Nonlinear Perturbations and Abstract Evolution Equations in General Banach Spaces (*).

GABRIELLA DI BLASIO (Roma)

Summary. – *Part I deals with the problem of determining sufficient conditions under which the sum of two m -accretive operators on a closed convex set Q_1 is m -accretive on Q_1 . Part II is concerned with the initial value problem: $u' + Au + g(u) = v$, $u(0) = u_0$. Applications are given to the Boltzmann equation.*

Introduction.

Let X_1 be a real (possibly non reflexive) Banach space with the norm $\|\cdot\|_1$ and $Q_1 \subset X_1$ a closed convex set. Part I of this paper is primarily intended to determine sufficient conditions under which the sum of two operator which are m -accretive on Q_1 is m -accretive on Q_1 . Then main result in this direction is the following: let $X_2 \hookrightarrow X_1$ with the norm $\|\cdot\|_2$ and $Q_2 = Q_1 \cap X_2$. Let f be m -accretive on Q_1 , $g: D_g \supset Q_2 \rightarrow X_1$ be accretive on Q_1 and assume that there exists a family of approximating functions $\{g_\beta\}$ such that

- (i) g_β is uniformly continuous and bounded on bounded subsets of Q_1 ;
- (ii) for each $r > 0$ there exists r' independent of β such that:

$$(I + \alpha f + \alpha g_\beta)(Q_2^{r'}) \supset Q_2^r$$

where $Q_2^r = \{u \in Q_2, \|u\|_2 \leq r\}$;

- (iii) $\lim \|g_\beta(u) - g(u)\|_1 = 0$ uniformly on bounded subsets of Q_2 in X_2 ;

then $\overline{f + g}$ is m -accretive on Q_1 .

We continue with some regularity results for the case in which X_1 and X_2 are supposed to be reflexive: namely we will prove that if f is demiclosed then the solution of

$$u + \alpha \overline{f + g}(u) = v, \quad v \in Q_2$$

belongs to $D_f \cap D_g$.

Finally we give an example of how the preceding theory may be used in the study of certain elliptic partial differential equations.

(*) Entrata in Redazione il 2 luglio 1975.

Part II is devoted to the study of the initial value problem

$$(*) \quad \begin{cases} \frac{du}{dt} + Au + g(u) = v \\ u(0) = u_0 \end{cases} \quad t \in [0, T],$$

where A is a linear operator such that $-A + cI$ is the infinitesimal generator of a linear contraction semigroup on Q_1 for some $c \in \mathbf{R}$.

We start with an existence theorem providing local strong solutions of $(*)$ for the case in which g is uniformly continuous and bounded on bounded subsets of Q_1 and the condition that g is m -accretive on Q_1 is weakened to the following:

- (i) for each $r > 0$ there exists $\omega(r)$ such that $\omega(r)I + g$ is accretive on Q_1^r and we have

$$(I + \omega(r)I + \alpha g)(aQ_1^r) \supset aQ_1^r$$

for each $a < 1$.

(If A is continuous then $(*)$ has a unique strict solution.)

Furthermore we prove that if there exists σ such that

$$\|u\|_1 \leq \|u + \alpha \sigma u + \alpha g(u)\|_1$$

for each $u \in Q_1$ then the solution of $(*)$ exists for all $T > 0$.

If g is not supposed to be continuous we shall prove the following result: let $D_\sigma \supset Q_2$ and assume that there exists a family of approximating functions $\{g_\beta\}$ such that

- (i) g_β is uniformly continuous and bounded on bounded subsets of Q_1 ;
(ii) for each $r > 0$ there exists $\omega(r)$ such that $\omega(r)I + g_\beta$ is accretive on Q_2^r and we have

$$(I + \alpha \omega(r)I + \alpha g_\beta)(aQ_2^r) \supset aQ_2^r, \quad \alpha > 0$$

for each $a \leq 1$;

- (iii) we have $\lim \|g_\beta(u) - g(u)\|_1 = 0$ uniformly on bounded subsets of Q_2 in X_2 ;
(iv) $-A + cI$ is the infinitesimal generator of a linear contraction semigroup on Q_1 for some $c \in \mathbf{R}$ and we have

$$\|u\|_2 \leq \|u + \alpha cu + \alpha Au\|_2, \quad \alpha > 0$$

for each $u \in D_A$;

then there exists a unique local solution in some generalized sense of problem $(*)$ (if A is continuous then $(*)$ has a strong solution).

In addition, if there exists σ such that

$$\|u\|_2 \leq \|u + \alpha\sigma u + \alpha g_\beta(u)\|_2$$

for each $u \in Q_2$, then the solution exists for all $T > 0$.

We conclude with some sufficient conditions concerning regularity and give a simple illustrative example.

Finally as an application of the preceding theory we study the spatially homogeneous Boltzmann equation for the case in which the extrinsic force is not supposed to be zero and obtain results which extend those of [2], [10] and [20].

Some of the results given in this paper were presented in [11].

PART I

ABSTRACT RESULTS

1.1. Preliminaries.

Let X_1 be a real Banach space with the norm $\|\cdot\|_1$ and let $Q_1 \subset X_1$ be a closed convex set. By $G(X_1)$ we denote the class of multivalued functions which we view as subsets of $X_1 \times X_1$. If $f \in G(X_1)$ we define:

$$\begin{aligned} f(x) &= \{y \in X_1 : (x, y) \in f\}, \\ D_f &= \{x \in X_1 : f(x) \neq \emptyset\}, \\ f(D_f) &= \bigcup \{f(x) : x \in D_f\}, \\ D_f - 1 &= f(D_f), \\ f^{-1}(y) &= \{x \in D_f : y \in f(x)\}. \end{aligned}$$

If $f, g \in G(X_1)$ and α is real one sets

$$\begin{aligned} D_{\alpha f} &= D_f, \\ \alpha f(x) &= \{\alpha y : y \in f(x)\}, \\ D_{f+g} &= D_f \cap D_g, \\ (f+g)(x) &= \{y = y_1 + y_2 : y_1 \in f(x), y_2 \in g(x)\}. \end{aligned}$$

Finally we denote by $G_0(X_1) \subset G(X_1)$ the class of singlevalued functions and by

$\text{Lip}(Q_1)$ the class of all $f \in G_0(X_1)$ such that $D_f = Q_1$ and

$$\|f\|_{1,L} = \sup_{x \neq y} \left\| \frac{f(x) - f(y)}{x - y} \right\|_1 < +\infty.$$

If $f \in G(X_1)$ one sets

$$\begin{aligned} \varrho_{Q_1}(f) &= \{\alpha \in \mathbf{R} \text{ and } \alpha > 0: (I + \alpha f)^{-1} \in G_0(X_1), (I + \alpha f)(D_f \cap Q_1) \supset Q_1, \\ &\quad (I + \alpha f)^{-1}/Q_1 \in \text{Lip}(Q_1)\}^{(1)}, \\ J_{Q_1}(\alpha, f) &= (I + \alpha f)^{-1}/Q_1, \quad \alpha \in \varrho_{Q_1}(f). \end{aligned}$$

$\varrho_{Q_1}(f)$ and $J_{Q_1}(\alpha, f)$ are called the resolvent set and the resolvent of f on Q_1 .

LEMMA 1.1.1. – If $f \in G(X_1)$, $\alpha, \beta \in \varrho_{Q_1}(f)$ and $\alpha \geq \beta$ then

$$J_{Q_1}(\alpha, f) = J_{Q_1}(\beta, f) \left(\frac{\beta}{\alpha} + \frac{\alpha - \beta}{\alpha} J_{Q_1}(\alpha, f) \right).$$

PROOF. – Let $x \in Q_1$, $y \in J_{Q_1}(\alpha, f)(x)$ and $u \in f(y)$ be such that $y + \alpha u = x$; we have:

$$y + \beta u = y - \frac{\beta}{\alpha} J_{Q_1}(\alpha, f)(x) + \frac{\beta}{\alpha} J_{Q_1}(\alpha, f)(x) + \beta u = \frac{\alpha - \beta}{\alpha} y + \frac{\beta}{\alpha} x \in Q_1$$

from which it follows that

$$J_{Q_1}(\alpha, f)(x) = y = J_{Q_1}(\beta, f)(y + \beta u) = J_{Q_1}(\beta, f) \left(\frac{\beta}{\alpha} + \frac{\alpha - \beta}{\alpha} J_{Q_1}(\alpha, f) \right)(x).$$

1.2. Accretive functions.

We begin this section with some basic definitions and elementary facts. Many of the results are standard and appear in the existing literature. See, e.g. [14] and [7].

DEFINITION 1.2.1. – $f \in G(X_1)$ is called accretive on Q_1 if for each $\alpha > 0$ and $(x, u), (y, v) \in f$ such that $x, y \in Q_1$ we have

$$(1) \quad \|x - y\|_1 \leq \|x - y + \alpha(u - v)\|_1.$$

The following lemmas hold:

LEMMA 1.2.1. – Let $f \in G(X_1)$; if (1) is verified for $\alpha = \alpha_0 > 0$ then (1) is verified for all $\alpha \in [\alpha_0, +\infty[$.

⁽¹⁾ If $h \in G(X_1)$ by h/Q_1 we denote the restriction of h to Q_1 .

PROOF. — Let $\alpha \in]\alpha_0, +\infty[$ and $(x, u), (y, v) \in f$; we have

$$\|x - y\|_1 \leq \|x - y + \alpha_0(u - v)\|_1 \leq \left\| \left(1 - \frac{\alpha_0}{\alpha}\right) (x - y) + \alpha_0(u - v) \right\|_1 + \frac{\alpha_0}{\alpha} \|x - y\|_1,$$

and this implies

$$\|x - y\|_1 \leq \left\| x - y + \frac{\alpha_0 \alpha}{\alpha - \alpha_0} (u - v) \right\|_1.$$

Setting $\varphi(\alpha) = (\alpha_0 \alpha) / (\alpha - \alpha_0)$ ($\alpha > \alpha_0$) we have $\varphi(]\alpha_0, +\infty[) =]\alpha_0, +\infty[$ and the result follows.

COROLLARY. — If f satisfies (1) for $\alpha \in]0, \alpha_0]$ then f is accretive on Q_1 .

LEMMA 1.2.2. — Let f satisfy (1) for $\alpha = \alpha_0$ and $\alpha_0 \in \varrho_{Q_1}(f)$. Then $[\alpha_0, +\infty[\subset \varrho_{Q_1}(f)$ and we have

$$J_{Q_1}(\alpha, f) = J_{Q_1}(\alpha_0, f) \left(\frac{\alpha}{\alpha_0} - \frac{\alpha - \alpha_0}{\alpha_0} J_{Q_1}(\alpha_0, f) \right)^{-1}$$

for all $\alpha \in]\alpha_0, +\infty[$.

PROOF. — Let $y \in Q_1$ and $\alpha > \alpha_0$; setting

$$g(x) = \frac{\alpha - \alpha_0}{\alpha} J_{Q_1}(\alpha_0, f)(x) + \frac{\alpha_0}{\alpha} y, \quad x \in Q_1$$

we have $g(Q_1) \subset Q_1$ and $\|g\|_{1,L} \leq (\alpha - \alpha_0) / \alpha < 1$. Since g is a contraction there exists a unique $x \in Q_1$ such that

$$\frac{\alpha}{\alpha_0} x - \frac{\alpha - \alpha_0}{\alpha_0} J_{Q_1}(\alpha_0, f)(x) = y$$

and this implies

$$\left(\frac{\alpha}{\alpha_0} - \frac{\alpha - \alpha_0}{\alpha_0} J_{Q_1}(\alpha_0, f) \right)^{-1} / Q_1 \in G_0(X_1)$$

and

$$\left(\frac{\alpha}{\alpha_0} - \frac{\alpha - \alpha_0}{\alpha_0} J_{Q_1}(\alpha_0, f) \right)^{-1} (Q_1) \subset Q_1.$$

Let $y \in Q_1$; by Lemma 1.2.1 there exists at last one $z \in D$, such that

$$(2) \quad z + \alpha f(z) \ni y;$$

moreover it is easy to verify that z satisfies (2) if and only if

$$(3) \quad z + \alpha f(z) \ni \frac{\alpha}{\alpha_0} x - \frac{\alpha - \alpha_0}{\alpha_0} J_{Q_1}(\alpha_0, f)(x)$$

where

$$x = \left(\frac{\alpha}{\alpha_0} - \frac{\alpha - \alpha_0}{\alpha_0} J_{Q_1}(\alpha_0, f) \right)^{-1} (y).$$

Set $\tilde{z} = J_{Q_1}(\alpha_0, f)(x)$; since

$$z + \alpha f(z) = \frac{\alpha}{\alpha_0} [z + \alpha_0 f(z)] - \frac{\alpha - \alpha_0}{\alpha_0} z,$$

we have that $\tilde{z}$ satisfies (3) and the result follows.

DEFINITION 1.2.1. — $f \in G(X_1)$ is called m -accretive on Q_1 if it is accretive on Q_1 and we have $(I + \alpha f)(D_f \cap Q_1) \supset Q_1$ for each $\alpha > 0$.

