before it follows from (6.1) and Lemma 3.5 that

$$w = \lim_{n\to\infty} v_n \quad \text{in } X \,, \tag{6.2}$$

and clearly

$$s(v_n) \subseteq s(u_n) \,, \qquad n = 1, 2, \ldots \,. \tag{6.3}$$

For a fixed n let $v_n^\varepsilon \in C^\infty(\Omega)$ be defined, for $\varepsilon > 0$, by

$$v_n^\varepsilon(x) = (J_\varepsilon v_n)(x) = \int_\Omega j_\varepsilon(x-y)\, v_n(y)\, dy \,,$$

where j_ε is a mollifier defined as in [1]. From (6.3) it follows that $v_n^\varepsilon \in C_0^\infty(\Omega)$ when ε is sufficiently small. Moreover, since v_n, $\nabla v_n \in L^\infty(\Omega)$ (by Lemma 2.1c),

$$|v_n^\varepsilon|, \ |\nabla v_n^\varepsilon| \leqslant C \quad \text{on } \Omega, \tag{6.4}$$

where, for n fixed, C is independent of ε. Finally, for n fixed

$$v_n^{1/m} \to v_n\,, \qquad \nabla v_n^{1/m} \to \nabla v_n\,, \qquad \text{a.e. on } \Omega, \tag{6.5}$$

as $m \to \infty$. In view of the fact that the $v_n^{1/m}$ all have their supports in some fixed bounded set, it follows from (6.4) and (6.5) and the dominated convergence theorem that

$$\lim_{m\to\infty} v_n^{1/m} = v_n \qquad \text{in } X\,,$$

and thus $v_n \in H_0(A, b, \Omega)$ for all n; it is then immediate from (6.2) that $w \in H_0(A, b, \Omega)$.

THEOREM 6.1. – *Let (X, Y) be an admissible pair, and let X' be a subspace of X which is such that (X', Y) is admissible and X' is solid relative to X. Then*

$$k_{X,Y} \geqslant k_{X',Y} \tag{6.6}$$

in the sense that

$$k_{X,Y} f \geqslant k_{X',Y} f$$

whenever $f \in Y$ and $f \geqslant 0$ on Ω.

PROOF. – Let $f \in Y$, $f \geqslant 0$ on Ω and put

$$u = i'^* f\,, \qquad w = i^* f$$

where i', i denote the inclusions $X' \subseteq Y$, $X \subseteq Y$ respectively. Then $u \in X'$, $w \in X$ and, by Lemma 4.3, $u, w \geqslant 0$ on Ω. Since $(w-u)_- \leqslant u$ and X' is full relative to X it follows that $(w-u)_- \in X'$. We have, moreover

$$\langle w-u, z\rangle = (f, iz - i'z) = 0 \qquad \text{for all } z \in X',$$

and thus, by Lemma 6.1.

$$w \geqslant u \qquad \text{a.e. on } \Omega.$$

Since f was an arbitrary non-negative element of Y the result follows.

COROLLARY. – *Let X, X' and Y be as in Theorem 6.1. If $\lambda_1^{-1}(X', Y)$ is an eigenvalue of $k_{X',Y}$ and $X' \neq X$ then*

$$\lambda_1(X', Y) > \lambda_1(X, Y)\,. \tag{6.7}$$

PROOF. – For brevity let $k' = k(X', Y)$, $k = k(X, Y)$, $\lambda_1' = \lambda_1(X', Y)$, $\lambda_1 = \lambda_1(X, Y)$ By the Corollary to Theorem 4.1, k' then has an eigenfunction u with

$$k' u = \lambda_1'^{-1} u\,,$$

and $u > 0$ a.e. on Ω. It then follows from Theorem 6.1 that

$$ku \geqslant k' u$$

and equality holds a.e. on Ω only if $k = k'$. Indeed if $k \neq k'$ then there exists an $f \geqslant 0$ in Ω such that $(kf)(x) > (k'f)(x)$, on a set of positive measure in Ω, but then

$$0 < \big((k - k')f, u\big) = \big(f, (k - k')u\big),$$

so that $ku \neq k'u$. On the other hand if $k = k'$, then clearly $i^* = i'^*$. Since the ranges of i^* are dense in X and X' respectively, equality of i^* and i'^* implies equality of X and X'. Thus if $X \neq X'$ we must have $\|ku\|_Y > \|k'u\|_Y$, and this implies (6.7).

