# A Function Theoretic Method for $\Delta_4^2 u + Q(x)u = 0$ (*) (**).

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**Summary.** – *The approach used in this paper generalizes Colton's treatment* [4] *of certain second order elliptic equations in four independent variables to the fourth order case. This method is essentially a function theoretic one that is based on the earlier work of Tjong* [11]. *An integral operator is found that permits one to construct a complete family of solutions with respect to uniform convergence in compact sets of $E^4$. Consequently, one is provided with a useful numerical procedure for solving the associated boundary value problems.*

## 1. – Introduction.

BERGMAN [1] and VEKUA [12] developed the function theoretic approach for solving elliptic equations in two variables with analytic coefficients. Recently, progress has been made in extending these techniques to higher dimension, notably the work of TJONG [11], GILBERT and LO [8], COLTON and GILBERT [5], and COLTON [2], [3], [4]. These works have dealt with second order equations of the form

$$\Delta_k u + \sum_{l=1}^{k} A_l(\boldsymbol{x}) \frac{\partial u}{\partial x_l} + Q(\boldsymbol{x})u = 0\,, \qquad k = 3, 4 \tag{1.1}$$

and ($\boldsymbol{x} = (x_1, x_2, x_3)$ or $(x_1, x_2, x_3, x_4)$, depending on whether $k = 3$, or 4, and special instances where the coefficients $A_l(\boldsymbol{x}) \equiv 0$. GILBERT and KUKRAL [7] then showed that similar techniques permitted the treatment of the fourth order equation with three variables,

$$\Delta_3^2 u + Q(\boldsymbol{x})\, u = 0\,. \tag{1.2}$$

This was the first time a function theoretic method was devised which generated all real solutions of a higher order, elliptic equation in three variables with an arbitrary coefficient. Furthermore, a scheme was given by which one could obtain complete families of solutions in starlike regions.

In the present note we shall develop the analogous theory for the four dimensional equation

$$\Delta_4^2 u + \hat{Q}(\boldsymbol{x}) u = 0\,. \tag{1.3}$$

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$\hat{Q}(\boldsymbol{x})$ is assumed to be an entire function of the real variables $x_1$, $x_2$, $x_3$, and $x_4$, i.e. $\hat{Q}(\boldsymbol{x})$ is an entire function for $\boldsymbol{x} \in \mathbb{C}^4$ and is real valued for $x \in \mathbb{R}^4$.

In the case when $\hat{Q}(\boldsymbol{x}) \equiv 0$, one obtains an operator which generates solutions to the biharmonic equations, and bears a formal resemblence to the operator $\boldsymbol{G}_4$ [6], namely

$$u(\boldsymbol{x}) = \boldsymbol{G}_4\big(f_1(\mu, \zeta, \eta) + Y^* f_2(\mu, \zeta, \eta)\big) . \tag{1.4}$$

where $Y^* = x_1 + ix_2$, $\mu = \frac{1}{2}(x_1 + ix_2) + \frac{1}{2}(x_3 + ix_4)\zeta^{-1} + \frac{1}{2}(-x_3 + ix_4)\eta^{-1} + \frac{1}{2}(x_1 - ix_2)y^{-1}$, and where the $f_1$, $f_2$ are holomorphic functions of their arguments.

## 2. – The integral operator.

We first establish an elementary result which will be quite useful to us later on

LEMMA 2.1. – *Let* $Y = (x_1 + ix_2)/2$, $Y^* = (x_1 - ix_2)/2$, $Z = (x_3 + ix_4)/2$, $Z^* = (-x_3 + ix_4)/2$, *and let* $u(\boldsymbol{x})$ *be a real valued* $\mathbb{C}^4$ *solution of equation* (1.3) *in a neighborhood of the origin. Then* $U(Y, Y^*, Z, Z^*) \equiv u(\boldsymbol{x})$ *is an analytic function of* $Y$, $Y^*$ $Z$, $Z^*$ *in some neighborhood of the origin in* $\mathbb{C}^4$ *and is uniquely determined by the functions* $F(Y, Z, Z^*) \equiv U(Y, 0, Z, Z^*)$ *and* $G(Y, Z, Z^*) \equiv U_{Y^*}(Y, 0, Z, Z^*)$.