By Lemmas 1.2.1 and 1.2.2 we get the following result:

LEMMA 1.2.3. — f is m -accretive on Q_1 if and only if there exists $\alpha_0 > 0$ such that $\varrho_{Q_1}(f) \supset]0, \alpha_0]$ and we have:

$$\begin{aligned} \|x - y\|_1 &\leq \|x - y + \alpha(u - v)\|_1, \\ \forall \alpha \in]0, \alpha_0] \quad \text{and} \quad (x, u), (y, v) &\in f: x, y \in Q_1. \end{aligned}$$

REMARK 1. — If f is accretive on Q_1 we have $\|(I + \alpha f)^{-1}(x) - (I + \alpha f)^{-1}(y)\|_1 \leq \|x - y\|_1$ for each $x, y \in (I + \alpha f)(D_f \cap Q_1)$ and $\alpha > 0$.

THEOREM 1.2.1. — If h is accretive on Q_1 , then the following conditions are equivalent:

- (i) there exists $\alpha_0 > 0$ such that $\overline{(I + \alpha h)(D_h \cap Q_1)} \supset Q_1$ for each $\alpha \in]0, \alpha_0]$,
- (ii) $\bar{h}$ is m -accretive on Q_1 (here $\bar{h}$ denotes the closure of h in $X_1 \times X_1$).

Assume in addition that $h \in G_0(X_1)$ and that there exists $\{h_\beta\}_{\beta \in [0, 1]} \subset G_0(X_1)$ such that h_β is m -accretive on Q_1 and $\lim_{\beta \rightarrow 0^+} h_\beta(x) = h(x)$ for each $x \in D_h$. Then if (i) or (ii) is fulfilled we have:

$$(iii) \quad J_{Q_1}(\alpha, \bar{h})(y) = \lim_{\beta \rightarrow 0^+} J_{Q_1}(\alpha, h_\beta)(y) \quad \text{for each } y \in Q_1.$$

PROOF. — (i) $\Rightarrow$ (ii): by the accretiveness of h it follows that $\bar{h}$ is accretive thus, by Lemma 1.2.3, it suffices to show that $\varrho_{Q_1}(\bar{h}) \supset]0, \alpha_0]$. Let $\alpha \in]0, \alpha_0]$; for each $y \in Q_1$ there exists $\{(x_n, u_n)\} \subset \bar{h}$ such that $y_n = x_n + \alpha u_n$ and $y_n \rightarrow y$. Since $\|x_n - x_m\|_1 \leq \|y_n - y_m\|_1$ we obtain that there exists $x \in Q_1$ such that $x_n \rightarrow x$ and $x_n + \alpha u_n \rightarrow y$ so that $x \in D_{\bar{h}}$ and $x + \alpha \bar{h}(x) = y$.

(ii) $\Rightarrow$ (i). – Let $\alpha > 0$ and $y \in Q_1$. As we have $J_{Q_1}(\alpha, \bar{h})(y) \in D_{\bar{h}}$ there exists $\{(x_n, u_n)\} \subset h$ such that $x_n \rightarrow J_{Q_1}(\alpha, \bar{h})(y)$, $x_n + \alpha u_n \rightarrow y$ and the result follows.

Finally let h_β satisfy the above conditions and let (i) or (ii) be fulfilled; setting

$$\begin{aligned} x &= J_{Q_1}(\alpha, \bar{h})(y) \\ x_\beta &= J_{Q_1}(\alpha, h_\beta)(y) \end{aligned} \quad \text{for } y \in Q_1 \cap (I + \alpha h)(D_h)$$

we obtain $x - x_\beta + \alpha h_\beta(x) - \alpha h_\beta(x_\beta) = \alpha h_\beta(x) - \alpha h(x)$, from which it follows that $\|x - x_\beta\|_1 \leq \alpha \|h_\beta(x) - h(x)\|_1 \rightarrow 0$. Since $(I + \alpha h)(D_h \cap Q_1) \supset Q_1$ the conclusion follows.

DEFINITION 1.2.2. – Let f be m -accretive on Q_1 ; for each $\lambda > 0$ set

$$f_\lambda = \frac{1}{\lambda} (I/Q_1 - J_{Q_1}(\lambda, f)) .$$

The functions f_λ are called Yosida's approximating functions on Q_1 .

The following lemma collects some elementary facts about the functions f_λ :

LEMMA 1.2.4. – If f is m -accretive on Q_1 , then the following statements hold:

- (i) if $x \in Q_1$ we have $f_\lambda(x) \in f J_{Q_1}(\lambda, f)(x)$;
- (ii) $f_\lambda \in \text{Lip}(Q_1)$ and $\|f_\lambda\|_{1,L} \leq 2/\lambda$;
- (iii) f_λ is m -accretive on Q_1 ;
- (iv) if $x \in D_f$ we have $\|f_\lambda(x)\|_1 \leq \inf \{\|y\|_1, y \in f(x)\}$.

PROOF. – Assertions (i) and (ii) are elementary. In order to prove (iii) observe that

$$\|J_{Q_1}(\lambda, f)\|_{1,L} \leq 1$$

implies that for each $y \in Q_1$ there exists a unique $x \in Q_1$ such that

$$x - \frac{\alpha}{\lambda + \alpha} J_{Q_1}(\lambda, f)(x) = \frac{\lambda}{\lambda + \alpha} y ,$$

so $(I + \alpha f_\lambda)(Q_1) \supset Q_1$ is proved. Now let $x_1, x_2 \in Q_1$; setting

$$\begin{aligned} x_1 + \alpha f_\lambda(x_1) &= y_1 \\ x_2 + \alpha f_\lambda(x_2) &= y_2 \end{aligned}$$

we have

$$\|x_1 - x_2\|_1 \leq \frac{\lambda}{\lambda + \alpha} \|y_1 - y_2\|_1 + \frac{\alpha}{\lambda + \alpha} \|y_1 - y_2\|_1$$

from which it follows that f_λ is accretive. Finally we show (iv). If $x \in D_f$ and $u \in f(x)$ we have

$$(4) \quad \|x - J_{Q_1}(\lambda, f)(x)\|_1 \leq \|(I + \lambda f)^{-1}(x + \lambda u) - (I + \lambda f)^{-1}(x)\|_1 \leq \|u\|_1 \lambda.$$

Since $u \in f(x)$ was arbitrary, the result follows

LEMMA 1.2.5. – If $f \in G_0(X_1)$ is continuous and m -accretive on Q_1 then

$$(i) \quad \lim_{\lambda \rightarrow 0^+} f_\lambda(x) = f(x) \text{ for each } x \in D_f.$$

Moreover if f is uniformly continuous and bounded on bounded subsets of Q_1 then the limit (i) is uniform on bounded subsets of Q_1 .

PROOF. – Using Lemma 1.2.4 (i) we have

$$\|f(x) - f_\lambda(x)\|_1 = \|f(x) - J_{Q_1}(\lambda, f)(x)\|_1$$

and the result follows from (4).

THEOREM 1.2.2. – If f and g are m -accretive on Q_1 then $f + g_\lambda$ is m -accretive on Q_1 and we have

$$J_{Q_1}(\alpha, f + g_\lambda) = J_{Q_1}\left(\frac{\alpha\lambda}{\alpha + \lambda}, f\right) \left(\frac{\alpha + \lambda}{\lambda} I - \frac{\alpha}{\lambda} J_{Q_1}(\lambda, g) J_{Q_1}\left(\frac{\alpha\lambda}{\alpha + \lambda}, f\right) \right)^{-1}.$$

PROOF. – Let $\alpha > 0$; by assumption

$$\frac{\alpha}{\alpha + \lambda} J_{Q_1}(\lambda, g) J_{Q_1}\left(\frac{\alpha\lambda}{\alpha + \lambda}, f\right)$$

is a contraction, so there exists a unique $z \in Q_1$ such that

$$z - \frac{\alpha}{\alpha + \lambda} J_{Q_1}(\lambda, g) J_{Q_1}\left(\frac{\alpha\lambda}{\alpha + \lambda}, f\right)(z) = \frac{\lambda}{\alpha + \lambda} y, \quad y \in Q_1.$$

Setting $x = J_{Q_1}(\alpha\lambda/(\alpha + \lambda), f)(z)$ we have $x \in D_f$ and

$$x + \frac{\alpha\lambda}{\alpha + \lambda} f(x) - \frac{\alpha}{\alpha + \lambda} J_{Q_1}(\lambda, g)(x) \ni \frac{\lambda}{\alpha + \lambda} y$$

from which it follows that $x + \alpha f(x) + \alpha g_\lambda(x) \ni y$ and $(I + \alpha f + \alpha g_\lambda)(D_f) \supset Q_1$.

Moreover we have

$$x = J_{Q_1} \left(\frac{\alpha\lambda}{\alpha + \lambda}, f \right) \left(\frac{\alpha + \lambda}{\lambda} I - \frac{\alpha}{\lambda} J_{Q_1}(\lambda, g) J_{Q_1} \left(\frac{\alpha\lambda}{\alpha + \lambda}, f \right) \right)^{-1} (y).$$

Now let $(x_1, u_1), (x_2, u_2) \in f$; setting

$$x_1 + \alpha u_1 + \alpha g_\lambda(x_1) = y_1, \quad x_2 + \alpha u_2 + \alpha g_\lambda(x_2) = y_2$$

and

$$\begin{aligned} x_1 + \frac{\alpha\lambda}{\alpha + \lambda} u_1 &= z_1, \\ x_2 + \frac{\alpha\lambda}{\alpha + \lambda} u_2 &= z_2, \end{aligned}$$

we have

$$\begin{aligned} z_1 - \frac{\alpha}{\alpha + \lambda} J_{Q_1}(\lambda, g) \left(I + \frac{\alpha\lambda}{\alpha + \lambda} f \right)^{-1} (z_1) &= \frac{\lambda}{\alpha + \lambda} y_1, \\ z_2 - \frac{\alpha}{\alpha + \lambda} J_{Q_1}(\lambda, g) \left(I + \frac{\alpha\lambda}{\alpha + \lambda} f \right)^{-1} (z_2) &= \frac{\lambda}{\alpha + \lambda} y_2. \end{aligned}$$

Thus $\|x_1 - x_2\|_1 \leq \|z_1 - z_2\|_1$ and

$$\|z_1 - z_2\|_1 \leq \frac{\lambda}{\alpha + \lambda} \|y_1 - y_2\|_1 + \frac{\alpha}{\lambda + \alpha} \|z_1 - z_2\|_1$$

combine to yield $\|x_1 - x_2\|_1 \leq \|y_1 - y_2\|_1$ and the proof is complete.

THEOREM 1.2.3. — Let f and g be m -accretive on Q_1 ; if g is uniformly continuous and bounded on bounded subsets of Q_1 then $\bar{f} + g$ is m -accretive on Q_1 ⁽²⁾.

PROOF. — By Theorem 1.2.2 $f + g_\lambda$ is accretive, hence by Lemma 1.2.5, $f + g$ is accretive on Q_1 . Now let $\alpha > 0$ and $y \in Q_1$ and consider the set

$$(5) \quad \{J_{Q_1}(\alpha, f + g_\lambda)(y), \lambda \in]0, 1]\}.$$

Setting $x_1 = J_{Q_1}(\alpha, f + g_1)(y)$, $x = J_{Q_1}(\alpha, f + g_\lambda)(y)$ we have

$$x_1 - x + \alpha[f(x_1) - f(x) + g_\lambda(x_1) - g_\lambda(x)] \ni \alpha[g_\lambda(x_1) - g_1(x_1)]$$

hence by the accretiveness of $f + g_\lambda$ and Lemma 1.2.4 (ii)

$$\|x - x_1\|_1 \leq \alpha \|g_\lambda(x_1) - g_1(x_1)\|_1 \leq 2\alpha \|g(x_1)\|_1$$

⁽²⁾ In [1] BARBU obtains a similar result in case $Q_1 = X_1$.

which implies that (5) is bounded. Using Lemma 1.2.5 and

$$(I + \alpha f + \alpha g)J_{Q_1}(\alpha, f + g_\lambda)(y) \ni y + \alpha(g - g_\lambda)J_{Q_1}(\alpha, f + g_\lambda)(y)$$

we obtain that $(I + \alpha f + \alpha g)(D_f \cap D_g \cap Q_1)$ is dense in Q_1 . Setting $h = \bar{f} + g$ we have $\bar{h} = h$ and the result follows from Theorem 1.2.1.

1.3. Regular functions.

DEFINITION 1.3.1. — $f \in G_0(X_1)$ is called regular on Q_1 if there exists $\{f_\beta\} \subset G_0(X_1)$, $\beta \in]0, 1]$, such that

- (i) f_β is uniformly continuous and bounded on bounded subsets of Q_1 ;
- (ii) f_β is m -accretive on Q_1 ;
- (iii) we have $\|f_\beta(x)\|_1 \leq \|f(x)\|_1$ and $\lim f_\beta(x) = f(x)$ for each $x \in D_f$.

If f is regular on Q_1 and f_β satisfies (i), (ii) and (iii) the functions f_β are called approximating functions of f on Q_1 .

REMARK 2. — It is easy to verify that if $f_\beta \in \text{Lip}(Q_1)$, f_β is accretive on Q_1 and there exists $\alpha:]0, 1] \rightarrow \mathbf{R}^+$, $\beta \rightarrow \alpha(\beta)$ such that $(I - \alpha f_\beta)(Q_1) \subset Q_1$ for each $\alpha \in]0, \alpha(\beta)]$ then assertions (i) and (ii) are fulfilled. In fact let $y \in Q_1$ and $\alpha \in]0, \alpha(\beta)]$; if $\alpha \|f_\beta\|_{1,L} < 1$ there exists a unique $x \in Q_1$ such that $x = \frac{1}{2}y + \frac{1}{2}(x - \alpha f_\beta(x))$ and the result follows from Lemma 1.2.3.

