Interpolation and Non-Commutative Integration (*).

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Summary. – *We extend the interpolation theory of a previous publication to the case of non-commutative L_p spaces in the sense of Segal. As illustrations we give some simple concrete applications (Fourier transform on unimodular groups, Weyl transform, spinor transform).*

0. – Introduction.

This is in a way a continuation of an earlier work (PEETRE-SPARR [21]). There among other things we noticed a close connection between the two cases: 1° interpolation of the usual L_p spaces over a measure space and 2° interpolation of the trace classes $\mathfrak{S}_p$ of compact operators in a Hilbert space (¹). There arises the question whether one cannot treat both cases within one and the same framework. We now show that this is indeed possible if we use the theory of non-commutative integration, in particular the theory of non-commutative L_p spaces over a (regular) gage space, as developped by Segal and his students (see SEGAL [23], KUNZE [6], STINESPRING [29]). While writing [21] we simply were not aware that such a theory existed. Accordingly we now try to fill in this gap. We notice however that in the meantime there has appeared also a paper by OVČINNIKOV [13] where somewhat related ideas can be found.

The plan of the paper reads as follows. There are five Sections. In Section 1 we briefly review some basic facts about gage spaces. In Section 2 we then carry over the interpolation theory of [21] to the case of non-commutative L_p spaces. In Section 3 we apply the results of Section 2 to the Fourier (-Segal) transform on unimodular groups. In doing this we cover anew and improve somewhat on the results of KUNZE [6]. In Section 4 we likewise treat the Weyl transform and also, more briefly, the spinor transform. Hereby we generalize some results by LAVINE [7] and STREATER [30]. We remark however that since the gage spaces involved are of the trivial type we could here have used [21] directly. Finally in Section 5 we treat in a somewhat more general framework part of the material of Section 2.

Whenever applicable we use the notation and terminology of [21].

(*) Entrata in Redazione il 6 giugno 1973.

(¹) This parallelism is referred to also in PEETRE [14], [15], and is probably known to many authors (cf. notably MITJAGIN [11], GOH'BERG and KREIN [5], TRIEBEL [31], COTLAR [3], and, for a more recent study, MÉRUCCI and PHAM THE LAI [10] as well as other works by the same authors quoted there).

Since most of the proofs are routine repetition of known arguments, we have chosen to cut down the details to a minimum. (We do ascertain that all results stated here are correct.) We hope that nevertheless this compilation might be of some use to those who might be tempted to apply interpolation in non-commutative situations, say, in problems of quantum field theory.

We would like to express our gratitude to prof. I. E. SEGAL for most valuable information pertaining to non-commutative integration, and several stimulating conversations.

1. – Basic facts about gage spaces.

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{A}$ a ring of bounded operators on $\mathcal{H}$, in the von Neumann sense (i.e. in particular weakly and/or strongly closed in $\mathcal{L}(\mathcal{H})=$ the ring of all bounded operators in $\mathcal{H}$). By a (regular) *gage* on $\mathcal{A}$ we mean a mapping m: projections of $\mathcal{A}\to\mathbf{R}_+$ such that the following axioms hold:

1) $m(P)>0$ if $P\neq 0$, $m(0)=0$.

2) $m(\bigcup\limits_\alpha P_\alpha)=\sum\limits_\alpha P_\alpha$ if $P_\alpha P_\beta=0$, $\alpha\neq\beta$ (orthogonality).

3) $m(UPU^{-1})=m(P)$ if $U^{-1}=U^*$ (unitarity).

4) every projection in $\mathcal{A}$ is the $\bigcup$ of m-finite projections.

The triple $\Gamma=(\mathcal{H},\mathcal{A},m)$ is termed (regular) *gage space*. One can define the notion of measurable operator on Γ, and one can extend m to positive such. If T has the spectral resolution $T=\int\limits_0^\infty\lambda\,dP(\lambda)$ then holds the formula

$$m(T)=\int\limits_0^\infty\lambda\,dm(P(\lambda))\,.$$

If $m(T)<\infty$ then T is termed positive integrable. By linearity one can extend m to general integrable operators, i.e. those which belong to the hull of the positive integrable ones.

The following examples clarify what we are attempting at.

EXAMPLE 1.1 (the commutative case). – Let M be an ordinary measure space, i.e. a triplet $(X,\mathcal{B},m)$ where X is a space, $\mathcal{B}$ a Boolean ring of sets of X, m a measure on $\mathcal{B}$. (Usually one writes abusively X is place of $(X,\mathcal{B},m)$). Then M can be identified with the gage space $(L_2(M),L_\infty(M),m)$, an element of L_∞ being identified with the corresponding multiplicative operator.

EXAMPLE 1.2. – The triple $(\mathcal{H}, \mathcal{L}(\mathcal{H}), \mathrm{tr})$ is a gage space. Here tr stands for the ordinary (von Neumann) trace. Note that $\mathrm{tr}(P) = \mathrm{rank}(P)$ if P is a projection, so that here $m(P) < \infty$ iff $\mathrm{rank}(P) < \infty$.

EXAMPLE 1.3. – One can combine ex. 1.1 and ex. 1.2 (i.e. the trivial cases) as follows. Let there be given a Hilbert bundle $\mathcal{K} = \{\mathcal{K}_x\}$ over the measure space X, i.e. for each $x \in X$ there is attached a Hilbert space $\mathcal{K}_x$. We consider the operator buncle $\mathcal{L}(\mathcal{K}) = \{\mathcal{L}(\mathcal{K}_x)\}_{x\in X}$. We get a gage space $\Gamma = (\mathcal{H}, \mathcal{A}, m)$ by taking:

$$\begin{cases} \mathcal{H} = \text{sections } f = \{f_x\} \text{ of } \mathcal{K} \text{ with } \displaystyle\int_X \|f_x\|^2\, dx < \infty\,, \\ \mathcal{A} = \text{sections } T = \{T_x\} \text{ of } \mathcal{L}(\mathcal{K}) \text{ with } \displaystyle\sup_{x\in X} \|T_x\| < \infty\,, \\ m(T) = \displaystyle\int_X \mathrm{tr}\,(T_x)\, dm\,. \end{cases}$$

EXAMPLE 1.4. – Let G be a locally compact unimodular group provided with a Haar measure dg, and let L be the left regular representation (defined by $L_g f(\cdot) = f(g^{-1}\cdot)$). Then we obtain a gage space (the dual of G) $\hat{G} = \Gamma = (\mathcal{H}, \mathcal{A}, m)$ by taking:

$$\begin{cases} \mathcal{H} = L_2(G), \\ \mathcal{A} = \text{the ring generated by the operators } \{L_g\}_{g\in G}, \\ m(P) = \|f\|^2 \text{ if } P \text{ is a projection of the form } P = L_f, \\ \left(\text{with } L_f\varphi = f * \varphi = \int L_g f \varphi(g)\, dg\right). \end{cases}$$

(The unimodularity is need for the verification of axiom 2°!)