PROOF. – Since $\hat{Q}(\boldsymbol{x})$ is entire in $\mathbb{C}^4$, the solution $u(\boldsymbol{x})$ is analytic and $U(Y, Y^*, Z, Z^*)$ is analytic. Hence locally we can write,

$$U(Y, Y^*, Z, Z^*) = \sum_{i,j,k,l=0}^{\infty} \alpha_{i,j,k,l} Y^i Y^{*j} Z^k Z^{*l} , \tag{2.1}$$

$$U(Y, 0, Z, Z^*) = \sum_{i,k,l=0}^{\infty} \alpha_{i,0,k,l} Y^i Z^k Z^{*l} , \tag{2.2}$$

$$Y_{Y^*}(Y, Y^*, Z, Z^*) = \sum_{i,j,k,l=0}^{\infty} j\alpha_{i,j,k,l} Y^i Y^{*j-1} Z^k Z^{*l} , \tag{2.3}$$

$$U_{Y^*}(Y, 0, Z, Z^*) = \sum_{i,k,l=0}^{\infty} \alpha_{i,1,k,l} Y^i Z^k Z^{*l} , \tag{2.4}$$

$$U(0, Y^*, Z, Z^*) = \sum_{j,k,l=0}^{\infty} \alpha_{0,j,k,l} Y^{*j} Z^k Z^{*l} , \tag{2.5}$$

$$U_Y(Y, Y^*, Z, Z^*) = \sum_{j,k,l=0}^{\infty} i\alpha_{i,j,k,l} Y^{i-1} Y^{*j} Z^k Z^{*l} , \tag{2.6}$$

$$U_Y(0, Y^*, Z, Z^*) = \sum_{j,k,l=0}^{\infty} \alpha_{1,j,k,l} Y^{*j} Z^k Z^{*l} . \tag{2.7}$$

Since $u(\boldsymbol{x})$ is real valued, we have for $x_1, x_2, x_3, x_4$ real,

$$U(Y, Y^*, Z, Z^*) = \overline{U(Y, Y^*, Z, Z^*)}\,, \tag{2.8}$$

where the bar denotes complex conjugation. This implies that for $x_1, x_2, x_3, x_4$ real

$$\sum_{i,j,k,l=0}^{\infty} \alpha_{i,j,k,l} Y^i Y^{*j} Z^k Z^{*l} = \sum_{i,j,k,l=0}^{\infty} \overline{\alpha_{i,j,k,l}}\, Y^j Y^{*i} (-Z^*)^k (-Z)^l\,. \tag{2.9}$$

Hence, it follows that

$$\alpha_{i,j,k,l} = (-1)^{k+l}\,\overline{\alpha_{j,i,l,k}}\,. \tag{2.10}$$

Equations (2.2), (2.5), and (2.10) now show that $U(0, Y^*, Z, Z^*)$ is determined uniquely by $U(Y, 0, Z, Z^*)$. Also, equations (2.4), (2.7), and (2.10) show that $U_Y(0, Y^*, Z, Z^*)$ is uniquely determined by $U_{Y^*}(Y, 0, Z, Z^*)$. In the $Y$, $Y^*$, $Z$, $Z^*$ variables equation (1.3) becomes the equation

$$U_{YYY^*Y^*} = 2U_{YY^*ZZ^*} - U_{ZZZ^*Z^*} + Q(Y, Y^*, Z, Z^*)\,U = 0\,, \tag{2.11}$$

where

$$Q(Y, Y^*, Z, Z^*) = \hat{Q}(\boldsymbol{x})\,. \tag{2.12}$$

From Hormander's generalized Cauchy-Kowalewski theorem [9] we have that $U(Y, Y^*, Z, Z^*)$ is uniquely determined by $U(0, Y^*, Z, Z^*)$, $U_Y(0, Y^*, Z, Z^*)$ $U(Y, 0, Z, Z^*)$ and $U_{Y^*}(Y, 0, Z, Z^*)$ which we have shown to be determined by $U(Y, 0, Z, Z^*)$ and $U_{Y^*}(Y, 0, Z, Z^*)$ alone. So the lemma is proven.

We use the following coordinates introduced by COLTON [3] for the second order case,

$$\begin{aligned} (2.12)\qquad & \xi_1 = Y^*/\eta\zeta\,, && \xi_2 = Y^*/\eta\zeta + Z^*/\eta\,,\\ & \xi_3 = Z^*/\eta + Y\,, && \xi_4 = Z/\zeta + Y\,,\\ & \mu = \xi_2 + \xi_4 = Y + Z/\zeta + Z^*/\eta + Y^*/\zeta\eta\,, &&\\ & \omega = (1 - t^2)\mu\,. && \end{aligned}$$

THEOREM 2.2. – Let $D$ be a neighborhood of the origin in the $\mu$ plane, $B = \{(\zeta, \eta)|1 - \varepsilon < |\zeta| < 1 + \varepsilon, 1 - \varepsilon < |\eta| < 1 + \varepsilon\}$, $G$ a neighbordhood of the origin in the $\xi_1$, $\xi_2$, $\xi_3$, $\xi_4$ space and $T = \{t\colon |t| \leqslant 1\}$. Let $f(\mu, \zeta, \eta)$ be a function of three complex variables analytic in the product domain $D \times B$, and

$$E^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) = E(Y, Y^*, Z, Z^*, \zeta, \eta, t)$$