REMARK 3. — If f is m -accretive and continuous on Q_1 then f is regular on Q_1 . In fact it is easy to verify (see Remark 2 and Lemma 1.2.5) that f_λ satisfy (i), (ii) and (iii) of Definition 1.3.1.

By Theorem 1.2.3 we have

THEOREM 1.3.1. — Let f be m -accretive on Q_1 and g be regular on Q_1 , if g_β are approximating functions of g on Q_1 then $f + g_\beta$ is m -accretive on Q_1 .

THEOREM 1.3.2. — Let f be m -accretive on Q_1 and g be regular on Q_1 . The following conditions are equivalent:

- (i) there exists $\alpha_0 > 0$ such that $\overline{(I + \alpha f + \alpha g)(D_f \cap D_g \cap Q_1)} \supset Q_1$ for each $\alpha \in]0, \alpha_0]$,
- (ii) $\overline{f + g}$ is m -accretive on Q_1 .

Moreover if (i) or (ii) is fulfilled and g_β are approximating functions of g we have

$$\lim_{\beta \rightarrow 0^+} J_Q(\alpha, \bar{f} + g_\beta)(y) = J_{Q_1}(\alpha, \overline{f + g})(y).$$

PROOF. — Theorem 1.3.1 and Definition 1.3.1 show that $f+g$ is accretive; thus, setting $h=f+g$ the assertion (i) $\Leftrightarrow$ (ii) follows from Theorem 1.2.1. The last argument in the proof of Theorem 1.2.1 can be repeated here and the proof is complete.

Let $X_2 \subset X_1$ be a Banach space with the norm $\|\cdot\|_2$ and let $Q_2 \subset X_2$ be a closed convex set dense in Q_1 . The following theorem provides conditions which are sufficient to guarantee that assumption (i) of Theorem 1.3.2 holds.

THEOREM 1.3.3. — Let $f \in G(X_1)$ and $g \in G_0(X_1)$ such that $D_g \supset Q_2$. If the following assumptions hold:

- (i) there exists $\{g_\beta\} \subset G_0(X_1)$ such that $D_{g_\beta} \supset Q_2$ and $\lim g_\beta(x) = g(x)$ uniformly on bounded subsets of Q_2 in X_2 ;
- (ii) there exist $\alpha_0 > 0$ and a function $k: [0, +\infty[\rightarrow [0, +\infty[$ such that for each $y \in Q_2$ and $\beta \in]0, 1]$ there is $(x_\beta, u_\beta) \in f$ satisfying

$$x_\beta + \alpha u_\beta + \alpha g_\beta(x_\beta) = y, \quad \alpha \in]0, \alpha_0], \quad x_\beta \in Q_2$$

and

$$\|x_\beta\|_2 \leq k(\|y\|_2).$$

Then $\overline{(I + \alpha f + \alpha g)(D_f \cap D_g \cap Q_1)} \supset Q_1$.

PROOF. — Let $y \in Q_2$ and (x_β, u_β) satisfying (ii); we have:

$$x_\beta + \alpha u_\beta + \alpha g(x_\beta) = y + \alpha(g - g_\beta)(x_\beta)$$

hence using (i)

$$\lim (x_\beta + \alpha u_\beta + \alpha g(x_\beta)) = y.$$

Since X_2 is dense in X_1 the result follows.

Theorem 1.3.3 extends a similar result for the case $Q_1 = X_1$ and f and g linear. See [8].

DEFINITION 1.3.2. — $x \in X_1$ is called a strong solution of the problem

$$(6) \quad x + \alpha f(x) + \alpha g(x) \ni y, \quad y \in Q_1$$

if there exists $(x_\beta, u_\beta) \in f+g$ such that $x_\beta \rightarrow x$ and $x_\beta + \alpha u_\beta \rightarrow y$.

By Theorems 1.3.2 and 1.3.3 we get the following results:

THEOREM 1.3.4. — Let f be m -accretive and g be regular on Q_1 . If assumption (i) of Theorem 1.3.2 holds and g_β are approximating functions of g then (6) has a unique strong solution $x = \lim_{\beta \rightarrow 0^+} J_{Q_1}(\alpha, \tilde{f} + g_\beta)(y)$.

THEOREM 1.3.5. – Let f and g be accretive on Q_1 and let the assumptions of Theorem 1.3.3 hold. Then (6) has a unique strong solution x ; moreover if $y \in Q_2$ we have $x = \lim_{\beta \rightarrow 0^+} J_{Q_1}(\alpha, f + g_\beta)(y)$.

PROOF. – Let $y \in Q_2$ and let (x_β, u_β) satisfy condition (ii) of Theorem 1.3.3; we have

$$x_\beta - x_\gamma + \alpha(u_\beta - u_\gamma) + \alpha[g_\beta(x_\beta) - g_\beta(x_\gamma)] = \alpha[g_\gamma(x_\gamma) - g_\beta(x_\gamma)].$$

Hence, using Theorem 1.3.3 (i), there exists $x = \lim_{\beta \rightarrow 0^+} x_\beta$ and the result follows.

1.4. M -accretive functions in reflexive Banach spaces.

In this section we require X_1 and X_2 to be reflexive Banach spaces.

DEFINITION 1.4.1. – $f \in G(X_1)$ is called demiclosed if the following statement hold

- (i) if $\{x_n\} \subset D_f$, $y_n \in f(x_n)$, $x_n \rightarrow x$ and $y_n \rightharpoonup y \in f$ (here $\rightharpoonup$ refers to convergence in the weak topology).

THEOREM 1.4.1. – Let f and g be accretive functions satisfying the assumptions of Theorem 1.3.3. If in addition:

- (i) f is demiclosed;
(ii) $\|g(x)\|_1$ is uniformly bounded on bounded subsets of Q_2 in X_2 ;

then for any $y \in Q_2$ (6) has a unique strong solution x and we have $x = \lim_{\beta \rightarrow 0^+} J_{Q_1}(\alpha, f + g_\beta)(y)$ and $x \in D_f \cap Q_2$.

PROOF. – Let $y \in Q_2$ and $x_\beta = J_{Q_1}(\alpha, f + g_\beta)(y)$; by Theorem 1.3.5 $x = \lim_{\beta \rightarrow 0^+} x_\beta$ is the strong solution of (6) and we have $\|x_\beta\|_2 \leq k(\|y\|_2)$.

Since $\{x_\beta\}$ is bounded in X_2 there exists $\{x_{\beta_k}\}$ such that $x_{\beta_k} \rightharpoonup \tilde{x}$ in X_2 ; therefore we have $\tilde{x} = x$ and $\tilde{x} \in Q_2$. Now let $u_\beta \in f(x_\beta)$ be such that $x_\beta + \alpha u_\beta + \alpha g_\beta(x_\beta) = y$; by assumption $\{g_\beta(x_\beta)\}$ is bounded in X_1 so that $\{u_\beta\}$ is bounded in X_1 . It follows that there exists $\{u_{\beta_k}\}$ such that $u_{\beta_k} \rightharpoonup u$ hence, by the demiclosedness of f , $(x, u) \in f$.

COROLLARY. – In addition to the condition of Theorem 1.4.1 suppose that g is demiclosed. Then for any $y \in Q_2$ (6) has a unique strict solution $x = \lim_{\beta \rightarrow 0^+} J_{Q_1}(\alpha, f + g_\beta)(y)$.

1.5. Evolution equations.

Let Y_1 be a real Banach space with the norm $\|\cdot\|_1$ and let $K_1 \subset Y_1$ be a closed convex set. By X_1^p we denote the Banach space $L^p(0, T; Y_1)$ ($1 \leq p < +\infty$) with the usual norm $\|\cdot\|$ and by $W^{1,p}(0, T; Y_1)$ we denote the Sobolev space of all functions in X_1^p all of whose distributional derivatives also belong to X_1^p .

Let $g \in G(Y_1)$ and $w \in X_1^p$; as an application of the results obtained in Section 1.3 we shall study the initial value problem

$$(7) \quad \begin{cases} \frac{du}{dt} + g(u) \ni w & t \in [0, T], \\ u(0) = u_0 & u_0 \in K_1. \end{cases}$$

DEFINITION 1.5.1. — A function $u \in X_1^p$ is called a strong solution of (7) if there exists $\{u_\beta\}$ such that

- (i) $u_\beta \in W^{1,p}(0, T; Y_1)$, $u_\beta(0) = u_0$, $u_\beta \rightarrow u$;
- (ii) $u_\beta(t) \in D_g$ for a.e. $t \in]0, T[$;
- (iii) there exists $\{v_\beta\} \subset X_1^p$ such that $v_\beta(t) \in g(u_\beta(t))$ for a.e. $t \in]0, T[$ and

$$\frac{du_\beta}{dt} + v_\beta \rightarrow w$$

Set $Q_1^p = \{u \in X_1^p, u(t) \in K_1 \text{ and } |u(t)|_1 \leq M \text{ a.e. } t \in]0, T[\}$ where $M > |u_0|_1$, and

$$\begin{cases} D_{f^p} = \{u \in W^{1,p}(0, T; Y_1), u(0) = u_0\}, \\ f^p(u) = \frac{du}{dt}. \end{cases}$$

If g is regular on K_1 with approximating functions $\{g_\beta\}$ we define

$$\begin{cases} D_{\tilde{g}^p} = \{u \in X_1^p, u(t) \in D_g \text{ a.e. } t \in]0, T[, t \rightarrow g(u(t)) \in X_1^p\} \\ (\tilde{g}^p(u))(t) = g(u(t)) \end{cases}$$

and

$$\begin{cases} D_{\tilde{g}_\beta^p} = \{u \in X_1^p, t \rightarrow g_\beta(u(t)) \in X_1^p\}, \\ (\tilde{g}_\beta^p(u))(t) = g_\beta(u(t)). \end{cases}$$

The following lemma holds:

LEMMA 1.5.1. — Let g be regular on K_1 with approximating functions $\{g_\beta\}$; if $g(0) \ni 0$ and g_β is Lipschitz continuous on bounded subsets of K_1 then $\tilde{g}^p$ is regular on Q_1^p with approximating functions $\{\tilde{g}_\beta^p\}$.

PROOF. — Since g_β is continuous and bounded the function $t \rightarrow g_\beta(u(t))$ is measurable and we have

$$\int_0^T |g_\beta(u(t))|_1^p dt < +\infty, \quad u \in Q_1^p$$

so that $D_{\tilde{g}_\beta^p} \supset Q_1^p$. Next we show that for each $\alpha > 0$ $(I + \alpha g_\beta)(Q_1^p) \supset Q_1^p$. Let $v \in Q_1^p$, for a.e. $t \in]0, T[$ there exists $u(t) \in K_1$ such that

$$u(t) = (I + \alpha g_\beta)^{-1}(v(t))$$

hence $u \in Q_1^p$ by the Lipschitz continuity of $(I + \alpha g_\beta)^{-1}$ and the assumption $g(0) \ni 0$. Since the accretiveness of $\tilde{g}_\beta^p$ on Q_1^p is evident we have that $\{\tilde{g}_\beta^p\}$ satisfies (ii) of Definition 1.3.1. It remains to verify (i) and (ii). It follows directly from the Lipschitz continuity of g_β on bounded subsets of K_1 that $\tilde{g}_\beta^p$ satisfies (i). Moreover we have

$$\lim_{\beta \rightarrow 0^+} |g_\beta(u(t)) - g(u(t))|_1 = 0, \quad u \in D_{\tilde{g}^p}$$

and

$$|g_\beta(u(t))|_1 \leq |g(u(t))|_1$$

a.e. $t \in]0, T[$. Hence using the dominated convergence theorem the conclusion follows.

REMARK 4. - It can be easily checked that the conclusion of Lemma 1.5.1 can be established without requiring the condition $g(0) \ni 0$ in the case $g_\beta \in \text{Lip}(K_1)$ and replacing Q_1^p by

$$\tilde{Q}_1^p = \{u \in X_1^p, u(t) \in K_1 \text{ a.e. } t \in]0, T[\}.$$

By the m -accretiveness of f^p on Q_1^p and by Theorem 1.3.4 we get the following results

THEOREM 1.5.1. - In addition to the conditions of Theorem 1.3.4 suppose that

$$(I + \alpha_0 f^p + \alpha_0 \tilde{g}_\beta^p)(D_{f^p} \cap D_{\tilde{g}_\beta^p} \cap Q_1^p) \supset Q_1^p$$

for some $\alpha_0 > 0$. Then for any $\alpha > 0$ and $v \in Q_1^p$ there exists a unique strong solution of

$$(8) \quad u + \alpha f^p(u) + \alpha \tilde{g}_\beta^p(u) = v$$

and we have $u = \lim_{\beta \rightarrow 0^+} J_{Q_1}(\alpha, f^p + \tilde{g}_\beta^p)(v)$.

THEOREM 1.5.2. - Let the assumptions of Theorem 1.5.1 hold; if $w \in L^p(0, T; K_1)$ then (8) has a unique strong solution.

PROOF. - If $w \in L^\infty(0, T; K_1)$ it suffices to take $M > \|w\|_\infty$ and the result follows from Theorem 1.5.1. Since $L^\infty(0, T; K_1)$ is dense in $L^p(0, T; K_1)$ the conclusion follows.