EXAMPLE 1.5. – In certain cases $\hat{G}$ can be presented in a more explicit equivalent form. This is the case when G is compact (and so automatically unimodular). Then we take $\hat{G}$ to be the space of all equivalence classes of irreducible unitary representations of G provided with the discrete measure m which with a given irreducible unitary representation U^x, corresponding to a point $x \in X$, in a (finite dimensional) Hilbert space $\mathcal{V}^x$ associates the mass $\dim \mathcal{V}^x$.

We return to the case of a general gage space $\Gamma = (\mathcal{H}, \mathcal{A}, m)$.

In what follows we shall usually put ourselves on a purely formal level, leaving out all technicalities related to measurability.

Let $0 < p < \infty$. We introduce the spaces $L_p = L_p(\Gamma)$ (non-commutative L_p spaces) of measurable operators T by the condition

$$\|T\|_{L_p} = \|T\|_{L_p(\Gamma)} = m(|T|^p)^{1/p} < \infty$$

where $|T|$ denotes a positive measurable operator equivalent (²) to T (e.g. $|T| = \sqrt{TT^*}$ will do). Note that if T is positive then

$$\|T\|_{L_p} = \left(\int_0^\infty \lambda^p \, dm(P(\lambda))\right)^{1/p}.$$

If $1 \leqslant p < \infty$ then $\|T\|_{L_p}$ is a norm but if $0 < p < 1$ only a quasi-norm. One can show that L_p is complete, i.e. a Banach space in the first case, a quasi-Banach space space in the second case. We also introduce the spaces $\tilde{L}_p = \tilde{L}_p(\Gamma)$ by putting

$$\tilde{L}_p = L_p^{[\tilde{p}]}, \qquad \|T\|_{\tilde{L}_p} = \|T\|_{L_p}^{\tilde{p}} \text{ with } \tilde{p} = \min(1, p) \; (^3).$$

If $1 \leqslant p < \infty$ nothing changes. However if $0 < p < 1$ we get a normed Abelian group in the sense of [21] (or, what is usually called a p-normed vector space (p-norm $\equiv p$-homogeneous norm)). Finally we introduce the spaces $\tilde{L}_0$ and $\tilde{L}_\infty = L_\infty$ by passing to the limit $p \to 0$ or $p \to \infty$ respectively. Or, spelled out, $\tilde{L}_0$ is the space corresponding to the (0-homogeneous) norm

$$\|T\|_{\tilde{L}_0} = m(\text{supp.}\, T)$$

Where supp. T (support of T) is the smallest projection $P \in \mathcal{A}$ such that $PT = T$, and $\tilde{L}_\infty = L_\infty$ is the space corresponding to the (1-homogeneous) norm

$$\|T\|_{\tilde{L}_\infty} = \|T\|_{L_\infty} = \|T\|_{\mathfrak{L}(\mathcal{H})} = \sup \|Tf\|_{\mathcal{H}} / \|f\|_{\mathcal{H}},$$

i.e. the restriction to $\mathcal{A}$ of the norm in $\mathfrak{L}(\mathcal{H})$.

EXAMPLE 1.6. – In the case of ex. 1.1 we obtain the usual (commutative) L_p spaces. In the case of ex. 1.2 we obtain the trace classes $\mathfrak{S}_p = \mathfrak{S}_p(\mathcal{H})$ (as well as $\tilde{\mathfrak{S}}_p$).

2. – Interpolation of noncommutative L_p spaces.

As we have already told we use throughout the notation and terminology of [21]. For the following discussion see in particular [21], Sub-Section 6.1 and Sub-Section 7.1.

2.1. - Let again $\Gamma = (\mathcal{H}, \mathcal{A}, m)$ be a gage space. We consider first the normed Abelian couple $\{\tilde{L}_0, \tilde{L}_\infty\}$. We shall use the notation

$$T^\star(t) = E(t, T; \{\tilde{L}_0, \tilde{L}_\infty\}) = \inf_{\|S\|_{\tilde{L}_0} \leqslant t} \|T - S\|_{\tilde{L}_\infty}$$

(²) Two operators S_1 and S_2 in a Hilbert space $\mathcal{H}$ are equivalent if $S_2 = U' S_1 U''$ with U' and U'' unitary.

(³) The brackets $^{[\,]}$ are used conformally with the notation of [21].

(decreasing rearrangement of T). Note that

$$(T_1 + T_2)^\star (t_1 + t_2) \leq T_1^\star (t_1) + T_2^\star (t_2) .$$

We define

$$L_{pq}(\Gamma) = L_{pq} = (\tilde{L}_0, \tilde{L}_\infty)_{\alpha q;E} \quad \text{with } \alpha = 1/p ,$$

(non-commutative Lorentz [4] space), i.e. we have $T \in L_{pq}$ iff

$$\|T\|^{L_{pq}} = \left(\int_0^\infty (t^{1/p} T^\star(t))^q \frac{dt}{t} \right)^{1/q} < \infty . \tag{2.1}$$

Consider the special case $p = q$. Then (2.1) becomes

$$\left(\int_0^\infty (T^\star(t))^p \, dt \right)^{1/p} < \infty$$

or since $t \mapsto T^\star(t)$ is the inverse of the function $\lambda \mapsto m(P(\lambda))$ where $\lambda \mapsto P(\lambda)$ gives the spectral resolution of the operator $|T|$, simply

$$\int_0^\infty \lambda^p \, dm(P(\lambda)) < \infty .$$

In other words we have

$$L_{pq} = L_p \quad \text{if} \quad p = q .$$

Using [21], th. 5.10 (equivalence of E- and K-spaces) follows now

THEOREM 2.1. – Let $\theta = p/(p+1)$, $r = \theta q$. Then holds

$$(\tilde{L}_0, \tilde{L}_\infty)^{[1/\theta]}_{\theta q;K} = L_{pr} .$$

In particular holds if $p = \theta q$

$$(\tilde{L}_0, \tilde{L}_\infty)^{[1/\theta]}_{\theta q;K} = L_p . \quad \square$$

Using [21], th. 5.11 (reiteration theorem for K-spaces) follows

THEOREM 2.2. – Let $1/p = ((1-\theta)/p_0 + \theta/p_1)$ Then holds

$$(L_{p_0 r_0}, L_{p_1 r_1})_{\theta q;K} = L_{pq} .$$

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(4) G. G. LORENTZ, *not* H. A. LORENTZ!

In particular holds

$$(L_{p_0}, L_{p_1})_{\theta p;K} = L_p\,. \quad \square$$

Thus th. 2.1 tells us that all the spaces L_{pq} and in particular the L_p can be reconstructed from the couple $\{\tilde{L}_0, \tilde{L}_\infty\}$. Th. 2.2 is useful if we want to obtain interpolation theorems. We state the typical.