*be a regular solution of the partial differential equation*

$$(2.14)\qquad \begin{aligned} &E^*_{1133}+E^*_{1144}+E^*_{2233}+E^*_{3344}+2E^*_{1134}+2E^*_{1233}\\ &-2E^*_{1334}+2E^*_{1234}-2E^*_{1344}-2E^*_{2334}\\ &-\frac{1}{t^2\mu}\{E^*_{113}+E^*_{114}+E^*_{123}-E^*_{134}\}\\ &+\frac{(1-t^2)}{t\mu}\{E^*_{113t}+E^*_{114t}+E^*_{123t}-E^*_{134t}\}\\ &\qquad+\frac{3}{4}\frac{E^*_{11}}{t^4\mu^2}-\frac{3(1-t^4)E^*_{11t}}{4t^3\mu^2}+\frac{E^*_{11tt}(1-t^2)^2}{4t^2\mu^2}+\eta^2\zeta^2Q^*E^*=0\,, \end{aligned}$$

in $G\times B\times T$ *where* $Q^*(\xi_1,\xi_2,\xi_3,\xi_4,\zeta,\eta)=Q(Y,Y^*,Z,Z^*)$ *and the subscripts denote differentiation with respect to* $\xi_1,\xi_2,\xi_3,\xi_4$, or $t$. *Then*

$$(2.15)\qquad U(Y,Y^*,Z,Z^*)=\frac{-1}{4\pi^2}\int_{|\zeta|=1}\int_{|\eta|=1}\int_{\gamma}E(Y,Y^*,Z,Z^*,\zeta,\eta,t)\, f(\omega,\zeta,\eta)\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}\,,$$

*where* $\gamma$ *is a path in* $T$ *joining* $t=-1$ *and* $t=+1$, *is a (complex valued) solution of equation* (2.11) *which is regular in aneighborhood of the origin in* $Y$, $Y^*$, $Z$, $Z^*$ *space.*

PROOF. – Since the Jacobian of the transformation (2.13) is equal to $-1/(\eta\zeta)^2\neq 0$ we can conclude that $U(Y,Y^*,Z,Z^*)$ is regular in a neighborhood of the origin in the $Y$, $Y^*$, $Z$, $Z^*$ space.

Completely straightforward differentiation together with integration by parts in the $t$ variable shows that if $E=E^*$ is a solution of equation (2.14) then equation (2.15) defines $U$ as a solution of (2.11), which proves the theorem.

We must show that solutions of (2.14) do indeed still exist.

THEOREM 2.3. – If we formally define

$$(2.16)\qquad E^*=1+\sum_{n=1}^{\infty}t^{2n}\mu^n p^{(n)}(\xi_1,\xi_2,\xi_3,\xi_4,\zeta,\eta)\,,$$

$$(2.17)\qquad \hat{E}^*=\zeta\eta\xi_1+\sum_{n=1}^{\infty}t^{2n}\mu^n q^{(n)}(\xi_1,\xi_2,\xi_3,\xi_4,\zeta,\eta)\,,$$

*then a sufficient condition that* $E^*$ *and* $\hat{E}^*$ *formally satisfy equation* (2.14) *is that*

$$(2.18)\qquad \begin{aligned} p^{(1)}_{11}&=0=q^{(1)}_{11}\\ p^{(2)}_{11}&=-\tfrac{4}{3}\eta^2\zeta^2Q^*\\ q^{(2)}_{11}&=-\tfrac{4}{3}\eta^3\zeta^3\xi_1Q^* \end{aligned}$$

*and both the* $p^{(n)}$ *and* $q^{(n)}$ *satisfy the following equation for* $n \geq 1$,

$$(2.19) \qquad p_{11}^{(n+2)} = \frac{-4}{(2n+3)(2n+1)}\{(2n+1)(p_{113}^{(n+1)} + p_{114}^{(n+1)} + p_{123}^{(n+1)} - p_{134}^{(n+1)}) + \\ + \zeta^2\eta^2 Q^* p^{(n)} + p_{1133}^{(n)} + p_{1144}^{(n)} + p_{2233}^{(n)} + p_{3344}^{(n)} + 2p_{1134}^{(n)} + \\ + 2p_{1233}^{(n)} + 2p_{1234}^{(n)} - 2p_{1334}^{(n)} - 2p_{1344}^{(n)} - 2p_{2334}^{(n)}\}\,.$$

PROOF. – The proof is again accomplished by completely straightforward differentiation.

THEOREM 2.4. – Let $D_r = \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_i| < r, i = 1, 2, 3, 4\}$, *where* $r$ *is an arbitrary positive number, and let* $B_{2\varepsilon} = \{(\zeta, \eta) : |\zeta - \zeta_0| < 2\varepsilon,\ |\eta - \eta_0| < 2\varepsilon,\ 0 < \varepsilon < \frac{1}{2}\}$, *where* $\zeta_0$ *and* $\eta_0$ *are arbitrary with* $|\zeta_0| = |\eta_0| = 1$. *Then for each* $n$, $n = 1, 2, \ldots$, *there exists unique functions* $p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$ *and* $q^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$ *which are regular in* $D_r \times B_{2\varepsilon}$ *and satisfy*

$$(2.20) \qquad \begin{aligned} p^{(1)} &= 0 = q^{(1)} \\ p_{11}^{(2)} &= -\tfrac{4}{3}\eta^2\zeta^2 Q^* \\ q_{11}^{(2)} &= -\tfrac{4}{3}\eta^3\zeta^3\xi_1 Q^* \end{aligned}$$