Now let $Y_2 \hookrightarrow Y_1$ be a real Banach space with the norm $\|\cdot\|_2$ and let $K_2 \subset Y_2$ be a closed convex set dense in K_1 . Set $X_2 = L^\infty(0, T; Y_2)$, $\tilde{Q}_1^p = L^p(0, T; K_1)$, $Q_2 =$

$= L^\infty(0, T; K_2)$ and for $\{g_\beta\} \subset G_0(Y_1)$

$$\begin{cases} D_{\tilde{g}_\beta^p} = \{u \in X_1^p, u(t) \in D_{g_\beta} \text{ a.e. } t \in]0, T[, t \rightarrow g_\beta(u(t)) \in X_1^p\}, \\ (\tilde{g}_\beta^p(u))(t) = g_\beta(u(t)). \end{cases}$$

The following lemma holds

LEMMA 1.5.2. – Let $D_g \supset K_2$ and let $|g(u)|_1$ be uniformly bounded on bounded subsets of K_2 in Y_2 . If there exists $\{g_\beta\} \subset G_0(Y_1)$ such that

(i) $D_{g_\beta} = K_1$, g_β is continuous;

(ii) we have $\lim |g_\beta(u) - g(u)|_1 = 0$ uniformly on bounded subsets of K_2 in Y_2 ;

then $D_{\tilde{g}_\beta^p}, D_{\tilde{g}_\beta^p} \supset Q_2$ and we have

(iii) $\lim \|\tilde{g}_\beta^p(u) - \tilde{g}^p(u)\|_1 = 0$

uniformly on bounded subsets of Q_2 in X_2 .

PROOF. – If $u \in Q_2$ we have $u(t) \in K_2$ and $|u(t)|_2 \leq \|u\|_\infty$ for a.e. $t \in]0, T[$.

Using (i) and (ii) we then obtain that $t \rightarrow g_\beta(u(t))$ and $t \rightarrow g(u(t))$ are measurable and $|g_\beta(u(t))|_1 \leq k$, $|g(u(t))|_1 \leq k$ for a.e. $t \in]0, T[$ so that $u \in D_{\tilde{g}_\beta^p} \cap D_{\tilde{g}^p}$. Assertion (iii) follows by noting that if u belongs to a bounded subset of Q_2 then $u(t)$ belongs to a bounded subset of K_2 for a.e. $t \in]0, T[$.

As a consequence of Lemma 1.5.2 and Theorem 1.3.5 we get the following result:

THEOREM 1.5.3. – Let the assumptions of Lemma 1.5.2 hold and g be accretive on K_1 . If in addition for each $\beta \in]0, 1[$ and $v \in Q_2$ there exists $u_\beta \in D_{f^p} \cap Q_2$ such that

$$u_\beta + \alpha_0 f(u_\beta) + \alpha_0 \tilde{g}_\beta^p(u_\beta) = v$$

and $\|u_\beta\|_2 \leq k(\|v\|_2)$ for some $\alpha_0 > 0$, then there is a unique strong solution of the equation

$$u + \alpha f^p(u) + \alpha \tilde{g}^p(u) = v, \quad v \in Q_1.$$

Moreover if $v \in Q_2$ we have $u = \lim_{\beta \rightarrow 0^+} J_{Q_1}(\alpha, f + \tilde{g}_\beta^p)(v)$.

THEOREM 1.5.4. – If the assumptions of Theorem 1.5.3 hold and Y_1 and Y_2 are reflexive then for each $v \in Q_2$ we have $\lim_{\beta \rightarrow 0^+} J_{Q_1}(\alpha, f + \tilde{g}_\beta^p)(v) \in D_{f^p} \cap L^p(0, T; Y_2)$ ($1 < p < +\infty$).

PROOF. – For $v \in Q_2$ set

$$u_\beta = J_{Q_1}(\alpha, f + \tilde{g}_\beta^p)(v), \quad u = \lim u_\beta;$$

by Theorem 1.5.3 we have $u_\beta(t) \in K_2$ and $|u_\beta(t)|_2 \leq k(\|v\|_2)$ for a.e. $t \in]0, T[$ so that

$\{u_\beta\}$ is bounded in $L^p(0, T; Y_2)$. Since Y_2 is reflexive there exists $u_{\beta_k} \rightharpoonup \tilde{u}$ in $L^p(0, T; Y_2)$ from which it follows that $u = \tilde{u} \in L^p(0, T; K_2)$. Since f^p is demiclosed the last argument in the proof of Theorem 1.4.1 can be repeated here and the result follows.

From Theorems 1.5.3 and 1.5.4 we have:

THEOREM 1.5.5. — Let the assumptions of Lemma 1.5.2 hold and g be accretive on K_1 . If in addition for each $\beta \in]0, 1]$ and $v \in L^\infty(0, T; K_2)$ there exists $u_\beta \in W^{1,p}(0, T; Y_1) \cap L^\infty(0, T; K_2)$ satisfying

$$\begin{cases} \frac{du_\beta}{dt} + g_\beta(u_\beta) = v, \\ u_\beta(0) = u_0, \end{cases} \quad \|u_\beta\|_2 \leq K(\|v\|_2).$$

Then for each $w \in L^p(0, T; K_1)$ the problem (6) has a unique strong solution u . Moreover if Y_1 and Y_2 are reflexive, if $|g(u)|_1$ is uniformly bounded on bounded subsets of K_2 in Y_2 and $w \in L^\infty(0, T; K_2)$ then $u \in W^{1,p}(0, T; Y_1) \cap L^p(0, T; K_2)$ ($1 < p < +\infty$).

1.6. Examples.

In this section we give an example in which the previous theory is applied to partial differential equations. Let $\Omega \subset \mathbf{R}^n$ be an open bounded domain with smooth boundary and by $H^m(\Omega), H_0^m(\Omega)$ the usual Sobolev spaces. On Ω we consider the differential operator with real coefficients

$$Eu = \sum_{|p|, |q| \leq m} (-1)^{|p|} D^p (a_{pq} D^q u)$$

where $a_{pq} \in C^m(\bar{\Omega})$ and for some positive constant γ

$$\sum_{|p|=|q|=m} a_{pq}(x) \xi^{p+q} \geq \gamma |\xi|^2.$$

Let $\varphi: \mathbf{R} \rightarrow \mathbf{R}$, accretive and continuous; we define:

$$\begin{cases} D_f = H^{2m}(\Omega) \cap H_0^m(\Omega) \\ f u = Eu \\ D_g = \{u \in L^2(\Omega), a\varphi \circ u \in L^2(\Omega)\} \\ g(u) = a\varphi \circ u \end{cases}$$

and for each $\lambda \in]0, 1]$

$$\begin{cases} D_{g_\lambda} = L^2(\Omega) \\ g_\lambda(u) = a\varphi_\lambda \circ u \end{cases}$$

where $a \in L^\infty(\Omega)$, $a \geq 0$ a.e. $x \in \Omega^n$ and φ_λ are the Yosida's approximating functions of φ . Furthermore we denote by X_1 and X_2 the Hilbert spaces $L^2(\Omega)$ and $H^m(\Omega)$ with the usual norm $\|\cdot\|_1$ and $\|\cdot\|_2$ and suppose $m > n/2$.

LEMMA 1.6.1. – The following statements hold:

- (i) g_λ is Lipschitz continuous and accretive on X_1 ;
- (ii) we have $\lim_{\lambda \rightarrow 0^+} \|g_\lambda(u) - g(u)\|_1 = 0$ uniformly on bounded subsets of X_2 .

PROOF. – (i) follows directly from Lemma 1.2.4. (ii) is a consequence of Sobolev's inequality and the uniform continuity of φ on compact subsets of $\mathbf{R}$.

LEMMA 1.6.2. – For each $\lambda \in]0, 1]$ and μ sufficiently large there exists a unique $u_\lambda \in D_f$ such that

$$(9) \quad \mu u_\lambda + f(u_\lambda) + g_\lambda(u_\lambda) = v, \quad v \in X_1$$

and we have

$$(i) \quad \|u_\lambda\|_2 \leq k \|v\|_1^2.$$

PROOF. – By Garding's inequality there exists $\mu_0 \geq 0$ such that $\mu_0 I + f$ is m -accretive on X_1 . Hence using Theorem 1.2.3 and Lemma 1.6.1 we obtain that (9) has a unique solution u_λ for $\mu > \mu_0$ and we have

$$\|u_\lambda\|_1 \leq \frac{1}{\mu - \mu_0} \|v\|_1.$$

By Garding's inequality there exists $\alpha > 0$ such that

$$\mu \|u_\lambda\|_1^2 + \alpha \|u_\lambda\|_2^2 - \mu_0 \|u_\lambda\|_1^2 + (g_\lambda(u_\lambda), u_\lambda)_1 \leq \|v\|_1 \cdot \|u\|_1.$$

Combining these results we have

$$\alpha \|u_\lambda\|_2^2 \leq \frac{1}{\mu_0 - \mu} \|v\|_1^2$$

and the proof is complete.

THEOREM 1.6.1. – For every $v \in X_1$ there exists a unique $u \in D_f$ such that

$$(10) \quad \mu u + f u + g(u) = v, \quad \mu > \mu_0.$$

PROOF. – By Theorem 1.3.5 there exists a unique strong solution of (10) and we have $u = \lim_{\lambda \rightarrow 0^+} u_\lambda$ with u_λ satisfying (9). Hence by Lemma 1.6.2 (i) and the dominated convergence theorem we obtain $\lim_{\lambda \rightarrow 0^+} g_\lambda(u_\lambda) = g(u)$ and the conclusion follows.

From the regularity results for elliptic equations we get the following theorem

THEOREM 1.6.2. — Let u be the solution of (10); if $v \in H^k(\Omega)$ and $a \in C^k(\bar{\Omega})$ then $u \in H^{2m+k}(\Omega)$.

PROOF. — Let $v \in H^1(\Omega)$, $a \in C^1(\bar{\Omega})$; by Theorem 1.6.1 there exists $u \in H^{2m}(\Omega) \cap H_0^m(\Omega)$ such that $\mu u + fu = v - g(u) \in H^1(\Omega)$ so that $u \in H^{2m+1}(\Omega)$. If $k > 1$ the result follows by induction.

PART II

EVOLUTION EQUATIONS

2.1. The continuous case.

Let Y_1 be a real Banach space with norm $|\cdot|_1$ and $g \in G_0(Y_1)$; we will consider the initial value problem

$$(1) \quad \begin{cases} \frac{du}{dt} + g(u) = w \\ u(0) = u_0 \end{cases} \quad t \in [0, T]$$

and will obtain criteria for the case in which g is not assumed to be m -accretive.

DEFINITION 2.1.1. — Let $w \in C(0, T; Y_1)$; a function $u: [0, T] \rightarrow Y_1$ is called a solution of (1) if

- (i) $u(t) \in D_g$ for each $t \in [0, T]$, $t \rightarrow g(u(t)) \in C(0, T; Y_1)$;
- (ii) $u \in C^1(0, T; Y_1)$ and satisfies (1).

Let $K_1 \subset Y_1$ a closed convex set containing the origin; for each $r > 0$, $a > 0$ we set $K_1^r = \{u \in K_1, |u|_1 \leq r\}$, $aK_1^r = \{u \in Y_1, (1/a)u \in K_1\}$.

The following theorems contain the main results concerning the case in which g is continuous. The non continuous case will be studied in section 2.5.

THEOREM 2.1.1. — Let $g: K_1 \rightarrow Y_1$ satisfy the following conditions

- (i) g is uniformly continuous and bounded on bounded subsets of K_1 ;
- (ii) for each $r > 0$ there exists $\omega(r)$ such that for each $\alpha > 0$ and $u, v \in K_1^r$ we have

$$|u - v|_1 \leq |u - v + \alpha[\omega(r)(u - v) + g(u) - g(v)]|_1$$

and

$$(I + \alpha g + \alpha \omega(r)I)(aK_1^r) \supset aK_1^r$$

for each $a \leq 1$.

In addition, if there exists $b > 1$ such that

(iii) $bu_0 \in K_1$, $bw(t) \in K_1$ for each $t \in [0, T]$;

then the initial value problem (1) admits a solution for T sufficiently small. Moreover there exists $\bar{r}$ such that

(iv) $u(t) \in K_1^-$ for each $t \in [0, T]$.

REMARK 1. — Assumption (iii) is satisfied if K_1 is a cone and $u_0 \in K_1$, $w(t) \in K_1$ for each $t \in [0, T]$.

THEOREM 2.1.2. — Let the assumptions of Theorem 2.1.1 hold. Then (1) has a unique solution satisfying (iv).

THEOREM 2.1.3. — In addition to the conditions of Theorem 2.1.1 assume that there exists σ such that for each $\alpha > 0$ and $u \in K_1$ we have

$$|u|_1 \leq |u + \alpha \sigma u + \alpha g(u)|_1.$$

Then if K_1 is a cone (1) has a unique solution satisfying (iv) for each $T > 0$.

The proof of Theorems 2.1.1, 2.1.2 and 2.1.3 require some preliminaries and will be given later.

2.2. Accretive functions.

In this section we recall some known result concerning accretive functions and the subdifferential of the norm which we present without proof (for a proof see e.g. [7]).