COROLLARY 2.1. (analogue of the M. Riesz interpolation theorem). – Let Γ and H be two gage spaces. Consider the quasi-Banach couples $\boldsymbol{A} = \{L_{p_0}(\Gamma), L_{p_1}(\Gamma)\}$ and $\boldsymbol{B} = \{L_{q_0}(H), L_{q_1}(H)\}$. If $\mathcal{T}\colon \boldsymbol{A} \to \boldsymbol{B}$ is a bounded linear mapping then holds $\mathcal{T}\colon L_p(\Gamma) \to \to L_q(H)$, provided

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \ \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \qquad (0 < \theta < 1)\,.$$

For the norms involved we have a convexity inequality of the type

$$\|\mathcal{T}\| \leqslant c \|\mathcal{T}\|_0^{1-\theta}\, \|\mathcal{T}\|_1^{\theta}\,, \tag{2.2}$$

with $\|\mathcal{T}\| = \|\mathcal{T}\|_{\mathfrak{L}(L_p(\Gamma), L_q(H))}$ etc., and $c \geqslant 1$ depending on θ. More generally (if $p_0 \neq p_1$, $q_0 \neq q_1$) holds $\mathcal{T}\colon L_{pr}(\Gamma) \to L_{qr}(H)$, with the same assumptions about the parameters and we have an analogous inequality.

PROOF. – It suffices to invoke [21], th. 5.2. $\square$

The fact that c in inequality (2.2) depends on θ is however a serious defect, at least for certain types of applications. We therefore must give a more careful analysis, which takes much more into account the special features of non-commutative integration.

2.2. – We start with the Banach couple $\{L_1, L_\infty\}$. Then holds the following formula

$$K(t, T; \{L_1, L_\infty\}) = \int_0^t T^\star(s)\, ds\,, \tag{2.3}$$

the special case of which in the case of ex. 1.1 and ex. 1.2 is well-known (see e.g. [14] for references); the general case has also independently been proven in [13]. It is convenient to give the proof in a somewhat more general framework so we postpone it to Section 5. From (2.3) follows easily:

THEOREM 2.3. – We have

$$(L_1, L_\infty)_{\theta q;K} = L_{pq}\,, \qquad \theta = 1 - \frac{1}{p}\,. \tag{2.4}$$

In particular holds

$$(L_1, L_\infty)_{\theta p;K} = L_p\,, \qquad \theta = 1 - \frac{1}{p}\,.$$

PROOF. - We rewrite (2.3) as

$$tT^\star(t) \leqslant K(t, T) = \int_0^1 \lambda t T^\star(\lambda t)\,\frac{d\lambda}{\lambda}\,. \tag{2.5}$$

With $\Phi = \Phi_{\theta q}$ as in [21], def. 5.3, we obtain from (2.5) if $1 \leqslant q \leqslant \infty$

$$\Phi[tT^\star(t)] \leqslant \Phi[K(t, T)] \leqslant \int_0^1 \Phi[\lambda t T^\star(\lambda t)]\,\frac{d\lambda}{\lambda} = \int_0^1 \lambda^\theta \frac{d\lambda}{\lambda}\,\Phi[tT^\star(t)] = \frac{1}{\theta}\,\Phi[tT^\star(t)]\,.$$

(This is essentially an implicite use of Hardy's inequality!) But

$$\Phi[T^\star(t)] = \|T\|_{L_{pq}}\,, \qquad \Phi[K(t, T)] = \|T\|_{(L_1, L_\infty)_{\theta q;K}}$$

and the proof of (2.4) is complete in this special case. If $0 < q < 1$ we have first to replace the integral in (2.5) by a discrete sum and then use the fact that $\Phi^{\tilde{q}}$, with $\tilde{q} = \min(1, q)$, satisfies the triangle inequality (see [21], (5.4)). □

We note that th. 2.3 is essentially a special case of th. 2.2. So we get also a new proof of a special case of cor. 2.1 ($p_0 = q_0 = 1$, $p_1 = q_1 = \infty$), with additional information on the constant c in (2.2), at least if $1 \leqslant q \leqslant \infty$. However (2.3) can also be used to prove more directly the following « exact » result.

THEOREM 2.4. - Let $1 < p < \infty$. Then L_p is with equality of norms a K-space, in particular an interpolation space, for the Banach couple $\{L_1, L_\infty\}$, i.e. there exists a functional Φ such that we have

$$\|T\|_{L_p} = \Phi[K(t, T)]\,. \tag{2.6}$$

PROOF. - By Hölder's inequality we have

$$\|T\|_{L_p} = \sup_{f\in F} \int_0^\infty T^\star(t)\,f(t)\,dt$$

where F denotes the set of positive functions f subject to the condition

$$\int_0^\infty (f(t))^{p'} dt = 1 \qquad \text{with } \frac{1}{p} + \frac{1}{p'} = 1\,.$$

But since $T^{\star}(t)$ is decreasing, we can in this special situation replace F by the subset F' of decreasing functions. Therefore, upon integrating by parts and using (2.3) we obtain

$$\|T\|_{L_p} = \sup_{f\in F'} -\int_0^\infty K(t, T)\, df(t), ,$$

which clearly is a proof of (2.6). □

From th. 2.4 follows an « exact » interpolation result corresponding to a special case of cor. 2.1 ($p_0 = q_0 = 1$, $p_1 = q_1 = \infty$), the precise formulation of which we omit.

REMARK 2.1. – The same argument as in the proof of th. 2.4 can be adapted so as to characterize the most general interpolation space for the couple $\{L_1, L_\infty\}$ (the analogue of the results of MITJAGIN [11] and COTLAR [3]; cf. MÉRUCCI and PHAM THE LAI [10] where the case $\{\mathfrak{S}_1, \mathfrak{S}_\infty\}$ is worked out).

2.3. – The considerations of Sub-Section 2.2 can in part be extended to the case of the Banach couple $\{L_p, L_\infty\}$ (see LORENTZ and SHIMOGAKI [9], BERGH [1], [2]). In particular holds the following partial analogue of (2.3):

$$\left(\int_0^{t^p} (T^{\star}(s))^p\, ds\right)^{1/p} \leqslant K(t, T; \{L_p, L_\infty\}) \leqslant 2^{1-1/p}\left(\int_0^{t^p} (T^{\star}(s))^p\, ds\right)^{1/p}. \tag{2.7}$$

The constant $2^{1-1/p}$ cannot be improved.

2.4. – Finally we consider briefly the case of the (quasi-) Banach couple $\{L_{p_0}, L_{p_1}\}$ with $0 < p_0$, $p_1 < \infty$. It is difficult to apraise $K(t, T; \{L_{p_0}, L_{p_1}\})$ directly. However for the modified couple $\{L_{p_0}^{[p_0]}, L_{p_1}^{[p_1]}\}$ this is easy. Indeed we have

$$K(t, T; \{L_{p_0}^{[p_0]}, L_{p_1}^{[p_1]}\}) = \int_0^\infty K(t, T^{\star}(s); \{\mathbf{C}^{[p_0]}, \mathbf{C}^{[p_1]}\})\, ds \tag{2.8}$$

(cf. PEETRE [16], PEETRE-SPARR [21], OLOFF [12]). From (2.8) follows at once

THEOREM 2.5. – We have with proportionality of norms

$$(L_{p_0}^{[p_0]}, L_{p_1}^{[p_1]})_{\eta 1;K} = L_p^{[p]}, \qquad p = (1-\eta)p_0 + \eta p_1 . \quad \square$$

From th. 2.5 again follows the « exact » version of the first half of cor. 2.1, i.e. we can afford $c = 1$ in the convexity inequality (2.2).