*and for* $n \geq 1$ *both* $p^{(n+2)}$ *and* $q^{(n+2)}$ *satisfy* (2.19) *such that for* $n = 1, 2, \ldots$

$$(2.21) \qquad p^{(n)}(0, \xi_2, \xi_3, \xi_4, \zeta, \eta) = 0 = q^{(n)}(0, \xi_2, \xi_3, \xi_4, \zeta, \eta)$$

$$(2.22) \qquad p_1^{(n)}(0, \xi_2, \xi_3, \xi_4, \zeta, \eta) = 0 = q_1^{(n)}(0, \xi_2, \xi_3, \xi_4, \zeta, \eta)\,.$$

Furthermore, the functions

$$(2.23) \qquad E^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) = 1 + \sum_{n=2}^{\infty} t^{2n}\mu^n p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$$

$$(2.24) \qquad \hat{E}^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) = \eta\zeta\xi_1 + \sum_{n=2}^{\infty} t^{2n}\mu^n q^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$$

*are solutions of the differential equation* (2.14) *which are regular in the product domain* $G_R \times B \times T$ *where* $R$ *is an arbitrary positive number and*

$$\begin{aligned} G_R &= \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_i| < R,\ i = 1, 2, 3, 4\}\,, \\ B &= \{(\zeta, \eta) : 1 - \varepsilon < |\zeta| < 1 + \varepsilon,\ 1 - \varepsilon < |\eta| < 1 + \varepsilon,\ 0 < \varepsilon < \tfrac{1}{2}\}\,, \\ T &= \{t : |t| \leq 1\}\,, \end{aligned}$$

Also

(2.25) $$E^*(0, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) = 1$$

(2.26) $$E_1^*(0, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) = 0$$

(2.27) $$\hat{E}^*(0, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) = 0$$

(2.28) $$\hat{E}_1^*(0, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) = \eta\zeta\,.$$

PROOF. – For $n = 1$, $p^{(1)} \equiv 0 \equiv q^{(1)}$ satisfy the requirements. For $n = 2$, equations (2.20) have the solutions which satisfy (2.21) and (2.22)

(2.29) $$p^{(2)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta) = -\tfrac{4}{3}\eta^2\zeta^2\int_0^{\xi_1}\int_0^{\xi_1'} Q^*(\xi_1'', \xi_2, \xi_3, \zeta, \eta)\,d\xi_1''\,d\xi_1'$$

$$q^{(2)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta) = -\tfrac{4}{3}\eta^2\zeta^2\int_0^{\xi_1}\int_0^{\xi_1'} \xi_1'' Q^*(\xi_1'', \xi_2, \xi_3, \xi_4, \zeta, \eta)\,d\xi_1''\,d\xi_1'$$

and are uniquely determined and regular in $\overline{D_r}\times\overline{B_{2\varepsilon}}$ (for arbitrary $(\zeta_0, \eta_0)$). By induction $p^{(n)}$ and $q^{(n)}$ are uniquely determined and regular in $\overline{D_r}\times\overline{B_{2\varepsilon}}$.

Now consider the formal series defined by equations (2.23) and (2.24). Theorem 2.3 showed that if $p^{(n)}$ and $q^{(n)}$ satisfy equations (2.19) through (2.22) then $E^*$ and $\hat{E}^*$ formally satisfy the differential equation (2.14).

It remains to be shown that $E^*$ and $\hat{E}^*$ are regular in $G_R\times B\times T$, i.e. that series (2.23) and (2.24) converge absolutely and uniformly in this region. In these proofs the only essential differences between the series for $E^*$ and $\hat{E}^*$ is the appearance of « $\eta^2\zeta^2 Q^*$ » in one case and « $\eta^3\zeta^3\xi_1 Q$ » in another case. Since we use only the fact that « $\eta^2\zeta^2Q^*$ » is regular in $\overline{D_r}\times\overline{B_{2\varepsilon}}$, these proofs are essentially identical and we will present only the proof for $E^*$.

Since $\bar{B}$ is a compact subset of the $(\zeta, \eta)$ space, there are finitely many points $(\zeta_j, \eta_j)$ with $|\zeta_j| = |\eta_j| = 1$, $j = 1, 2, \ldots, N$ such that $\bar{B}$ is covered by the union of the sets $N_j = \{(\zeta, \eta)\colon |\zeta - \zeta_j| < -\varepsilon, |\eta - \eta_j| < -\varepsilon\}$ $j = 1, 2, \ldots, N$. So it is sufficient to show that the series converges absolutely and uniformly in $\overline{G_R}\times\overline{N_j}\times T$. To this end we majorize the $p^{(n)}$ in $D_r\times B_{2\varepsilon}$ (where « $(\xi_0, \eta_0)$ » is taken to represent any $(\xi_j, \eta_j)$).