Let X be a real Banach space with norm $\| \cdot \|$ and let X^* be its dual; the value of $x^* \in X^*$ at $x \in X$ will be denoted by $\langle x, x^* \rangle$.

DEFINITION 2.2.1. — With each $x \in X$ we associate the set

$$\partial \|x\| = \{x^* \in X^*: \|x + y\| \geq \|x\| + \langle y, x^* \rangle \text{ for each } y \in X\}.$$

The function $X \rightarrow 2^{X^*}$, $x \rightarrow \partial \|x\|$ is called the subdifferential of the norm.

The following lemmas hold:

LEMMA 2.2.1. — Conditions (i) and (ii) below on a function $f \in G(X)$ are equivalent:

(i) for each $(x, u) \in f$ and $\alpha > 0$ we have

$$\|x\| \leq \|x + \alpha u\|$$

(ii) there exists $x^* \in \partial\|x\|$ such that

$$\langle u, x^* \rangle \geq 0, \quad \text{for each } (x, u) \in f.$$

LEMMA 2.2.2. – Conditions (i) and (ii) below on a function $f \in G(X)$ are equivalent:

(i) for every $(x, u), (y, v) \in f$ and $\alpha > 0$ we have

$$\|x - y\| \leq \|x - y + \alpha(u - v)\|$$

(ii) there exists $x^* \in \partial\|x - y\|$ such that

$$\langle u - v, x^* \rangle \geq 0, \quad \text{for each } (x, u), (y, v) \in f.$$

LEMMA 2.2.3. – If $u: [0, T] \rightarrow X$ has the left derivative then $t \rightarrow \|u(t)\|$ has the left derivative and we have:

$$D_s\|u(t)\| \leq \langle D_s u(t), z^* \rangle, \quad \text{for each } z^* \in \partial\|u(t)\|.$$

2.3. Proof of Theorems 2.1.1 and 2.1.2.

The conditions (i) and (ii) of Theorem 2.1.1 are assumed to hold throughout this section.

Set $X_1 = C(0, T; Y_1)$, $Q_1 = C(0, T; K_1)$, $Q_1' = C(0, T; K_1')$ and assume that $bu_0 \in K_1$ for some $b > 1$. We define

$$\begin{cases} D_f = \left\{ u \in X_1, \frac{du}{dt} \in X, u(0) = u_0 \right\}, \\ f(u) = \frac{du}{dt}, \\ \\ \begin{cases} D_g = Q_1 \\ (\tilde{g}(u))(t) = \exp[-(1 + \omega_0)t] g(u(t) \exp[(1 + \omega_0)t]) \end{cases} \end{cases}$$

where $r_0 > |u_0|_1 b$ and $\omega_0 = \omega(r_0)$ is given by Theorem 2.1.1 (i).

LEMMA 2.3.1. – If

(i) $\exp[(1 + \omega_0)T] < b$

then $\omega_0 I + \tilde{g}: Q_1 \rightarrow X_1$ is m -accretive on $b^{-1}Q_1^*$.

PROOF. - Let $u, v \in b^{-1}Q_1^{r_0}$; setting

$$\tilde{u}(t) = \exp[(1 + \omega_0)t]u(t)$$

$$\tilde{v}(t) = \exp[(1 + \omega_0)t]v(t)$$

we have $\tilde{u}(t), \tilde{v}(t) \in Q_1^{r_0}$ for each $t \in [0, T]$. Using Theorem 2.1.1 we obtain

$$\begin{aligned} |[u(t) - v(t)](1 + \alpha\omega_0) + \alpha \exp[-(1 + \omega_0)t][g(\tilde{u}(t)) - g(\tilde{v}(t))]|_1 &= \\ &= \exp[-(1 + \omega_0)t] |[\tilde{u}(t) - \tilde{v}(t)](1 + \alpha\omega_0) + \alpha[g(\tilde{u}(t)) - g(\tilde{v}(t))]|_1 \geq \\ &\geq \exp[-(1 + \omega_0)t] |\tilde{u}(t) - \tilde{v}(t)|_1 = |u(t) - v(t)|_1 \end{aligned}$$

from which it follows that $\omega_0 I + \tilde{g}$ is accretive on $b^{-1}Q_1^{r_0}$. Now let $v \in b^{-1}Q_1^{r_0}$; for each $t \in [0, T]$ we have $\tilde{v}(t) = \exp[(1 + \omega_0)t]v(t) \in \exp[(1 + \omega_0)t]b^{-1}K_1^{r_0}$. Hence using Theorem 2.1.1 (ii) there exists $\tilde{u}(t) \in \exp[(1 + \omega_0)t]b^{-1}K_1^{r_0}$ such that

$$\tilde{u}(t) + \alpha\omega_0\tilde{u}(t) + \alpha g(\tilde{u}(t)) = \tilde{v}(t).$$

Setting $u(t) = \exp[-(1 + \omega_0)t]\tilde{u}(t)$ we have $u \in b^{-1}Q_1^{r_0}$ and

$$u + \alpha\omega_0 u + \alpha \tilde{g}(u) = v$$

which yields $(I + \alpha\omega_0 I + \alpha \tilde{g})(b^{-1}Q_1^{r_0}) \supset b^{-1}Q_1^{r_0}$ and the proof is complete.

As a consequence of Lemma 2.3.1 and Theorem 1.2.3 we obtain the following result:

THEOREM 2.3.1. - If $\exp[(1 + \omega_0)T] < b$ then for every $v \in b^{-1}Q_1^{r_0}$ there exists $u \in b^{-1}Q_1^{r_0} \cap D_f$ such that

$$u + \alpha f(u) + \alpha\omega_0 u + \alpha \tilde{g}(u) = v.$$

PROOF OF THEOREM 2.1.1. - Let $r_0 > \|w\|_1 b$ and T satisfy Lemma 2.3.1 (i). By Theorem 2.3.1 there exists $\tilde{u}$ satisfying the initial value problem

$$\begin{cases} \tilde{u} + \frac{d\tilde{u}}{dt} + \omega_0 \tilde{u} + \exp[-(1 + \omega_0)t]g(\exp[(1 + \omega_0)t]\tilde{u}(t)) = \exp[-(1 + \omega_0)t]w(t), \\ \tilde{u}_0(0) = u_0 \end{cases}$$

and we have $\tilde{u}(t) \in b^{-1}K_1^{r_0}$ for each $t \in [0, T]$. Hence $u = \exp[(1 + \omega_0)t]\tilde{u}$ satisfies (1) and we have

$$u(t) \in K_1^{r_0}, \quad \text{for each } t \in [0, T].$$

THEOREM 2.3.1. - Let $u, \bar{u}$ be solutions of (1) with initial datum $u_0, \bar{u}_0$. If $u(t),$

$\bar{u}(t) \in K_1^r$ for each $t \in [0, T]$ then

$$|u(t) - \bar{u}(t)|_1 \leq \exp[\omega(\bar{r})t] |u_0 - \bar{u}_0|_1.$$

PROOF. — By assumption we have

$$\frac{d}{dt}(u - \bar{u}) + g(u) - g(\bar{u}) = 0, \quad u(0) - \bar{u}(0) = u_0 - \bar{u}_0.$$

By Lemma 2.2.2 and Theorem 2.1.1 (ii) there exists $z^*(t) \in \partial|u(t) - \bar{u}(t)|_1$ such that

$$\begin{aligned} \left\langle \frac{d}{dt}(u - \bar{u}), z^*(t) \right\rangle_1 - \omega(r)|u(t) - \bar{u}(t)|_1 &\leq \\ &\leq \left\langle \frac{d}{dt}(u - \bar{u}), z^*(t) \right\rangle_1 + \langle g(u(t)) - g(\bar{u}(t)), z^*(t) \rangle_1 = 0. \end{aligned}$$

Hence using Lemma 2.2.3

$$D_s|u(t) - \bar{u}(t)|_1 - \omega(r)|u(t) - \bar{u}(t)|_1 \leq 0$$

and the result follows.

PROOF OF THEOREM 2.1.2. — It follows directly from Theorem 2.3.1 by assuming $u_0 = \bar{u}_0$.

2.4. Proof of Theorem 2.1.3.

LEMMA 2.4.1. — Let $\bar{u}$ be a solution of problem (1); if there exists σ such that

$$|u|_1 \leq |u + \alpha\sigma u + \alpha g(u)|_1$$

for each $u \in K_1$ then

$$|\bar{u}(t)|_1 \leq \exp[\sigma t] |u_0|_1 + \int_0^t \exp[\sigma(t-s)] |w(s)|_1 ds$$

for each $t \in [0, T]$.

PROOF. — By Lemma 2.2.1 there exists $z^* \in \partial|u(t)|_1$ such that

$$\left\langle \frac{d\bar{u}}{dt}, z^*(t) \right\rangle_1 - \sigma|u(t)|_1 \leq \left\langle \frac{d\bar{u}}{dt}, z^*(t) \right\rangle_1 + \langle g(\bar{u}(t)), z^*(t) \rangle_1 \leq |w(t)|_1.$$

Hence using Lemma 2.2.3

$$D_s|u(t)|_1 - \sigma|u(t)|_1 \leq |w(t)|_1$$

and the result follows.

PROOF OF THEOREM 2.1.3. – Let T^* be supremum of the set of all T such that (1) has a unique solution and let u be such that

$$\begin{cases} \frac{du}{dt} + g(u) = w, \\ u(0) = u_0 \end{cases} \quad t \in [0, T^*[.$$

We have ($s < t$)

$$|u(t) - u(s)|_1 \leq \int_s^t |g(u(\tau))|_1 d\tau + \int_s^t |w(\tau)|_1 d\tau \rightarrow 0$$

where we used Lemma 2.4.1 and assumption (i) of Theorem 2.1.1. Let $T^* < +\infty$; setting $u(T^*) = \lim_{t \rightarrow T^*} u(t)$ we have

$$\lim_{t \rightarrow T^*} |g(u(t)) - g(u(T^*))|_1 = 0$$

from which it follows that T^* is a maximum so, using standard arguments, that $T^* = +\infty$. Since the uniqueness follows by Theorem 2.3.1 the proof is complete.

2.5. The noncontinuous case.

We are interested in the solution of the initial value problem (1) if g is not supposed to be continuous. To state our results more precisely it is convenient to introduce some notations and definitions.

DEFINITION 2.5.1. – Let $w \in X_1$; a function $u \in X_1$ is called a strong solution of (1) if there exists $\{u_\beta\}$ such that

- (i) $u_\beta \in C^1(0, T; Y_1)$, $u_\beta(0) = u_0$, $\lim_{\beta \rightarrow 0^+} u_\beta = u$ in X_1 ,
- (ii) $u_\beta(t) \in D_g$ for each $t \in [0, T]$, $t \rightarrow g(u_\beta(t)) \in X_1$,
- (iii) we have $\lim_{\beta \rightarrow 0^+} du_\beta/dt + g(u_\beta) = w$ in X_1 .

Let $Y_2 \subset Y_1$ be a real Banach space with norm $|\cdot|_2$ and $K_2 = K_1 \cap Y_2$. For each $r > 0$, $a > 0$ we set

$$K_2^r = \{u \in K_2, |u|_2 \leq r\}, \quad aK_2^r = \left\{u \in Y_2: \frac{1}{a}u \in K_2^r\right\}$$

and suppose that Y_1 and Y_2 have the property that K_2 is closed in Y_1 .

The following theorems contain the main results concerning the non continuous case:

THEOREM 2.5.1. – Let $D_g \supset K_2$ and assume that there exists $\{g_\beta\}$ such that

- (i) g_β is uniformly continuous and bounded on bounded subsets of K_1 ;
- (ii) for each $r > 0$ there exists $\omega(r)$ such that $\omega(r)I + g_\beta$ is accretive on K_2^r and we have

$$(I + \alpha\omega(r)I + \alpha g_\beta)(\alpha K_2^r) \supset \alpha K_2^r, \quad \alpha > 0$$

for each $\alpha \leq 1$;

- (iii) we have $\lim_{\beta \rightarrow 0} |g_\beta(u) - g(u)|_1 = 0$ uniformly on bounded subsets of K_2 in Y_2 .

In addition, if $u_0 \in K_2$, $w(t) \in K_2$ for each $t \in [0, T]$ and there exists $b > 1$ such that

- (iv) $bu_0 \in K_1$, $bw(t) \in K_1$ for each $t \in [0, T]$;

then for T sufficiently small the initial value problem (1) admits a strong solution u ; moreover there exists $\bar{r}$ such that

- (v) $u(t) \in K_2^{\bar{r}}$ for each $t \in [0, T]$.

THEOREM 2.5.2. – Let the assumptions of Theorem 2.5.1 hold. Then (1) has a unique strong solution u such that u is limit of functions satisfying (v).

THEOREM 2.5.3. – Let the assumptions of Theorem 2.5.1 hold. If K_1 is a cone and there exists σ such that

$$|u|_2 \leq |u + \alpha\sigma u + \alpha g_\beta(u)|_2, \quad \alpha > 0$$

for each $u \in K_2$ then for each $T > 0$ the problem (1) has a unique strong solution satisfying (v).

REMARK 2. – The conclusion of Theorem 2.5.3 can be established without requiring that K_2 is closed in Y_1 .

2.6. Proof of Theorems 2.5.1 and 2.5.2.

The assumptions of Theorem 2.5.1 are assumed to hold throughout this section.