3. – Applications to the Fourier transform on unimodular groups.

We place ourselves in the situation of ex. 1.4. More precisely, G is locally compact unimodular group with Haar measure dg, and $\hat{G} = \Gamma = (\mathcal{H}, \mathcal{A}, m)$ (the dual of G) is the gage space defined by

$$\begin{cases} \mathcal{H} = L_2(G), \\ \mathcal{A} = \text{the ring generated by the left translation operators } \{L_g\}_{g\in G}, \\ m(f) = \|f\|^2 \text{ if } P \text{ is a projection of the form } P = L_f. \end{cases}$$

We define the Fourier (- Segal) transform $\mathcal{F}$ on G formally by assigning to a function f on G the convolution operator L_f. In the case of a compact group, with the identification of $\hat{G}$ made in ex. 1.5, we can identify L_f with a function $\hat{f}$ on $\hat{G}$, in such a way that $\widehat{L_f \varphi} = \hat{f}\hat{\varphi}$, so that this definition (due to Segal) agrees, in this case, with the usual definition of the Fourier (-Peter-Weyl) transform. Directly from the definition we obtain

$$\|L_f\|_{L_2(\hat{G})} = \|f\|_{L_2(G)} \qquad \text{(analogue of Parseval's relation).}$$

In other words holds

$$\mathcal{F}: L_2(G) \to L_2(\hat{G})\,. \tag{3.1}$$

On the other hand, the well-known Minkowsky or Young inequality

$$\|f * \varphi\|_{L_2(G)} \leqslant \|f\|_{L_1(G)} \|\varphi\|_{L_2(G)}$$

can be rewritten as

$$\|L_f\|_{L_\infty(\hat{G})} \leqslant \|f\|_{L_1(G)}\,.$$

Thus holds also

$$\mathcal{F}: L_1(G) \to L_\infty(\hat{G})\,. \tag{3.2}$$

Now we interpolate between (3.1) and (3.2). Application of the «exact» version of cor. 2.1, which is lawful in view of th. 2.5, yields, with the constant 1,

$$\mathcal{F}: L_p(G) \to L_{p'}(\hat{G})\,, \qquad \frac{1}{p} + \frac{1}{p'} = 1\,, \quad 1 \leqslant p \leqslant 2 \tag{3.3}$$

(analogue of the Hausdorff-Young theorem).

This was first established by KUNZE [6] using complex variables (à la Thorin). We have thus obtained a real variable proof of his result. But invoking also non-commutative Lorentz spaces we can prove a in a sense sharper result. Indeed, application of the last part of cor. 2.1 yields

$$\mathcal{F}: L_p(G) \to L_{p'p}(\hat{G}), \qquad \frac{1}{p} + \frac{1}{p'} = 1, \qquad 1 \leqslant p \leqslant 2 \tag{3.4}$$

(analogue of the Paley theorem).

The connection between (3.3) and (3.4) is imbodied in the general relation

$$L_{pq_1} \subset L_{pq_2} \quad \text{if} \quad q_1 \leqslant q_2 .$$

REMARK 3.1. – For the discussion of (3.3) and (3.4) in their classical Fourier series context, i.e. the group $G = \mathbf{T}$ (torus), see ZYGMUND [32], chap. XII. From this classical case follows that (3.3) and (3.4) are about the best possible for general unimodular groups. However for semi-simple Lie groups drastically sharper results can be obtained, as has been demonstrated by Kunze and Stein in a series of works; we refer to the survey article of STEIN [27]. Here the basic tool, from the interpolation point of view, is the celebrated interpolation theorem of STEIN [28] (also discussed e.g. in [32], chap. XII). This is again a typical complex variable argument. It is tempting to ask how much one can achieve with the real methods only. Quite generally, the relation of Stein's theorem to the real methods ought to be clarified.

4. – Applications to the Weyl transform and the spinor transform.

As we already told in the Introduction, we could in this Section have used [21] directly, the gage spaces involved being of the trivial type (i.e. either of the type of ex. 1.1 or of ex. 1.2).

4.1. – We consider $\mathbf{R}^n$, with the general point $x = (x_1, \ldots, x_n)$, and $\mathbf{R}^{2n}$, with the general point $\zeta = (a, b) = (a_1, \ldots, a_n, b_1, \ldots, b_n)$, provided with the usual Haar measures $dx = dx_1, \ldots, dx_n$ and $d\zeta = da\, db = da_1, \ldots, da_n\, db_1, \ldots, db_n$. Via the natural mapping $\mathbf{R}^{2n} \to \mathbf{R}^n \times \mathbf{R}^n$ we may identify $L_2(\mathbf{R}^{2n})$ with $L_2(\mathbf{R}^n) \otimes L_2(\mathbf{R}^n)$ (with convenient interpretation of $\otimes$!). If $f \in L_2(\mathbf{R}^{2n})$ its Weyl transform $\mathcal{W}(f)$ is a Hilbert-Schmidt operator in $L_2(\mathbf{R}^n)$, $\mathcal{W}(f) \in \mathfrak{S}_2$. The formal definition reads

$$\mathcal{W}(f) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^{2n}} f(\zeta)\, \mathcal{W}(\zeta)\, d\zeta ,$$

where

$$\mathcal{W}(\zeta)\varphi(x) = \exp i \left(b \cdot x + \frac{a \cdot b}{2}\right) \varphi(x + a) \qquad \text{if } \varphi \in L_2(\mathbf{R}^n)$$

with the scalar products

$$b\cdot x=\sum_{j=1}^{n} b_j x_j\,, \qquad a\cdot b=\sum_{j=1}^{n} a_j b_j\,.$$

It is well-known (SEGAL [23]; cf. PEETRE [18] where further references can be found) that $\mathcal{W}$ is a unitary mapping from $L_2(\mathbf{R}^{2n})$ into $\mathfrak{S}_2$:

$$\mathcal{W}\colon L_2(\mathbf{R}^{2n})\to\mathfrak{S}_2\,.$$

Trivially (by Minkowsky or Young) we have

$$\mathcal{W}\colon L_1(\mathbf{R}^{2n})\to\mathfrak{S}_\infty\,. \tag{4.2}$$

Interpolating between (4.1) and (4.2) in the same way as in Section 3 we thus obtain analogues of Hausdorff-Young and Paley for $\mathcal{W}$ (cf. (3.3) and (3.4)). We leave it to the reader to contemplate over the particulars. Instead we turn to analogues of the classical theorems of Bernstein and Szasz (cf. ZYGMUND [32]).