Since $Q(Y, Y^*, Z, Z^*)$ is an entire function, it follows that $Q^*(\xi_1, \xi_2, \xi_3, \zeta, \eta)$ is regular in $\overline{D_r}\times\overline{B_{2\varepsilon}}$ and hence we have

(2.30) $$\zeta^2\eta^2 Q^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta) \ll C\left(1-\frac{\xi_1}{r}\right)^{-1}\left(1-\frac{\xi_2}{r}\right)^{-1}\cdot$$
$$\cdot\left(1-\frac{\xi_3}{r}\right)^{-1}\left(1-\frac{\xi_4}{r}\right)^{-1}\left(1-\frac{\zeta-\zeta_j}{2\varepsilon}\right)^{-1}\left(1-\frac{\eta-\eta_j}{2\varepsilon}\right)^{-1}$$

where, as in usual practice, « $\ll$ » means «is dominated by» or «is majorized by». Using a straightforward but lengthy procedure it can be shown by induction, using properties of dominants, that in $\overline{D_r}\times\overline{B_{2\varepsilon}}$ we have for $k\geqslant 2$

$$p_{11}^{(k)}\ll\frac{M(24+\delta)^{k-2}}{(2k-1)r^{k-2}}\left(1-\frac{\xi_1}{r}\right)^{-(k-1)}\left(1-\frac{\xi_2}{r}\right)^{-(k-1)}\cdot \cdot\left(1-\frac{\xi_3}{r}\right)^{-(k-1)}\left(1-\frac{\xi_4}{r}\right)^{-(k-1)}\left(1-\frac{\zeta-\zeta_j}{2\varepsilon}\right)^{-(k-1)}\left(1-\frac{\eta-\eta_j}{2\varepsilon}\right)^{-(k-1)} \tag{2.31}$$

where $M$ and $\delta$ are positive constants independent of $k$, and $\delta$ is independent of $r$.

Using properties of dominants it follows from equation (2.31) that

$$p^{(k)}\ll\frac{M(24+\delta)^{k-2}}{r^{k-4}(2k-1)k(k-1)}\left(1-\frac{\xi_1}{r}\right)^{-(k-1)}\cdot \cdot\left(1-\frac{\xi_2}{r}\right)^{-(k-1)}\left(1-\frac{\xi_3}{r}\right)^{-(k-1)}\left(1-\frac{\xi_4}{r}\right)^{-(k-1)}\cdot\left(1-\frac{\zeta-\zeta_j}{2\varepsilon}\right)^{-(k-1)}\left(1-\frac{\eta-\eta_j}{2\varepsilon}\right)^{-(k-1)} \tag{2.32}$$

and hence in $\overline{D_r}\times\overline{N_j}$ we have

$$|p^{(k)}|\leqslant\frac{M(24+\delta)^{k-2}}{r^{k-4}(2k-1)k(k-1)}\left(1-\frac{|\xi_1|}{r}\right)^{-(k-1)}\cdot \cdot\left(1-\frac{|\xi_2|}{r}\right)^{-(k-1)}\left(1-\frac{|\xi_3|}{r}\right)^{-(k-1)}\left(1-\frac{|\zeta-\zeta_i|}{2\varepsilon}\right)^{-(k-1)}\left(1-\frac{|\eta-\eta_j|}{2\varepsilon}\right)^{-(k-1)}. \tag{2.33}$$

Now consider $|t^{2n}\mu^n p^{(n)}|$ in $\overline{D_{\alpha r}}\times\overline{N_j}\times T$ where

$$D_{\alpha r}=\left\{(\xi_1,\xi_2,\xi_3,\xi_4):|\xi_i|<\frac{r}{\alpha}\,;\,\alpha>1,\ i=1,2,3,4\right\}$$

in $\overline{D_{\alpha r}}\times\overline{N_j}\times T$ we have

$$\begin{aligned}
&1-\frac{|\xi_i|}{r}\geqslant\frac{\alpha-1}{\alpha}, \qquad i=1,2,3,4\,,\\
&1-\frac{|\zeta-\zeta_j|}{2\varepsilon}\geqslant\frac{1}{4},\\
&1-\frac{|\eta-\eta_j|}{2\varepsilon}\geqslant\frac{1}{4},\\
&|\mu|=|\xi_2+\xi_4|\leqslant\frac{2r}{\alpha},\\
&|t|\leqslant 1\,.
\end{aligned} \tag{2.34}$$