Set $Q_2 = C(0, T; K_2)$, $Q_2^r = C(0, T; K_2^r)$ and assume that $bu_0 \in K_1$ for some $b > 1$. We define

$$\left\{ \begin{array}{l} D_{\tilde{g}_\beta} = Q_1 \\ (\tilde{g}_\beta(u))(t) = \exp[-(1 + \omega_0)t] g_\beta(\exp[(1 + \omega_0)t]u(t)) \\ D_{\tilde{g}} = \{u \in X_1 : \exp[(1 + \omega_0)t]u(t) \in D_g \\ \quad \text{for each } t \in [0, T], t \rightarrow g(\exp[(1 + \omega_0)t]u(t)) \in X_1\} \\ (\tilde{g}(u))(t) = \exp[-(1 + \omega_0)t] g(\exp[(1 + \omega_0)t]u(t)) \end{array} \right.$$

where $r_0 > |u_0|_2 b$ and $\omega_0 = \omega(r_0)$ is defined as in (ii) of Theorem 2.5.1.

We have

LEMMA 2.6.1. – If $\exp[(1 + \omega_0)T] < b$ and $D_\sigma \supset K_2$ then for each $r > 0$, $D_\sigma \supset b^{-1}Q_1 \cap Q_2^r$ and we have

$$(i) \lim_{\beta \rightarrow 0^+} \|\tilde{g}(u) - \tilde{g}_\beta(u)\|_1 = 0$$

uniformly in $b^{-1}Q_1 \cap Q_2^r$.

PROOF. – If $u \in b^{-1}Q_1$ then $\exp[(1 + \omega_0)t]u(t) \in K_1$ for each $t \in [0, T]$ so that $u \in b^{-1}Q_1 \cap Q_2^r$ yields $u(t) \exp[(1 + \omega_0)t] \in K_2^r \cap D_\sigma$ for each $t \in [0, T]$. Now using assumption (iv) of Theorem 2.5.1 we obtain

$$\lim_{\beta \rightarrow 0^+} |g_\beta(\exp[(1 + \omega_0)t]u(t)) - g(\exp[(1 + \omega_0)t]u(t))|_1 = 0$$

uniformly in t on $b^{-1}Q_1 \cap Q_2^r$. This implies that $t \rightarrow g(\exp[(1 + \omega_0)t]u(t)) \in X_1$ and (i).

The existence Theorem 2.3.1 can be extended to the functions g_β with unessential changes and we obtain the following result:

LEMMA 2.6.2. – If $\exp[(1 + \omega_0)T] < b$ then for every $v \in b^{-1}Q_2^r$ there exists $u_\beta \in b^{-1}Q_2^r \cap D_\sigma$ such that

$$u_\beta + \alpha f(u_\beta) + \alpha \omega_0 u_\beta + \alpha \tilde{g}_\beta(u_\beta) = v.$$

THEOREM 2.6.1. – If $\exp[(1 + \omega_0)T] < b$ and $D_\sigma \supset K_2$ then for each $v \in b^{-1}Q_2^r$ there exists $u \in b^{-1}Q_2^r$ strong solution of the equation

$$u + \alpha \omega_0 u + \alpha f(u) + \alpha \tilde{g}(u) = v$$

and we have $u = \lim_{\beta \rightarrow 0^+} u_\beta$ with u_β defined as in Lemma 2.6.2.

PROOF. – Let u_β be defined as in Lemma 2.6.2; we have

$$u_\beta + \alpha \omega_0 u_\beta + \alpha f(u_\beta) + \alpha \tilde{g}(u_\beta) = v + \alpha[\tilde{g}(u_\beta) - \tilde{g}_\beta(u_\beta)]$$

and

$$u_\beta - u_\gamma + \alpha[f(u_\beta) - f(u_\gamma)] + \alpha[\omega_0(u_\beta - u_\gamma) + \tilde{g}_\beta(u_\beta) - \tilde{g}_\beta(u_\gamma)] = \alpha[\tilde{g}_\gamma(u_\gamma) - g_\beta(u_\gamma)]$$

hence by the accretiveness of $f + \tilde{g}_\beta$

$$\|u_\beta - u_\gamma\|_1 \leq \alpha \|\tilde{g}_\gamma(u_\gamma) - \tilde{g}_\beta(u_\gamma)\|_1.$$

In view of Lemma 2.6.1 the proof is complete.

PROOF OF THEOREM 2.5.1. — Let $r_0 > \|w\|_2 b$ and $\exp[(1 + \omega_0)T] < b$. By Theorem 2.6.1 there exists $\tilde{u} \in b^{-1}Q_2^{r_0}$ strong solution of the Cauchy problem

$$\begin{cases} \tilde{u} + \frac{d\tilde{u}}{dt} + \omega_0 \tilde{u} + \exp[-(1 + \omega_0)t]g(\exp[(1 + \omega_0)t]\tilde{u}(t)) = \exp[-(1 + \omega_0)t]w(t), \\ \tilde{u}(0) = u_0. \end{cases}$$

Hence $u = \exp[(1 + \omega_0)t]\tilde{u}$ is strong solution of problem (1) and we have $u(t) \in K_2^{r_0}$ for each $t \in [0, T]$.

THEOREM 2.6.2. — Let $u = \lim_{\beta \rightarrow 0^+} u_\beta$ and $\bar{u} = \lim_{\beta \rightarrow 0^+} \bar{u}_\beta$ strong solutions of (1) with initial datum $u_0, \bar{u}_0$. If there exists $r > 0$ such that $u_\beta(t), \bar{u}_\beta(t) \in K_2^r$ for each $t \in [0, T]$ then

$$|u(t) - \bar{u}(t)|_1 \leq \exp[\omega(r)t]|u_0 - \bar{u}_0|_1.$$

PROOF. — By assumption we have

$$\begin{aligned} \lim_{\beta \rightarrow 0^+} u_\beta &= u, & \lim_{\beta \rightarrow 0^+} \bar{u}_\beta &= \bar{u}, \\ u_\beta(0) &= u_0, & \bar{u}_\beta(0) &= \bar{u}_0, \\ \lim_{\beta \rightarrow 0^+} \frac{d}{dt} u_\beta + g(u_\beta) &= \lim_{\beta \rightarrow 0^+} \frac{d}{dt} \bar{u}_\beta + g(\bar{u}_\beta) = w, \end{aligned}$$

so that

$$\begin{aligned} \frac{d}{dt}(u_\beta - \bar{u}_\beta) + g(u_\beta) - g(\bar{u}_\beta) &= w_\beta \rightarrow 0, \\ u_\beta(0) - \bar{u}_\beta(0) &= u_0 - \bar{u}_0. \end{aligned}$$

The arguments in the proof of Theorem 2.3.1 can be repeated here in an obvious way and the result follows.

PROOF OF THEOREM 2.5.2. — It follows directly from Theorem 2.6.2 by assuming $u_0 = \bar{u}_0$.

REMARK 3. — Let u be strong solution of problem (1); it is easy to verify that if the hypotheses of Theorem 2.5.1 hold then

$$u(t) = u_0 + \int_0^t [-g(u(s)) + w(s)] ds$$

for each $t \in [0, T]$.

2.7. – Proof of Theorem 2.5.3.

LEMMA 2.7.1. – Let u_β be a solution of the Cauchy problem

$$(2) \quad \begin{cases} \frac{du_\beta}{dt} + g_\beta(u_\beta) = w \\ u_\beta(0) = u_0 \end{cases} \quad t \in [0, T], \quad u_0 \in K_2, \quad w \in Q_2.$$

If there exists σ such that $|u|_2 \leq |u + \alpha\sigma u + \alpha g_\beta(u)|_2$ for each $u \in K_2$ then

$$|u_\beta(t)|_2 \leq \exp[\sigma t] |u_0|_2 + \int_0^t \exp[\sigma(t-s)] |w(s)|_2 ds$$

for each $t \in [0, T]$.

PROOF. – By Lemma 2.2.1 there exists $z^*(t) \in \partial|u_\beta(t)|_2$ such that

$$\langle g_\beta(u_\beta(t)), z^*(t) \rangle \geq -\sigma |u_\beta(t)|_2$$

so Lemma 2.2.3 yields

$$D_s |u_\beta(t)|_2 - \sigma |u_\beta(t)|_2 \leq |w(t)|_2$$

and the result follows.

LEMMA 2.7.2. – Let the hypotheses of Theorem 2.5.3 hold; then the initial value problem (2) has a unique solution u_β for each $T > 0$.

PROOF. – This follows from Lemma 2.7.1 by using the same arguments as in the proof of Theorem 2.1.3.

PROOF OF THEOREM 2.5.3. – Let u_β be defined as in Lemma 2.7.1; we have:

$$\frac{d}{dt}(u_\beta - u_\gamma) + g_\beta(u_\beta) - g_\beta(u_\gamma) = g_\gamma(u_\gamma) - g_\beta(u_\gamma).$$

By Lemma 2.7.1 there exists r such that $|u_\beta(t)|_2 \leq r$ for each, $t \in [0, T]$ hence, using assumption (ii) of Theorem 2.5.1 and Lemma 2.2.2, we obtain

$$\begin{aligned} \left\langle \frac{d}{dt}(u_\beta - u_\gamma), z^*(t) \right\rangle_1 - \omega(r) |u_\beta(t) - u_\gamma(t)|_1 &\leq \left\langle \frac{d}{dt}(u_\beta - u_\gamma), z^*(t) \right\rangle_1 + \\ &+ \langle g_\beta(u_\beta(t)) - g_\beta(u_\gamma(t)), z^*(t) \rangle_1 \leq |g_\gamma(u_\gamma(t)) - g_\beta(u_\gamma(t))|_1. \end{aligned}$$

Now Lemma 2.2.3 gives

$$D_s |u_\beta(t) - u_\gamma(t)|_1 - \omega(r) |u_\beta(t) - u_\gamma(t)|_1 \leq |g_\gamma(u_\gamma(t)) - g_\beta(u_\gamma(t))|_1$$

from which it follows that

$$|u_\beta(t) - u_\gamma(t)|_1 \leq \int_0^t \exp[\omega(r)(t-s)] |g_\gamma(u_\gamma(s)) - g_\beta(u_\gamma(s))|_1 ds.$$

In view of Lemma 2.7.1 and assumption (iv) of Theorem 2.5.1 there exists $u \in X_1$ such that $\lim_{\beta \rightarrow 0^+} u_\beta = u$ and we have

$$\lim_{\beta \rightarrow 0^+} \left(\frac{du_\beta}{dt} + g(u_\beta) \right) = \lim_{\beta \rightarrow 0^+} (w + g(u_\beta) - g_\beta(u_\beta)) = w.$$

Since the uniqueness follows from Theorem 2.6.2 the proof is complete.

2.8. – Local and global regularity.

In this section we will give conditions guaranteeing the differentiability of the strong solution of the problem (1).

As a first result we have:

THEOREM 2.8.1. – Let $Y_0 \hookrightarrow Y_1 \hookrightarrow Y_2$, $K_0 \subset Y_0$ a closed convex set containing the origin and $K_1 = K_0 \cap Y_1$, $K_2 = K_0 \cap Y_2$. Let $g \in G_0(Y_0)$ such that $D_g \supset K_1$ and assume that there exists $\{g_\beta\} \subset G_0(Y_0)$ such that

- (i) g_β is uniformly bounded and continuous on bounded subsets of K_1 ;
- (ii) for each $r > 0$ there exists $\omega(r)$ such that for each $\alpha > 0$ and $u, v \in K_2^r$ we have

$$|u - v|_1 \leq |u - v + \alpha[\omega(r)(u - v) + g_\beta(u) - g_\beta(v)]|_1$$

- (iii) for each $r > 0$ there exists $\sigma(r) \leq \omega(r)$ such that for each $u \in K_2^r$ and $a < 1$ we have

$$\begin{aligned} |u|_2 &\leq |u + \alpha\sigma(r)u + \alpha g_\beta(u)|_2 \\ (I + \alpha\omega(r)I + \alpha g_\beta)(aK_2^r) &\supset aK_2^r \end{aligned}$$

- (iv) $\lim |g_\beta(u) - g(u)|_1 = 0$ uniformly on bounded subsets of K_2 in Y_2 ;

- (v) if $u, u_\beta \in K_1$ and $\lim_{\beta \rightarrow 0^+} |u_\beta - u|_1 = 0$ then $\lim_{\beta \rightarrow 0^+} |g(u_\beta) - g(u)|_0 = 0$.

Assume in addition that $u_0 \in K_2$, $w(t) \in K_2$ for each $t \in [0, T]$ and that there ex-

ists $b > 1$ such that $bu_0 \in K_1$ and $bw(t) \in K_1$ for each $t \in [0, T]$; then if T is sufficiently small the initial value problem (1) has a unique solution $u \in C^1(0, T; Y_0) \cap C(0, T; K_2)$.

PROOF. — By Theorem 2.5.1 there exists $u = \lim u_\beta$ in $C(0, T; Y_1)$ strong solution of problem (1). Using (v) we have

$$\lim \frac{du_\beta}{dt} = \lim [-g(u_\beta) + w] = -g(u) + w$$

in $C(0, T; Y_0)$ which implies $u \in C^1(0, T; Y_0)$ and $du/dt + g(u) = w$.

The following theorem can be shown in the same way:

THEOREM 2.8.2. — Let the assumptions of Theorem 2.8.1 hold; if $\sigma(r)$ defined as in (iii) can be taken independent of r then the problem (1) has a unique solution $u \in C^1(0, T; Y_0) \cap C(0, T; K_2)$ for each $T > 0$.