We need some preliminaries related to $\mathcal{W}$. Let $\varphi_\nu=\varphi_{\nu_1\ldots\nu_n}\in L_2(\mathbf{R}^n)$ be the normalized Hermite functions in $\mathbf{R}^n$. They constitute an orthonormal basis in $L_2(\mathbf{R}^n)$. In $L_2(\mathbf{R}^{2n})$ we have correspondingly the basis $\varphi_{\nu\mu}$ of Laguerre type functions (cf. [18] if $\mu=\nu$). They are formally given by $\Pi_{\nu\mu}=\mathcal{W}(\varphi_{\mu\nu})$ where the operators (projections) $\Pi_{\nu\mu}$ are defined by

$$\Pi_{\nu\mu}\varphi_{\nu'}=\delta_{\nu\nu'}\varphi_\mu \qquad (\delta_{\nu\nu'}=\text{Kronecker-delta}).$$

(More explicitely, there holds the relation $\varphi_{\nu\mu}=\big(\mathcal{W}(-\zeta)\,\varphi_\mu|\varphi_\nu\big)$. We further introduce the «number operator» N by imposing

$$N\varphi_\nu=|\nu|\,\varphi_\nu \qquad \text{where } |\nu|=\nu_1+\ldots+\nu_n\,.$$

If $F=\mathcal{W}f$ then $NF=\mathcal{W}Kf$. (This we also may express as a transmutation relation $N\mathcal{W}=\mathcal{W}K$.) Here $K=K_{(l)}$ is the partial differential operator

$$K_{(l)}=\frac{1}{2}\left(-\Delta+\frac{1}{4}\zeta^2+\frac{i}{2}\sum_{j=1}^{n}\left(b_j\frac{\partial}{\partial a_j}-a_j\quad\frac{\partial}{\partial b_j}\right)-n\right),$$

with

$$\Delta=\sum_{j=1}^{n}\frac{\partial^2}{\partial a_j^2}+\sum_{j=1}^{n}\frac{\partial^2}{\partial b_j^2}=\text{Laplacian in } \boldsymbol{R}^{2n},$$

$$\zeta^2=\zeta\cdot\zeta=\sum_{j=1}^{n}a_j^2+\sum_{j=1}^{n}b_j^2\,.$$

If we note that $N\Pi_{\nu\mu} = |\mu|\Pi_{\nu\mu}$ we see that

$$K_{(l)}\varphi_{\nu\mu} = |\mu|\varphi_{\nu\mu}\,, \tag{4.3}$$

i.e. the $\varphi_{\nu\mu}$ are eigenfunctions of $K_{(l)}$.

REMARK 4.1. – Using instead of left multiplication NF right multiplication FN we find similarly

$$K_{(r)}\varphi_{\nu\mu} = |\nu|\varphi_{\nu\mu}\,, \tag{4.4}$$

with

$$K_{(r)} = \frac{1}{2}\left(-\Delta + \frac{1}{4}\zeta^2 - \frac{i}{2}\sum_{j=1}^{n}\left(b_j\frac{\partial}{\partial a_j} - a_j\frac{\partial}{\partial b_j}\right) - n\right).$$

Thus upon adding (4.3) and (4.4) we see that

$$J\varphi_{\nu\mu} = (|\mu| + |\nu|)\varphi_{\nu\mu}$$

with

$$J = K_{(l)} + K_{(r)} = -\Delta + \tfrac{1}{4}\zeta^2 - n\,.$$

We draw the conclusion that the $\varphi_{\nu\mu}$ are linear combinations of functions $\varphi_{\nu'}\otimes\varphi_{\mu'}$ with $|\mu'| + |\nu'| = |\mu| + |\nu|$. This yields another interesting connection between Hermite and Laguerre functions (cf. [18]), at least a special case of which is known, indeed due to Feldheim (cf. ERDÉLYI [4], vol. II, p. 195 [32]). The rightist theory will not be pursued in what follows.

After these preparations, we introduce the normed Abelian group of functions Λ such that

$$\|f\|_{\Lambda} \leqslant t \quad \text{iff} \quad f = \sum_{|\mu|+1\leqslant t} a_{\nu\mu}\varphi_{\nu\mu}\,,$$

i.e. f is a linear combination of eigenfunctions of $K_{(l)} + I$ belonging to eigenvalues $\leqslant t$.

If $f \in \Lambda$, with $\|f\|_{\Lambda} \leqslant t$, and $F = \mathfrak{W}f$, it is clear that

$$\operatorname{rank}(F) \leqslant \sum_{|\mu|+1\leqslant t} 1 \leqslant Ct^n\,, \tag{4.5}$$

with a suitable C. In other words for any $f \in \Lambda$ holds

$$\|f\|_{\tilde{\mathfrak{S}}_0} \leqslant C\|f\|_{\Lambda}^n\,,$$

and we have

$$\mathfrak{W}\colon \Lambda^{[n]} \to \tilde{\mathfrak{S}}_0\,. \tag{4.6}$$

We are now ready to interpolate between the previous (4.1) and the new (4.6). Using [21], th. 5.2, we get

$$\mathfrak{W}\colon (\Lambda^{[n]}, L_2)_{\theta q;K} \to (\tilde{\mathfrak{S}}_0, \tilde{\mathfrak{S}}_2)_{\theta q;K}\,. \tag{4.7}$$

We have to explicitize the interpolation spaces appearing in (4.7) We consider the right hand side of (4.7) first. Upon using [21], th. 5.8 and th 5.3 along with the definition of the Lorentz trace class $\mathfrak{S}_{pr}$ (cf. Sub-Section 2.1) we readily obtain

$$(\mathfrak{S}_0, \mathfrak{S}_2)_{\theta q;K} = \mathfrak{S}_{pr}^{(\theta)} \quad \text{with } \frac{1}{p} = \frac{1}{\theta} - \frac{1}{2}\,, \quad r = \theta q\,. \tag{4.8}$$

Next we turn to the left hand side of (4.7). If f has the expansion $f = \sum a_{\nu\mu}\varphi_{\nu\mu}$ we may write

$$E(t, f; \{\Lambda^{[n]}, L_2\}) = \Big(\sum_{|\mu|+1>t^{1/n}} |a_{\nu\mu}|^2\Big)^{\frac{1}{2}}$$

If follows that

$$\|f\|_{(\Lambda^{[n]},L_2)_{\alpha r;E}} = \left[\int_0^\infty t^{\alpha r}\Big(\sum_{|\mu|+1>t^{1/n}} |a_{\nu\mu}|^2\Big)^{r/2}\frac{dt}{t}\right]^{1/r}. \tag{4.9}$$