So from eqs. (2.23) and (2.34) we have in $\overline{D_{\alpha r}}\times\overline{N_j}\times T$

$$|t^{2k}\mu^k p^{(k)}|\leq\frac{M(24+\delta)^{k-2}}{r^{k-4}(2k-1)k(k-1)}\left(\frac{\alpha}{\alpha-1}\right)^{4(k-1)}4^{2(k-1)}\left(\frac{2r}{\alpha}\right)^k = \tag{2.35}$$

$$=\frac{Mr^4(\alpha-1)^4}{16(24+\delta)^2(2k-1)k(k-1)\alpha^4}\left\{\frac{32\alpha^3(24+\delta)}{(\alpha-1)^4}\right\}^k$$

so that if we choose $\alpha$ such that

$$32(24+\delta)\alpha^3(\alpha-1)^{-4}<1 \tag{2.36}$$

then the series (2.23) converges absolutely and uniformly in $\overline{D_{\alpha r}}\times\overline{N_j}\times T$. By taking $r=\alpha R$ we can conclude that $E^*(\xi_1,\xi_2,\xi_3,\xi_4,\zeta,\eta,t)$ is regular in $\overline{G_R}\times\overline{N_j}\times T$ for $j=1,2,\dots,N$ and hence in $G_R\times B\times T$, which completes the proof.

It follows from what has been shown in the previous theorems that since $\hat{Q}(\boldsymbol{x})$ is real valued (for $\boldsymbol{x}\in\mathbb{R}^4$) the operator

$$u(\boldsymbol{x})=U(Y,Y^*,Z,Z^*)$$
$$\equiv\operatorname{Re}\boldsymbol{P}_4^{(2)}\{f,\hat{f}\}$$

$$\equiv\operatorname{Re}\left\{\frac{-1}{4\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1}\int\limits_{\gamma}E(Y,Y^*,Z,Z^*,\zeta,\eta,t)f(\omega,\zeta,\eta)\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}-\right. \tag{2.37}$$

$$\left.-\frac{1}{4\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1}\int\limits_{\gamma}\hat{E}(Y,Y^*,Z,Z^*,\zeta,\eta,t)\hat{f}(\omega,\zeta,\eta)\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}\right\},$$

carries pairs of analytic functions in three variables to real valued solutions of the differential equation (1.3).

## 3. – Inversion of the integral operator $\boldsymbol{P}_4^{(2)}$.

We wish to show now that any $C^4$ solution of (1.3) defined in a neighborhood of the origin can be expressed locally in the form

$$U(\boldsymbol{x})=\operatorname{Re}\boldsymbol{P}_4^{(2)}\{f,\hat{f}\}, \tag{3.1}$$

where $f$ and $\hat{f}$ are determined by the Goursat data for $U(Y,Y^*,Z,Z^*)$.

THEOREM 3.1. – *Let $U(\boldsymbol{x})$ be a real valued $C^4$ solution of equation* (1.3) *in some neighborhood of the origin in $R^4$. Then there exists a pair of analytic functions of three complex variables.*

*$\{f(\mu, \zeta, \eta), \hat{f}(\mu, \zeta, \eta\}$, which are regular for $\mu$ in some neighborhood of the origin and $|\zeta| < 1 + \varepsilon$, $|\eta| < 1 + \varepsilon$, $\varepsilon > 0$, such that locally $u(x_1, x_2, x_3, x_4) = \mathrm{Re}\, \boldsymbol{P}_4^{(2)}\{f, \hat{f}\}$. In particular, denote by $U(Y, Y^*, Z, Z^*)$ the extension of $u(x_1, x_2, x_3, x_4)$ to the $Y, Y^*, Z, Z^*$ space and let*

$$F(Y, Z, Z^*) \equiv U(Y, 0, Z, Z^*)\,, \tag{3.2}$$

$$G(Y, Z, Z^*) \equiv U_{Y^*}(Y, 0, Z, Z^*)\,, \tag{3.3}$$

$$g(\mu, \zeta, \eta) \equiv \frac{\partial^2}{\partial\mu^2} \int_0^1 \int_0^1 \{2F(\mu t, \mu\zeta(1-t)s, \mu\eta(1-t)(1-s)) - \\ - F(0, \mu\zeta(1-t)s, \mu\eta(1-t)(1-s))\}\mu^2(1-t)\, dt\, ds\,, \tag{3.4}$$

$$\hat{g}(\mu, \zeta, \eta) \equiv \frac{\partial^2}{\partial\mu^2} \int_0^1 \int_0^1 \mu^2(1-t)\{2G(\mu t, \mu\zeta(1-t)s, \mu\eta(1-t)(1-s)) - \\ - 2G(0, (1-t)s\mu\zeta, (1-t)(1-s)\mu\eta) - \\ - \mu t G_1(0, (1-t)s\mu\zeta, (1-t)(1-s)\mu\eta)\}\, dt\, ds - \\ - \frac{1}{\zeta\eta}\frac{\partial}{\partial\mu}\{g(\mu, \zeta, \eta) - g(\mu, 0, \eta) - g(\mu, \zeta, 0) + g(\mu, 0, 0)\}\,, \tag{3.5}$$

then

$$f(\mu, \zeta, \eta) \equiv \frac{-1}{2\pi} \int_{\gamma'} g(\mu(1-t^2), \zeta, \eta)\, \frac{dt}{t^2}\,, \tag{3.6}$$

$$\hat{f}(\mu, \zeta, \eta) \equiv \frac{-1}{2\pi} \int_{\gamma'} \hat{g}(\mu(1-t^2), \zeta, \eta)\, \frac{dt}{t^2}\,, \tag{3.7}$$