2.9. — Linear perturbations.

Let A be a linear operator. In this section we shall consider the initial value problem:

$$(3) \quad \begin{cases} \frac{du}{dt} + Au + g(u) = w \\ u(0) = u_0 \end{cases} \quad t \in [0, T]$$

and give sufficient conditions assuring the existence of a solution in some generalized sense for problem (3); more precisely we shall investigate the existence of function u strong solution of the equation

$$(\overline{f + A})(u) + \tilde{g}(u) = w$$

where f is defined as in section 2.3 and

$$\begin{cases} D_{\tilde{g}} = \{u \in X_1, u(t) \in D_g \text{ for each } t \in [0, T], t \rightarrow g(u(t)) \in X_1\}, \\ (\tilde{g}(u))(t) = g(u(t)). \end{cases}$$

Replacing f by $\overline{f + A} + cI$ in the proof of Theorems 2.5.1, 2.5.2, 2.5.3, 2.8.1 and 2.8.2 we obtain the following results:

THEOREM 2.9.1. — In addition to the conditions of Theorem 2.5.1 suppose that

- (i) $-A + cI$ generates a linear contraction semigroup which leaves K_1 invariant for some $c \in R$;

(ii) for each $u \in D_A$ we have

$$|u|_2 \leq |u + \alpha cu + \alpha Au|_2.$$

Then if $u_0 \in D_A$ and T is sufficiently small the initial value problem (3) has a unique solution u in the generalized sense introduced above such that u is limit of functions satisfying condition (vi) of Theorem 2.5.1.

THEOREM 2.9.2. — Let the assumptions of Theorems 2.5.3 and 2.9.1 hold. Then for each $T > 0$ the Cauchy problem (3) has a unique solution defined as in Theorem 2.9.1.

THEOREM 2.9.3. — For $A \in G_0(Y_0)$ define

$$\begin{cases} D_{A_1} = \{u \in Y_1, Au \in Y_1\}, \\ A_1 u = Au. \end{cases}$$

In addition to the conditions of Theorem 2.8.1 suppose that $u_0 \in D_{A_1}$ and that A_1 satisfies assumptions (i) and (ii) of Theorem 2.9.1. Then if T is sufficiently small there is a unique u such that there exists $\{u_\beta\}$ verifying

- (i) $u_\beta \in C^1(0, T; Y_0) \cap C(0, T; K_2)$, $u_\beta(t) \in D_{A_1}$ for each $t \in [0, T]$,
- (ii) $u_\beta(0) = u_0$, $\lim_{\beta \rightarrow 0^+} u_\beta = u$ in $C(0, T; Y_1)$,
- (iii) $\lim_{\beta \rightarrow 0^+} [du_\beta/dt + Au_\beta] = -g(u) + w$ in $C(0, T; Y_0)$.

THEOREM 2.9.4. — Let the assumptions of Theorems 2.8.2 and 2.9.3 hold. Then for each $T > 0$ there exists u defined as in Theorem 2.9.3.

2.10. — Examples.

Let Y_i be the Banach space of measurable functions $u: R^n \rightarrow R$ such that

$$|u|_i = \int (1 + |x|^2)^{i-1} |u| dx < +\infty, \quad i = 1, 2$$

equipped with the norm $|\cdot|_i$. We consider the integral operator $(u, v) \rightarrow g^+(u, v) - g^-(u, v)$ defined as follows

$$\begin{aligned} g^+(u, v) &= \frac{1}{2} \left[u(x) \int a(y) v(y) dy + v(x) \int a(y) u(y) dy \right] \\ g^-(u, v) &= \frac{1}{2} \left[a(x) u(x) \int v(y) dy + a(x) v(x) \int u(y) dy \right] \end{aligned}$$

where $a(x) = \sqrt{1 + |x|^2}$, and for each $\beta \in]0, 1]$

$$\begin{aligned} g_\beta^+(u, v) &= \frac{1}{2} \left[u(x) \int a_\beta(y) v(y) dy + v(x) \int a_\beta(y) u(y) dy \right] \\ g_\beta^-(u, v) &= \frac{1}{2} \left[a_\beta(x) u(x) \int v(y) dy + a_\beta(x) v(x) \int u(y) dy \right] \end{aligned}$$

where $a_\beta(x) = (1 + \beta a(x))^{-1} a(x)$. Finally we denote by K_1 the closed convex set $K_1 = \{u \in Y_1, |u|_1 \leq 1\}$ and define

$$\begin{cases} D_g = \{u \in Y_1, g(u, u) \in Y_1\}, \\ g(u) = g(u, u), \\ D_{g_\beta} = Y_1, \\ g_\beta(u) = g_\beta(u, u). \end{cases}$$

LEMMA 2.10.1. – The following statements hold:

- (i) $2a_\beta I + g_\beta$ is Lipschitz continuous on bounded subsets of Y_1 ;
- (ii) if $u, v \in K_2^r$ then

$$|u - v|_1 \leq |u - v| + 2r\alpha(u - v) + 2\alpha a_\beta(u - v) + \alpha(g_\beta(u) - g_\beta(v))|_1$$

- (iii) if $u \in K_2^r$ and $a < 1$ then

$$\begin{aligned} |u|_2 &\leq |u + 2r\alpha u + 2\alpha a_\beta u + \alpha g_\beta(u)|_2 \\ (I + 2r\alpha I + \alpha 2a_\beta I + \alpha g_\beta)(aK_2^r) &\supset aK_2^r \end{aligned}$$

- (iv) we have $D_g \supset K_2$ and $\lim |a_\beta u + g_\beta(u) - au - g(u)|_1 = 0$ uniformly on bounded subsets of Y_2 .

PROOF. – Assertion (i) is evident. Assertion (ii) follows from Lemma 2.2.2 by noting that $\operatorname{sgn}(u - v) \in \partial|u - v|_1$ ⁽³⁾ and that if $u, v \in K_2^r$ we have

$$\begin{aligned} \langle 2a_\beta(u - v) + g_\beta(u) - g_\beta(v), \operatorname{sgn}(u - v) \rangle_1 &= 2 \int a_\beta |u - v| dx + \\ &+ \frac{1}{2} \left[\int |u - v| dx \int a_\beta(u + v) dy + \int (u + v) \operatorname{sgn}(u - v) dx \int a_\beta(u - v) dy \right] \\ &- \frac{1}{2} \left[\int a_\beta |u - v| dx \int (u + v) dy + \int a_\beta(u + v) \operatorname{sgn}(u - v) dx \int (u - v) dy \right] \geq \\ &\geq 2 \int a_\beta |u - v| dx - \int |u + v| dx \int a_\beta |u - v| dy - \int |u - v| dx \int a_\beta |u + v| dy \geq -2r|u - v|_1. \end{aligned}$$

⁽³⁾ $\operatorname{sgn} u$ is the function defined as follows

$$\operatorname{sgn} u \begin{cases} = 1 & \text{if } u > 0 \\ = 0 & \text{if } u = 0 \\ = -1 & \text{if } u < 0. \end{cases}$$

To prove (iii) take $v \in K_2^r$ and define

$$u_1 = v,$$

.....

$$u_n = \frac{1}{1 + \alpha a_\beta} \frac{v - g_\beta^+(u_{n-1})}{1 + (2r\alpha/(1 + \alpha a_\beta)) + \alpha((a_\beta - (u_{n-1})^{-1} g_\beta^-(u_{n-1})) / (1 + \alpha a_\beta))}.$$

We have

$$|u_1|_1 = |v|_1 \leq 1, \quad |u_1|_2 = |v|_2 \leq r,$$

.....

$$|u_n|_1 \leq \frac{1}{1 + \alpha 2r} (|v|_1 + \alpha |u_{n-1}|_1 |u_{n-1}|_2) \leq \frac{1 + \alpha r}{1 + \alpha 2r} \leq 1,$$

$$|u_n|_2 \leq \frac{1}{1 + \alpha 2r} (|v|_2 + \alpha |u_{n-1}|_2^2) \leq \frac{r + \alpha r^2}{1 + \alpha 2r} \leq r.$$

In view of (i) there exists $\lim u_n = u \in K_2^r$ and we have

$$u + 2r\alpha u + \alpha 2a_\beta u + \alpha g_\beta(u) = v$$

so that

$$(I + \alpha 2rI + \alpha 2a_\beta I + \alpha g_\beta)(K_2^r) \supset K_2^r.$$

Now let $u \in K_2^r$; we have

$$\begin{aligned} \langle 2a_\beta u + g_\beta(u), \operatorname{sgn} u \rangle_2 &= \int 2a_\beta |u| (1 + |x|^2) dx + \\ &\quad + \int |u| (1 + |x|^2) dx \int u a_\beta dy - \int a_\beta |u| (1 + |x|^2) dx \int u dy \geq -r |u|_2 \end{aligned}$$

hence combining these results and using Lemma 2.2.1 the conclusion follows. Finally we show (iv); if $u \in K_2^r$ then

$$|a_\beta u + g_\beta(u) - au - g(u)|_1 \leq \beta |u|_2 + 2\beta |u|_1 |u|_2$$

and the result follows.

As a consequence of Lemma 2.10.1, and Theorem 2.5.1 we have

THEOREM 2.10.1. - Assume that $|u_0|_1 < 1$ and $u_0 \in Y_2$. Then if T is sufficiently small for every $w \in C(0, T; K_2)$ such that $\sup |w(t)|_1 < 1$ the initial value problem

$$(4) \quad \begin{cases} \frac{du}{dt} + au + g(u) = w, \\ u(0) = u_0, \end{cases}$$

has a unique strong solution u such that u is limit of functions of $C(0, T; K_2^r)$ for some $r > 0$.

Now set $\tilde{K}_1 = \{u \in Y_1, |u|_1 \leq 1, u \geq 0 \text{ a.e. } x \in R^n\}$; we have:

LEMMA 2.10.2. – The following statements hold:

- (i) $2a_\beta I + g_\beta$ is Lipschitz continuous on bounded subsets of Y_1 ;
- (ii) if $u, v \in \tilde{K}_2^r$ then

$$|u - v|_1 \leq |u - v + \alpha a_\beta(u - v) + \alpha[g_\beta(u) - g_\beta(v)]|_1$$

- (iii) if $a < 1$ and $u \in K_2^r$ then

$$|u|_2 \leq |u + \alpha 2a_\beta u + \alpha g_\beta(u)|_2$$

$$(I + \alpha 2a_\beta I + \alpha g_\beta)(a\tilde{K}_2^r) \supset a\tilde{K}_2^r$$

- (iv) we have $D_g \supset \tilde{K}_2$ and $\lim_{\beta \rightarrow 0^+} |a_\beta u + g_\beta(u) - au - g(u)|_1 = 0$ uniformly on bounded subsets of Y_2 .

PROOF. – It suffices to show (ii) and (iii). Let $u, v \in \tilde{K}_2^r$; we have

$$\begin{aligned} & \langle 2a_\beta(u - v) + g_\beta(u) - g_\beta(v), \operatorname{sgn}(u - v) \rangle_1 = \\ & = 2 \int a_\beta |u - v| dx + \frac{1}{2} \left[\int |u - v| dx \int (u + v) a_\beta dy + \int (u + v) \operatorname{sgn}(u - v) dx \int (u - v) a_\beta dy \right] \\ & \quad - \frac{1}{2} \left[\int |u - v| a_\beta dx \int (u + v) dy + \int (u + v) \operatorname{sgn}(u - v) a_\beta dx \int (u - v) dy \right] \geq 0 \end{aligned}$$

which proves (ii).

To prove (iii) take $v \in K_2^r$ and define

$$\begin{aligned} u_1 &= v, \\ &\dots\dots\dots \\ u_n &= \frac{v}{1 + \alpha 2a_\beta + \alpha(u_{n-1})^{-1}[g_\beta^+(u_{n-1}) - g_\beta^-(u_{n-1})]}. \end{aligned}$$

We have $|u_n|_1 \leq |v|_1$, $|u_n|_2 \leq |v|_2$ hence, in view of (i) there exists $u = \lim u_n \in \tilde{K}_2^r$ and we have

$$u + \alpha 2a_\beta u + \alpha g_\beta(u) = v.$$

Now let $u \in K_2$; we have

$$\begin{aligned} \langle 2a_\beta u + g_\beta(u), \operatorname{sgn} u \rangle_2 &= \int |u|(1 + |x|^2) 2a_\beta dx + \\ &\quad - \int (1 + |x|^2) a_\beta |u| dx \int u dy + \int (1 + |x|^2) |u| dx \int a_\beta u dy \geq 0, \end{aligned}$$

hence combining these results the conclusion follows.

As a consequence of Lemma 2.10.2 and Theorem 1.3.5 we have:

THEOREM 2.10.2. – Assume that $u_0 \geq 0$ a.e. $x \in R^n$, $|u_0|_1 < 1$ and $u_0 \in Y_2$. Then for every $T > 0$ and $w \in C(0, T; K_2')$ such that $\sup_{t \in [0, T]} |w(t)|_1 \leq 1$ the initial value problem (4) has a unique strong solution u .

2.11. – The Boltzmann equation.