REMARK 11. – The result is false if we do not assume $\lambda_1^{-1}(X')$ is an eigenvalue of k'. This is easily seen from consideration of the operator $-d^2/dx^2 + 1 + p(x)$ with the boundary conditions $y'(0) = 0$ and $y(0) = 0$ respectively. To put these problems in the setting of this paper one takes $\Omega = [0, \infty)$, $A = 1$, $b = 1 + p$,

$$X = H(A, b, \Omega) = H\big(1, 1 + p, [0, \infty)\big) \quad \text{and} \quad X' = H_0(A, b, \Omega) = H_0\big(1, 1 + p, [0, \infty)\big).$$

One can choose $p(x)$ in such a way that the problem

$$-y'' + \big(1 + p(x)\big)y = \lambda y \quad \text{on } (0, \infty) \tag{6.8}$$

$$y'(0) = 0 \tag{6.9}$$

has a positive eigenfunction corresponding to the eigenvalue 1 and has spectrum $[1, \infty)$, while the boundary value problem

$$y(0) = 0, \tag{6.10}$$

for (6.8) has the same spectrum and [by the above Corollary then necessarily] has no eigenfunction which is positive in $(0, \infty)$. Thus the Green's functions for both problems will have norm 1 as operators in $L^2(0, \infty)$.

Indeed we define

$$y_0(x) = \frac{1}{x}, \qquad x \geqslant 1$$

and define $y_0(x)$ on $[0, 1]$ in such a way that $y_0 \in C^2[0, \infty)$, $y_0(x) > 0$ on $[0, \infty)$ and $y_0'(0) = 0$. We then take

$$p(x) = y_0''(x)/y_0(x),$$

so that

$$p(x) = \frac{2}{x^2}, \qquad x \geqslant 1.$$

Thus $y_0 \in L^2(0, \infty)$ and satisfies

$$y'' - p(x)y = 0\,, \qquad y'(0) = 0$$

i.e. y_0 is an eigenfunction of (6.8), (6.9) corresponding to $\lambda = 1$. Since $p \in L^1(0, \infty)$, it follows from [28, pp. 97-101], [32] that the spectrum of both (6.8), (6.9) and (6.8), (6.10) contains $[1, \infty)$. On the other hand, by Theorem 4.2, the spectrum of (6.8), (6.9) is contained in $[1, \infty)$, thus it follows from Theorem 6.1 and Lemma 6.2 that the spectrum of both problems is precisely $[1, \infty)$.

A simpler example is provided by taking $p \equiv 0$, however in this case neither problem has a dominant eigenfunction.

THEOREM 6.2. – *Let X, X', Y be as in Theorem 6.1. Let $\lambda \leqslant \lambda_1(X, Y)$ and let $u \in X$ satisfy*

$$u_- \in X'\,,$$

$$\langle u, v\rangle \geqslant \lambda(iu, iv)\,,$$

for all $v \in X'$ with $v \geqslant 0$ on Ω. Then either $X = X'$, $\lambda = \lambda_1(X, Y)$ and u is an eigenfunction of (5.2) or $u \geqslant 0$ in Ω.

PROOF. – If $\lambda < \lambda_1(X, Y) \leqslant \lambda_1(X', Y)$, the assertion has already been proved in Lemma 6.1. In any case, as in the proof of Lemma 6.1, $w - u_- \in X'$ satisfies

$$\langle w, w\rangle \leqslant \lambda_1(X, Y)(iw, iw) \tag{6.11}$$

and thus by the Corollary to Theorem 6.1 and (5.1), $u_- = w = 0$ if $X' \neq X$. Finally if $X' = X$ and (6.11) holds with $u_- = w \neq 0$, then $w > 0$ a.e. on Ω and $w - u_+ = -u$ is an eigenfunction of (5.2).

REMARK 12. – When X' is of the form X_Γ, then the condition $u_- \in X'$ can be interpreted as « $u \geqslant 0$ on Γ », compare Definition 1.1, p. 14, [26].

A partial converse to Theorem 6.2 is the following result.

THEOREM 6.3. – *If $u \in X$, $u \geqslant 0$ on Ω, $u \not\equiv 0$ and for some $\lambda > 0$*

$$\langle u, v\rangle \geqslant \lambda(iu, iv) \quad \textit{for all } v \in X,\ v \geqslant 0, \tag{6.12}$$

then

$$\lambda_1(X, Y) \geqslant \lambda\,.$$

PROOF. – In (6.12) let $f = iu$ and let $v = i^*g$, $g \in Y$, then (6.12) becomes

$$(f, g) \geqslant \lambda(kf, g) \quad \text{for all } g \in Y,\ g \geqslant 0$$

which implies (4.15), with $\mu = \lambda^{-1}$. The hypothesis and the definition of f imply that f is non-negative and not identically zero. It then follows from Theorem 4.1 and (4.15) that f is positive a.e. on Ω. The result then follows from Theorem 4.2 and the definition of $\lambda_1(X, Y)$.

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