Now we specialize, taking $r = 2$. Then we may interchange $\int$ and $\sum$ in (4.9). We get

$$\|f\|_{(\Lambda^{[n]},L_2)_{\alpha 2;E}} = \left[\sum |a_{\nu\mu}|^2 \int_0^{(|\mu|+1)^n} t^{\alpha 2}\frac{dt}{t}\right]^{\frac{1}{2}} =$$

$$= C[\sum(|\mu|+1)^{\alpha n 2}|a_{\nu\mu}|^2]^{\frac{1}{2}} = C\|(K+I)^s f\| = C\|f\|_{W^s} \quad \text{with } s = \alpha n\,,$$

where we have introduced

$$W^s = W^s_{(l)} = D((K+I)^s) \quad \text{(analogue of Sobolev spaces)}.$$

Using finally [21], th. 5.10, we find

$$(\Lambda^{[n]}, L_2)_{\theta q;K} = (W^s)^{[\theta]} \quad \text{with } s = \alpha n\,, \quad \theta = \frac{1}{1+\alpha}\,, \tag{4.10}$$

$$r = \theta q\,, \quad \textit{provided } r = 2\,.$$

Upon inserting (4.8) and (4.10) in (4.7) we get

$$\mathfrak{W}\colon W^s \to \mathfrak{S}_{p2} \quad \text{with } \frac{1}{p} - \frac{1}{2} = \frac{s}{n}\,. \tag{4.11}$$

Since $W^{s_1} \supset W^{s_2}$ if $s_1 \leqslant s_2$ it follows from (4.11) that

$$\mathfrak{W}: W^s \to \mathfrak{S}_p \quad \text{with } \frac{1}{p} - \frac{1}{2} < \frac{s}{n}. \tag{4.12}$$

Let us further introduce

$$B^{sr} = B^{sr}_{(l)} = (L_2, W^{\sigma}_{(l)})_{\theta r;K} \quad \text{with } s = \theta\sigma,\ \sigma > s \text{ (analogue of Besov space)}$$

(Using PEETRE [20] we can put B^{sr} in a rather explicite form). Then another interpolation helps us (apply [21], th. 5.8) to improve (4.11) to

$$\mathfrak{W}: B^{sq} \to \mathfrak{S}_{pq} \quad \text{with } \frac{1}{p} - \frac{1}{2} = \frac{s}{n}. \tag{4.13}$$

This formula (4.13) is the desired analogue of Bernstein-Szasz. As noted in the Introduction our result improves in several respects upon the one obtained by LAVINE [7].

REMARK 4.2. – The above method is of course also applicable in the classical case of the Fourier transform $\mathcal{F}$ on $\mathbf{R}^n$ (cf. PEETRE [19]) or $\mathbf{T}^n$ (torus) (cf. LÖFSTRÖM [8]). Generally speaking it can be applied for a variety of « transforms » as soon as one has defined an analogue of the Laplacian. (In the present case it was the operator K). With the considerations of Section 3 in mind we ask in particular: What is the Laplacian on a general unimodular group G (or, dually, on $\hat{G}$)?

4.2. – We now sketch an extension of the depelopments of SubSection 4.1 to the case of an infinite number of dimensions, which might be of interest from the point of quantum field theory. This case has previously also been considered by SEGAL [26]. The step $n \to \infty$ cannot be performed without making several adjustments of the original setup of Subsection 4.1. In the first place Haar measures have to be replaced by (normalized) Gauss measures. Thus in place of $L_2(\mathbf{R}^n)$ and $L_2(\mathbf{R}^{2n})$ we should take $L_2(\mathcal{H}', \gamma')$ and $L_2(\mathcal{H}, \gamma)$ where $\mathcal{H}$ is a given complex Hilbert space and $\mathcal{H}'$ any one of its real forms, and γ and γ' the corresponding (normalized) Gauss measures. An expression for the anlogue of the Weyl transform can now readily be written down (cf. SEGAL [25]). It is however much more convenient to work with the formalism of the (boson) Fock space $\mathcal{F} = \mathcal{F}_b$ built on $\mathcal{H}$, i.e. formally we have

$$\mathcal{F}_b = \sum_{\nu=0}^{\infty} \underbrace{\mathcal{H} \odot \mathcal{H} \odot \ldots \odot \mathcal{H}}_{\nu \text{ times}}.$$

(In an analogous fashion we could also have treated the case of fermions, i.e. $\mathcal{F}_b$ has to be replaced by

$$\mathcal{F}_f = \sum_{\nu=0}^{\infty} \underbrace{\mathcal{H} \wedge \mathcal{H} \wedge \ldots \wedge \mathcal{H}}_{\nu \text{ times}}\Big).$$

If $\varphi_1, \varphi_2, \ldots$ is any orthonormal basis in $\mathcal{H}$ we obtain an orthonormal basis $\varphi_\nu = \varphi_{\nu_1\nu_2\ldots} = \varphi_{\nu_1} \odot \varphi_{\nu_2} \odot \ldots$ in $\mathcal{F}$, which will play the role of the Hermite functions. The Weyl transform can then easily be expressed in terms of the creation and annihilation operators, as defined by their action on φ_ν. The number operator N can also be defined as before. However the fault is that the basic estimate (4.5) will break down; it is not true that F will have finite rank if $f \in \Lambda$. Therefore we use instead of N the « Hamiltonian operator » H formally defined by

$$H\varphi_\nu = (\lambda_1 \nu_1 + \lambda_2 \nu_2 + \ldots)\varphi_\nu$$

where λ_j (= the energy of the j-th particle state) is a given sequence of positive numbers with $\lambda_j \to \infty$ as $j \to \infty$. Let us put

$$\omega(t) = \sum_{\lambda_1\nu_1 + \lambda_2\nu_2 + \ldots \leqslant t} 1 .$$

Then obviously $\omega(t) < \infty$ for all t, and we have, replacing (4.5)

$$\text{rank}(F) \leqslant \omega(t) . \tag{4.14}$$

It is now clear how the argument of Sub-Section 4.1 can be carried through. However the final result will not be at all so clean as previously, unless we know a priori that $\omega(t)$ can be estimated in terms of a power of t, a not very natural assumption. E.g. corresponding to (4.11) we have

$$\mathcal{W}\colon D\big((\omega(K+I)\big)^\alpha \to \mathfrak{S}_{p2} \quad \text{with } \frac{1}{p} - \frac{1}{2} = \alpha . \tag{4.15}$$