*where $\gamma'$ is a rectifiable arc joining the points $t = -1$ and $t = +1$ but not passing through the origin.*

REMARK. – It can be shown that $\hat{g}(\mu, \zeta, \eta)$ and $g(\mu, \zeta, \eta)$ can be expressed in terms of $f(\mu, \zeta, \eta)$ and $\hat{f}(\mu, \zeta, \eta)$ by

$$g(\mu, \zeta, \eta) = \int_{\gamma} f(\mu[1-t^2], \zeta, \eta)\, \frac{dt}{\sqrt{1-t^2}}\,, \tag{3.8}$$

$$\hat{g}(\mu, \zeta, \eta) = \int_{\gamma} \hat{f}(\mu[1-t^2], \zeta, \eta)\, \frac{dt}{\sqrt{1-t^2}}\,. \tag{3.9}$$

PROOF. – For the purpose of exposition we present here only the barest outline of the proof. We hasten to add that the proofs of this and the other theorems of this paper are presented in great detail in [10] which is available on microfilm.

Since $u(\boldsymbol{x})$ is a $\mathbf{C}^4$ solution of equation (1.3) in some neighborhood of the origin and $\hat{Q}(\boldsymbol{x})$ is analytic, we can conclude that $u(\boldsymbol{x})$ is an analytic function of its four variables in a neighborhood of the origin. Also, since $\hat{Q}(\boldsymbol{x})$ is real valued, $\operatorname{Re}\boldsymbol{P}_4^{(2)}\{f,\hat{f}\}$ is a real valued solution (for $\boldsymbol{x}\in\mathbb{R}^4$) of equation (1.3) for any functions $f$, $\hat{f}$ which are analytic in the product fomain $D\times B$ (see theorem 2.2).

Using the notation $\bar{g}(\mu,\zeta,\eta)=\overline{g(\bar{\mu},\bar{\bar{\zeta}},\bar{\eta})}$, $\bar{f}(\mu,\zeta,\eta)\equiv\overline{f(\bar{\mu},\bar{\bar{\zeta}},\bar{\eta})}$, etc., and letting $E$ and $\bar{\hat{E}}$ be the generating functions corresponding to the equation $\Delta_4^2 U+\bar{Q}U=0$, we can write for $x_1, x_2, x_3, x_4$ real,

$$
\begin{aligned}
(3.10)\quad \operatorname{Re}\boldsymbol{P}_4^{(2)}\{f,\hat{f}\} &= \\
&=\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1}\int\limits_{\gamma} E(Y,Y^*,Z,Z^*,\zeta,\eta,t)\,f(\omega,\zeta,\eta)\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta} \\
&\quad\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1}\int\limits_{\gamma} \bar{E}(Y^*,Y,-Z^*,-Z,\zeta,\eta,t)\,\hat{f}(\bar{\omega},\zeta,\eta)\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}+ \\
&\quad\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1}\int\limits_{\gamma} \hat{E}(Y,Y^*,Z,Z^*,\zeta,\eta,t)\hat{f}(\omega,\zeta,\eta)\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta} \\
&\quad\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1}\int\limits_{\gamma} \bar{\hat{E}}(Y^*,Y,-Z^*,-Z,\zeta,\eta,t)\bar{\hat{f}}(\bar{\omega},\zeta,\eta)\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta},
\end{aligned}
$$

where

$$\omega=\mu[1-t^2]\qquad\text{and}\qquad\bar{\omega}=\left(Y^*+\frac{Y}{\eta\zeta}-\frac{Z^*}{\zeta}-\frac{Z}{\eta}\right)[1-t^2].$$

By theorem (2.1) we know that $U(Y,Y^*,Z,Z^*)$ is uniquely determined by its Goursat data along $Y^*=0$ so we attempt to determine the pair $\{f,\hat{f}\}$ in terms of $F(Y,Y^*,Z^*)$ and $G(Y,Z,Z^*)$. From the initial conditions (2.25) through (2.28), the relations (3.8) and 3.9), and some rather involved arguments, we are able to reduce equation (3.10) to the required condition,

$$
\begin{aligned}
(3.11)\qquad F(Y,Z,Z^*) &= \frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1} g(\mu_1,\zeta,\eta)\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}+ \\
&\quad\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1} \hat{g}(\mu_2,\zeta,\eta)\frac{d\eta}{\eta}\frac{d\zeta}{\zeta} \\
&\quad\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1} Y\bar{\hat{g}}(\mu_2,\zeta,\eta)\frac{d\eta}{\eta}\frac{d\zeta}{\zeta},
\end{aligned}
$$