4.3. – Next we turn to the application to the spinor transforn. Let us start by briefly reviewing the necessary back-ground (cf. STREATER [30] and the works quoted there for more details). Let $\mathcal{H}$ be a complex Hilbert space and pick up any of its real forms, say, $\mathcal{H}'$. By $\mathrm{so}(\mathcal{H}')_0$ we denote the Lie algebra of skew-symmetric operators T of finite rank in $\mathcal{H}'$, and by $\mathrm{spin}(\mathcal{H}')_0$ the Lie algebra generated by all commutators $[h_1, h_2] = h_1h_2 - h_2h_1$, $h_1, h_2 \in \mathcal{H}'$, considered as elements of the Clifford algebra $\mathrm{Cliff}(\mathcal{H}')$ over $\mathcal{H}'$. The spinor transform $\mathcal{S}\colon \mathrm{so}(\mathcal{H}')_0 \to \mathrm{spin}(\mathcal{H}')_0$ is then the canonical isomorphism defined by

$$\mathcal{S}(T) = [h_1, h_2] \quad \text{if} \quad Th = [[h_1, h_2], h] .$$

Next we have to define so and spin analogues of the Lorentz spaces. We define $\mathrm{so}(\mathcal{H}')_{pq}$ as the closure of $\mathrm{so}(\mathcal{H}')_0$ in $\mathfrak{S}_{pq} = \mathfrak{S}_{pq}(\mathcal{H})$, and $\mathrm{so}(\mathcal{H}')_p$ as the closure of $\mathrm{so}(\mathcal{H}')_0$ in $\mathfrak{S}_p = \mathfrak{S}_p(\mathcal{H})$; clearly $\mathrm{so}(\mathcal{H}')_{pp} = \mathrm{so}(\mathcal{H}')_p$. Using the (unique central state ω in the complex Clifford algebra $\mathrm{Cliff}(\mathcal{H})$ and the Gel'fand-Naimark-Segal construction with this ω, we can represent Cliff($\mathcal{H}$), and thus Cliff($\mathcal{H}'$), as operators in some complex Hilbert space $\mathcal{K}$. We can now similarly define $\mathrm{spin}(\mathcal{H}')_{pq}$ and $\mathrm{spin}\,(\mathcal{H}')_p$ by using $\mathfrak{S}_{pq} = \mathfrak{S}_{pq}(\mathcal{K})$ and $\mathfrak{S}_p = \mathfrak{S}_p(\mathcal{K})$. In particular holds thus

$$\|h\|_{\mathrm{spin}(\mathcal{H}')_p} = \omega\big((h^* h)^{p/2}\big)^{1/p} \quad \text{if} \quad h \in \mathrm{spin}(\mathcal{H}')_0 .$$

We can extend $\mathcal{S}$ to $\mathrm{so}(\mathcal{H}')_{pq}$ by continuity. In particular holds (STREATER [30])

$$\mathcal{S}\colon \mathrm{so}(\mathcal{H}')_2 \to \mathrm{spin}(\mathcal{H}')_2 \quad \text{(analogue of Parseval's relation)} \tag{4.16}$$

Using (4.16), and the analogue of Minkowsky-Young, along with complex interpolation, he [30] proves the analogue of the Hausdorff-Young theorem. It is clear that we can recapture this by our methods, and now also prove the analogue of the Paley theorem, exactly as in Section 3 in the case of a unimodular group (see (3.3) and (3.4)). However we shall again restrict us to a sort of analogue of Bernstein-Szasz, as in Sub-Section 4.1 and 4.2 for the Weyl transform.

To this end, let $T \in \mathrm{so}(\mathcal{H}')_0$ and let $\{\pm 2i\lambda_k\}_{k=1}^n$, with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$, be the non-zero eigenvalues of T. Then there exists a set of unit elements $\{e_j\}_{j=1}^{2n}$ in $\mathcal{H}'$ such that

$$Te_{2k-1} = i\lambda_k e_{2k+2}, \qquad Te_{2k+2} = -i\lambda_k e_{2k-1} .$$

If follows that

$$\mathcal{S}(T) = \sum_{k=1}^{n} \lambda_k e_{2k-1} e_{2k} .$$

A simple calculation [30] shows that $\mathcal{S}(T)$ has the 2^n eigenvalues $\pm\lambda_1 \pm \lambda_2 \pm \ldots \pm \lambda_n$. Hence we have

$$\|\mathcal{S}(T)\|_{\mathrm{spin}(\mathcal{H}')_0} \leq 2^n . \tag{4.17}$$

It is now natural to introduce the Abelian group Λ corresponding to the functional

$$\|T\|_{\bar{\Lambda}} = 2^{\frac{1}{2}\,\mathrm{rank\,supp}\,T} .$$

(Cf. the definition of Λ in Sub-Section 4.1; supp T is defined in Section 1.) This is *not* a norm (nor a quasi-norm) in the strict sense of [21]. Now (4.17) can be rewritten as

$$\mathcal{S}\colon \bar{\Lambda} \to \mathrm{spin}(\mathcal{H}')_0 . \tag{4.18}$$

By interpolation ([21], th. 5.2, which result still can be adapted to our present situation) we get from (4.18) and (4.16)

$$\mathcal{S}\colon (\bar{\Lambda}, \mathrm{so}(\mathcal{H}')_2)_{\theta q:K} \to (\mathrm{spin}(\mathcal{H}')_0, \mathrm{spin}(\mathcal{H}')_2)_{\theta q:K} . \tag{4.11}$$

It is easy to see (cf. (4.8))

$$\text{(4.20)}\qquad \big(\operatorname{spin}(\mathcal{H}')_0, \operatorname{spin}(\mathcal{H}')_2\big)_{\theta q;E} \subset \operatorname{spin}(\mathcal{H}')_{pr} \quad \text{if } \frac{1}{p}=\frac{1}{\theta}-\frac{1}{2}\,,\ r=\theta q\,.$$

(Since $\operatorname{spin}(\mathcal{H}')_{pr}$ is a retract (cf. [17]) of $\mathfrak{S}_{pr}$ we have indeed $=$ in place of $\subset$ in (4.20)). Further, since

$$\|T\|_{\operatorname{so}(\mathcal{H}')_2} = \big(\textstyle\sum 2(2\lambda_k)^2\big)^{\frac{1}{2}} = 2^{\frac{3}{2}}\big(\textstyle\sum \lambda_k^2\big)^{\frac{1}{2}}$$

we have

$$E(t, T; \{\bar{A}, \operatorname{so}(\mathcal{H}')_2\}) = 2^{\frac{3}{2}}\Big(\sum_{2^k>t} \lambda_k^2\Big)^{\frac{1}{2}}$$

so

$$\text{(4.21)}\qquad \|T\|_{(\bar{A},\operatorname{so}(\mathcal{H}))_{\alpha r;E}} = 2^{3/2}\left(\int_0^\infty t^{\alpha r}\Big(\sum_{2^k>t}\lambda_k^2\Big)^{r/2}\frac{dt}{t}\right)^{1/r} \quad \text{with } \alpha=\frac{1}{\theta}-1\,,\quad r=\theta q\,.$$

Finally, we introduce the « Sobolev class » $\overline{W}^\alpha$ corresponding to the functional

$$\|T\|_{\overline{W}^\alpha} = \big(\textstyle\sum (2^{k\alpha}\lambda_k)^2\big)^{\frac{1}{2}}\,;$$

this is not a norm nor is $\overline{W}^\alpha$ a vector space. Using (4.20) and (4.21), with $r=2$, we get from (4.19), exactly as in Sub-Section 4.1,

$$\text{(4.22)}\qquad S\colon \overline{W}^\alpha \to \operatorname{spin}(\mathcal{H}')_{p2} \quad \text{if} \quad \alpha = 1/p - 1/2\,,$$

which is our desired result and should be compared to (4.11). In the same way, corresponding to (4.12), we can prove

$$\text{(4.23)}\qquad S\colon \overline{W}^\alpha \to \operatorname{spin}(\mathcal{H}')_p \quad \text{if } \alpha > \frac{1}{p}-\frac{1}{2}\,.$$

If one wants to one can also define a « Besov space » $B^{\alpha q}$ and prove a result of the type (4.13). Because of the essentially non-linar character, the analogy with Bernstein-Szacz is of course on a very superficial level.