where

$$(3.12)\qquad \mu_1=Y+\frac{Z}{\zeta}+\frac{Z^*}{\eta},$$

$$(3.13)\qquad \mu_2=\frac{Y}{\eta\zeta}-\frac{Z^*}{\zeta}-\frac{Z}{\eta}.$$

Equation (3.10) must remain true for the analytic extension of $x_1, x_2, x_3, x_4$ to complex values, so we may deduce the following statement

$$G(Y, Z, Z^*) = U_{Y^*}(Y, 0, Z, Z^*) = \tag{3.14}$$
$$\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1}\int\limits_{\gamma} E_2(Y, 0, Z, Z^*, \zeta, \eta, t) f(\mu_1[1-t^2], \zeta, \eta)\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}$$
$$\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1}\int\limits_{\gamma} E(Y, 0, Z, Z^*, \zeta, \eta, t) f_\omega(\mu_1[1-t^2], \zeta, \eta)\frac{\sqrt{1-t^2}}{\eta\zeta}dt\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}$$
$$\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1}\int\limits_{\gamma} \bar{E}_1(0, Y, -Z^*, -Z, \zeta, \eta, t)\bar{f}(\mu_2[1-t^2], \zeta, \eta)\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}$$
$$\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1}\int\limits_{\gamma} \bar{E}(0, Y, -Z^*, -Z, \zeta, \eta, t)\bar{f}_{\bar{\omega}}(\mu_2[1-t^2], \zeta, \eta)\sqrt{1-t^2}\,dt\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}$$
$$\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1}\int\limits_{\gamma} \hat{E}_2(Y, 0, Z, Z^*, \zeta, \eta, t)\hat{f}(\mu_1[1-t^2], \zeta, \eta)\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}$$
$$\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1}\int\limits_{\gamma} \hat{E}(Y, 0, Z, Z^*, \zeta, \eta, t)\hat{f}_\omega(\mu_1[1-t^2], \zeta, \eta)\frac{\sqrt{1-t^2}}{\zeta\eta}dt\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}$$
$$\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1}\int\limits_{\gamma} \bar{\hat{E}}_1(0, Y, -Z^*, -Z, \zeta, \eta, t)\bar{\hat{f}}(\mu_2[1-t^2], \zeta, \eta)\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}$$
$$\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1}\int\limits_{\gamma} \bar{\hat{E}}(0, Y-Z^*, -Z, \zeta, \eta, t)\bar{\hat{f}}_{\bar{\omega}}(\eta_2[1-t^2], \zeta, \eta)\sqrt{1-t^2}\,dt\frac{d\eta}{\eta}\frac{d\zeta}{\zeta},$$

which, by the same type of arguments used for (3.10), may be reduced to

$$G(Y, Z, Z^*) = \frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1} g_{\mu_1}(\mu_1, \zeta, \eta)\frac{d\eta}{\eta^2}\frac{d\zeta}{\zeta^2} \tag{3.15}$$
$$\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1} \bar{g}_{\mu_2}(\mu_2, \zeta, \eta)\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}$$
$$\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1} \hat{g}(\mu_1, \zeta, \eta)\frac{d\eta}{d}\frac{d\zeta}{\zeta}$$
$$\frac{-1}{8\pi^2}\int\limits_{|\zeta|=1}\int\limits_{|\eta|=1} Y\bar{\hat{g}}_{\mu_2}(\mu_2, \zeta, \eta)\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}.$$

By writing the functions involved in eqs. (3.11) and (3.15) in terms of their local Taylor series and operating with these series, we are able to arrive at the sufficient conditions (3.4) and (3.5).

## 4. – Complete families of solutions.

The integral operator (3.1) can be used to generate complete families of solutions of (1.3). The proof of the following theorem closely parallels that of a proof of COLTON [3] for the second order equation.

THEOREM 4.1. – *Let $G$ be a bounded simply connected domain in* $\mathbb{R}^4$ *and define,*

$$\begin{aligned} u_{4n,m,l} &= \mathrm{Re}\,\boldsymbol{P}_4^{(2)}\{\mu^n\zeta^m\eta^l, 0\} \\ u_{4n+1,m,l} &= \mathrm{Re}\,\boldsymbol{P}_4^{(2)}\{0, \mu^n\zeta^m\eta^l\} \\ u_{4n+2,m,l} &= \mathrm{Im}\,\boldsymbol{P}_4^{(2)}\{0, \mu^n\zeta^m\eta^l\} \\ u_{4n+3,m,l} &= \mathrm{Im}\,\boldsymbol{P}_4^{(2)}\{\mu^n\zeta^m\eta^l, 0\} \end{aligned} \tag{4.1}$$

*where $n$, $m$, and $l$ are non-negative integers and $m + l \leq n$.* Re (*resp.* « Im ») *denotes « take the Real Part » (resp. « take the Imaginary Part »). Then the set $\{u_{n,m,l}\}$ is a complete family of solutions for equation* (1.3) *in the space of real valued $C^4$ solutions of* (1.3) *defined in $G$.*

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