5. – A class of Banach couples.

In this Section we want to treat the considerations of Sub-Section 2.2 in a somewhat more general framework. In particular we will prove the basic formula (2.3).

Let us recall the notion of pseudoretract (PEETRE [17]). Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be any two Banach couples. We say that $\boldsymbol{A}$ is a (bl-) *pseudoretract* of $\boldsymbol{B}$ if 1° there exists a bounded (b) (not necessary linear!) map $\alpha\colon \boldsymbol{A}\to\boldsymbol{B}$, which *by definition* means that

$$K(t, \alpha a; \boldsymbol{B}) \leqslant K(t, a; \boldsymbol{A}) \quad \text{for all } a\in\Sigma(\boldsymbol{A}) \tag{5.1}$$

and 2° there exists for any $a\in\Sigma(\boldsymbol{A})$ a linear (l) map $\beta_a\colon \boldsymbol{B}\to\boldsymbol{A}$, which *implies* that

$$K(t, \beta_a b; \boldsymbol{A}) \leqslant K(t, b; \boldsymbol{B}) \quad \text{for all } b\in\Sigma(\boldsymbol{B}), \tag{5.2}$$

such that there holds

$$a = \beta_a(\alpha(a)) . \tag{5.3}$$

(If β_a does not depend on a, $\beta_a=\beta$, we may write (5.3) simply as $\beta\circ\alpha=\mathrm{id}$, and we say that $\boldsymbol{A}$ is a (bl-) *retract* of $\boldsymbol{B}$.) If we combine (5.1) and (5.2), with $b=\alpha a$, we obtain

$$K(t, a; \boldsymbol{A}) = K(t, \alpha a; \boldsymbol{B}) \quad \text{for all } a\in\Sigma(\boldsymbol{A}) \tag{5.4}$$

We now show that formula (2.3) may be conceived as a special case of (5.4), with a convenient choice of $\boldsymbol{B}$. Denote by $\Xi(w)$ the space of Radon measures ξ on $[0,\infty)$ corresponding to the norm

$$\|\xi\|_{\Xi(w)} = \int_0^\infty w(s)|d\xi(s)| ;$$

$w=w(s)$ is a given positive weight function. We choose $\boldsymbol{B}=\boldsymbol{\Xi}=\{\Xi(s),\Xi(1)\}$. It is known that (cf. e.g. [2])

$$K(t, \xi; \{\Xi(s),\Xi(1)\}) = \int_0^\infty \min(s,t)|d\xi(s)| . \tag{5.5}$$

Let $\boldsymbol{A}=\{L_1,L_\infty\}=\{L_1(\Gamma),L_\infty(\Gamma)\}$ where Γ is any gage space. How shall we define α and β_T? We observe that, since $T^\star(t)$ is a decreasing function, we may write

$$T^\star(t) = \int_t^\infty d\xi(s)$$

with a positive ξ. Accordingly we define $\alpha(T)=\xi$, i.e.

$$T^\star(t) = \int^\infty d\alpha(s, T) .$$

It is now easy to establish condition 1°. Let us take for granted that $T \mapsto \int_0^t T^\star(s)\,ds$ is a norm, for fixed t, which may be established in one way or other. If $T = T_0 + T_1$ we therefore get

$$\int_0^t T^\star(s)\,ds \leqslant \int_0^t T_0^\star(s)\,ds + \int_0^t T_1^\star(s)\,ds \leqslant \|T_0\| + t\|T_1\| \,.$$

This proves the inequality $\geqslant$ in (2.3), which again by (5.5) is equivalent to (5.1), for a simple integration by parts shows that

$$\int_0^t T^\star(s)\,ds = \int_0^\infty \min(s, t)\,d\alpha(T, s)\,.$$

We proceed to the definition of β_T. For simplicity let us assume that T is positive. Consider the corresponding spectral resolution $T = \int_0^\infty \lambda\,dP(\lambda)$. If $t \to \mu(t, T)$ is the inverse of the function $t \to T^\star(t)$ we may write this as

$$T = \int_0^\infty T^\star(t)\,d\varphi(t) \tag{5.6}$$

with $\varphi(t) = P\big(\mu(t, T)\big)$. This leads us to define

$$\beta_T(\xi) = \int_0^\infty \int_t^\infty d\xi(s)\,d\varphi(t)\,.$$

Clearly β_T is linear; boundedness and thus (5.2) follows from

$$\|\beta_T\|_{L_1} \leqslant \|\xi\|_{\Xi(s)}\,, \qquad \|\beta_T\|_{L_\infty} \leqslant \|\xi\|_{\Xi(l)}$$

where we have used

$$(\beta_T(\xi))^\star = \xi \qquad \text{if } \xi \text{ positive;}$$

we need establish (5.7) for ξ positive only. Finally (5.3) is a restatement of (5.6), and we have verified condition 2° entirely. Thus in sum we have proven

Proposition 5.1. – For any gage space Γ, $\{L_1, L_\infty\}$ is a pseudoretract of Ξ. □

In particular (2.3) has been proven. There arises the obvious question whether the other developments of Sub-Section 2.2 can be treated on the same general level. (As another, more general instance to which our trentment generalizes we mention the case of the Lorentz couple $\{L_{p_0 1}, L_{p_1 1}\}$. If $p_0 = 1$, $p_1 = \infty$ we get back $\{L_1, L_\infty\}$). In this direction we can prove the analogue of th. 2.4.

PROPOSITION 5.2. – Let $\boldsymbol{A} = \{A_0, A_1\}$ be any pseudoretract of $\boldsymbol{\Xi}$, with α and β_a having the same meaning as above. Consider the space A corresponding to the norm

$$\|a\|_A = \left(\int_0^\infty \left(\int_t^\infty d\alpha(s, a)\right)^p ds\right)^{1/p}, \qquad 1 < p < \infty .$$

Then A is a K-space for $\boldsymbol{A}$.

PROOF. – Obvious adaptation of the proof of th. 2.4. □

REMARK 5.1. – SEDAEV and SEMENOV [22] recently proved the analogue of the Mitjagin-Cotlar result (cf. remark 2.1) for the couple Ξ (which incidently one of us conjectured already years ago). It is now natural to ask whether an analogous result holds for any pseudoretract of Ξ.

REMARK 5.2. – It appears likewise likely that a similar abstract treatment of the theory indicated in Sub-Section 2.3 is possible.

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