As an application of the results obtained in section 2.9 we shall study the Boltzmann equation in the space-independent case:

$$\frac{\partial u(\xi, t)}{\partial t} + c \cdot \nabla_{\xi} u + g(u, u) = 0, \quad \xi \in R^3, \quad t > 0,$$

where $c = (c_1, c_2, c_3)$ is constant and different from zero ⁽⁴⁾ and the bilinear function $(u, v) \rightarrow g(u, v)$ is defined as follows ⁽⁵⁾

$$g(u, v) = g^+(u, v) - g^-(u, v)$$

where

$$g^+(u, v) = \frac{1}{2} \int_{R^3} \int_0^{2\pi} \int_0^{\pi/2} [u(\xi)v(\xi_1) + u(\xi_1)v(\xi)] B(\theta, \xi, \xi_1) d\theta d\sigma d\xi_1$$

and

$$g^-(u, v) = \frac{1}{2} \int_{R^3} \int_0^{2\pi} \int_0^{\pi/2} [u(\xi')v(\xi'_1) + u(\xi'_1)v(\xi')] B(\theta, \xi, \xi_1) d\theta d\sigma d\xi_1.$$

Here the variables ξ' and ξ'_1 depend on ξ, ξ_1 through

$$\xi' = \xi + \alpha(\alpha \cdot (\xi - \xi_1))$$

$$\xi'_1 = \xi_1 - \alpha(\alpha \cdot (\xi - \xi_1))$$

where $\alpha = (\cos \theta, \sin \theta \cos \sigma, \sin \theta \sin \sigma)$. These equations imply the following equalities

$$\xi + \xi_1 = \xi' + \xi'_1,$$

$$\xi^2 + \xi_1^2 = \xi'^2 + \xi'^2_1.$$

⁽⁴⁾ The case $c = (0, 0, 0)$ is treated in [10].

⁽⁵⁾ By u, v we denote measurable functions.

$B(\theta, \xi, \xi_1):]0, \pi/2[\times \mathbb{R}^s \rightarrow \mathbb{R}$ is a given measurable function non negative a.e. such that

$$B(\theta, \xi, \xi_1) = B(\theta, \xi_1, \xi) ,$$

$$B(\theta, \xi, \xi_1) = B(\theta, \xi', \xi'_1) .$$

Finally we recall that g satisfies the following equalities (see [5] and [12]):

$$(5) \quad \begin{aligned} \int g(u, v) d\xi &= 0 , \\ \int \xi^2 g(u, v) d\xi &= 0 . \end{aligned}$$

We shall study the Boltzmann equation under the following assumption

$$(6) \quad \sup_{\xi, \xi_1 \in \mathbb{R}^s} \int_0^{\pi/2} \frac{B(\theta, \xi, \xi_1)}{1 + \sqrt{\xi^2 + \xi_1^2}} d\theta = B < +\infty .$$

For each $\beta \in]0, 1]$ we shall consider the approximating function

$$g_\beta(u, v) = g_\beta^+(u, v) - g_\beta^-(u, v)$$

where

$$g_\beta^+(u, v) = \frac{1}{2} \iiint [u(\xi)v\xi_1 + u(\xi_1)v(\xi)] \frac{B(\theta, \xi, \xi_1)}{1 + \beta \sqrt{\xi^2 + \xi_1^2}} d\theta d\sigma d\xi_1$$

and

$$g_\beta^-(u, v) = \frac{1}{2} \iiint [u(\xi')v(\xi'_1) + u(\xi'_1)v(\xi')] \frac{B(\theta, \xi, \xi_1)}{1 + \beta \sqrt{\xi^2 + \xi_1^2}} d\theta d\sigma d\xi_1 .$$

We now quote a number of properties of the functions g_β which are proved in [10]. By g_0 we denote the function g .

LEMMA 2.11.1. – For each $\beta \in]0, 1]$ we have

$$\int |g_\beta(u, v)|(1 + \xi^2)^i d\xi \leq M_\beta \int |u|(1 + \xi^2)^i d\xi \int |v|(1 + \xi^2)^i d\xi , \quad i = 0, 1, 2 ,$$

and

$$\int |g(u, v)|(1 + \xi^2)^i d\xi \leq M \int |u|(1 + \xi^2)^{i+1} d\xi \int |v|(1 + \xi^2)^{i+1} d\xi , \quad i = 0, 1$$

where

$$M_\beta = 8\pi \sup_{\xi, \xi_1 \in \mathbb{R}^2} \int_0^{\pi/2} \frac{B(\theta, \xi, \xi_1)}{1 + \beta \sqrt{\xi^2 + \xi_1^2}} d\theta$$

and $M = 8\pi B$.

LEMMA 2.11.2. — If u is non negative a.e. we have

$$\int (1 + \xi^2)^2 g_\beta(u, u) d\xi \geq -M \int (1 + \xi^2) u d\xi \int (1 + \xi^2)^2 u d\xi$$

for each $\beta \in [0, 1]$.

LEMMA 2.11.3. — If u and v are any functions non negative a.e. then

$$\int (1 + \xi^2) \operatorname{sgn}(u - v) [g_\beta(u, u) - g_\beta(v, v)] d\xi \geq -2M \int (1 + \xi^2)^2 (u + v) d\xi \int (1 + \xi^2) |u - v| d\xi$$

for each $\beta \in [0, 1]$.

LEMMA 2.11.4. — For each $\beta, \gamma \in [0, 1]$ we have

$$\int (1 + \xi^2)^i |g_\beta(u, u) - g_\gamma(u, u)| d\xi \leq 8M |\beta - \gamma| \left(\int (1 + \xi^2)^{i+1} |u| d\xi \right)^2,$$

where $i = 0, 1$.

We will study the following Cauchy problem

$$(7) \quad \begin{cases} \frac{\partial u(\xi, t)}{\partial t} + c \cdot \nabla_\xi u + g(u, u) = 0 \\ u(0, \xi) = u_0(\xi) \end{cases} \quad t \in [0, T]$$

and will give conditions on u_0 guaranteeing the existence of a unique strong solution of (7).

Let Y_i ($i = 0, 1, 2$) be the Banach space of measurable functions $u: \mathbb{R}^3 \rightarrow \mathbb{R}$ such that

$$|u|_i = \int (1 + \xi^2)^i |u| d\xi < +\infty$$

equipped with the norm $|\cdot|_i$. By $K_i \subset Y_i$ we denote the closed convex set

$$K_i = \{u \in Y_i, u \geq 0 \text{ a.e. } |u|_1 \leq N\}.$$

Let A_i be the function defined as follows

$$\begin{cases} D_{A_i} = \{u \in Y_i, c \cdot \nabla_\xi u \in Y_i\}, \\ A_i u = c \cdot \nabla_\xi u. \end{cases}$$

It is well known that $-A_1$ is the infinitesimal generator of the linear semi-group $\exp[-tA_1]$ given by

$$(\exp[-tA_1]u)(\xi) = u(\xi - ct);$$

moreover it is easy to see that

$$\begin{aligned} |\exp[-tA_1]u|_1 &\leq \exp[2|c|t]|u|_1, & \text{for each } u \in Y_1, \\ |\exp[-tA_1]u|_2 &\leq \exp[4|c|t]|u|_2, & \text{for each } u \in Y_2. \end{aligned}$$

For each $\beta \in [0, 1]$ let us denote by g_β^i ($i = 0, 1$) the function defined as follows

$$\begin{cases} D_{g_\beta^i} = K_{i+1}, \\ (g_\beta^i(u))(t) = g(u(t), u(t)), \\ D_{g_\beta^i} = K_i, \\ (g_\beta^i(u))(t) = g_\beta(u(t), u(t)), & \beta \in]0, 1]. \end{cases}$$

By Lemmas 2.11.1, 2.11.2, 2.11.3, and 2.11.4 we get the following result:

LEMMA 2.11.5. – The following statements hold:

- (i) for each $\beta \in]0, 1]$ g_β^i is Lipschitz continuous on bounded subsets of K_i ;
- (ii) for each $u, v \in K_2^r$ we have

$$|u - v|_1 \leq |u - v| + \alpha 8Mr(u - v) + \alpha g_\beta^1(u) - \alpha g_\beta^1(v)|_1$$

- (iii) for each $u \in K_2$ we have

$$|u|_2 \leq |u + \alpha MNu + \alpha g_\beta^1(u)|_2$$

- (iv) we have $\lim |g_\beta^1(u) - g^1(u)|_1 = 0$ uniformly on bounded subsets of K_2 in Y_2 ;
- (v) if $u_n, u \in K_1$ and $\lim |u_n - u|_1 = 0$ then $\lim |g^1(u_n) - g^1(u)|_0 = 0$.

The following lemma establish further properties of g_β^1 .

LEMMA 2.11.6. – For each $\beta \in]0, 1]$ there exists $\alpha(\beta)$ such that if $\alpha < \alpha(\beta)$ then

$$(1 - \alpha 8MrI - \alpha g_\beta^1)(K_2^r) \subset K_2^r.$$

PROOF. — Let $u \in K_2^r$, $\alpha(\beta) = (8Mr + rM_\beta)^{-1}$ and $\alpha < \alpha(\beta)$; we have

$$u - \alpha 8Mr u - \alpha g_\beta^1(u) = u \left(1 - \alpha 8Mr - \frac{\alpha}{u} g^+(u, u) \right) + \alpha g^-(u, u) \geq 0 \text{ a.e. .}$$

Thus using (5) and Lemma 2.11.2 we obtain

$$\begin{aligned} |u - \alpha 8Mr u - \alpha g_\beta^1(u)|_1 &= |u|_1 - \alpha 8Mr |u|_1 \leq N \\ |u - \alpha 8Mr u - \alpha g_\beta^1(u)|_2 &\leq |u|_2 - \alpha 8Mr |u|_2 + \alpha N M |u|_2 \leq r. \end{aligned}$$

By Lemmas 2.11.5 and 2.11.6 and Theorem 2.9.4 (see Remark 2, Part I) we obtain the following result:

THEOREM 2.11.1. — Let $u_0 \in K_2$ and let $N > |u_0|_1$. Then for each $T > 0$ there is a unique $u \in C(0, T; K_2)$ such that there exists $\{u_\beta\}$ verifying

- (i) $u_\beta \in C^1(0, T; Y_0)$, $u_\beta(t) \in D_{A_1}$ for each $t \in [0, T]$, $u_\beta(0) = u_0$,
- (ii) we have $\lim_{\beta \rightarrow 0^+} u_\beta = u$ in $C(0, T; Y_1)$ and $\lim_{\beta \rightarrow 0^+} \frac{du_\beta}{dt} + Au_\beta = -g(u)$ in $C(0, T; Y_0)$.

REFERENCES

- [1] V. BARBU, *Continuous perturbations of non-linear m-accretive operators in Banach spaces*, Boll. U.M.I., **6** (1972), pp. 270-278.
- [2] R. BODMER, *Zur Boltzmann-Gleichung*, Zürich, Seminar für Theor. Phys. E.T.H.
- [3] H. BRÉZIS, *Opérateurs maximaux monotones et semi-groupes de contraction dans les espaces de Hilbert*, Math. Studies 5, North-Holland (1973).
- [4] F. E. BROWDER, *Nonlinear accretive operators in Banach spaces*, Bull. Am. Soc., **73** (1967), pp. 470-476.
- [5] C. CERCIGNANI, *Mathematical methods in kinetic theory*, N. Y., Plenum Press (1969).
- [6] G. DA PRATO, *Somme d'applications non-linéaires*, Symposia Mathematica, VII (1971), pp. 233-268.
- [7] G. DA PRATO: *Applications croissantes et équations d'évolutions dans les espaces de Banach*, to appear.
- [8] G. DA PRATO - P. GRISVARD, *Somme d'opérateurs linéaires et équations différentielles opérationnelles*, J. Math. pures et appl., to appear.
- [9] G. DI BLASIO, *Strong solutions for Boltzmann equation in the spatially homogeneous case*, Boll. U.M.I., **8** (1973), pp. 127-136.
- [10] G. DI BLASIO, *Differentiability of spatially homogeneous solutions of the Boltzmann equation in the non Maxwellian case*, Comm. Math. Phys., **38** (1974), pp. 331-340.
- [11] G. DI BLASIO, *Somme d'opérateurs non linéaires et application à l'équation de Boltzmann*, C. R. Acad. Sc., **280**, série A (1975), pp. 1121-1123.
- [12] H. GRAD, *Principles of kinetic theory of gases*, Handbuch der Physik, **12**, Berlin - Göttingen - Heidelberg, Springer (1958).

-
- [13] M. IANNELLI, *A note on some non-linear non contraction semigroup*, Boll. U.M.I., **6** (1970), pp. 1015-1025.
 - [14] T. KATO, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan, **19** (1967), pp. 508-520.
 - [15] T. KATO, *Accretive operators and nonlinear evolution equations in Banach spaces*, Proc. Symp. Pure Math., **18**, Part I, A.M.S. Providence, R. I. (1968), pp. 138-161.
 - [16] J. L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod-Gauthier Villars (1969).
 - [17] R. H. MARTIN JR., *Differential equations on closed subsets of a Banach space*, Trans. Amer. Math. Soc., **179** (1973), pp. 399-414.
 - [18] R. H. MARTIN JR., *Invariant sets for perturbed semigroups of linear operators*, to appear.
 - [19] G. WEBB, *Nonlinear perturbations of linear accretive operators in Banach spaces*, Journal of Funct. An., **10** (1972), pp. 191-203.
 - [20] A. JA. POVZNER, *The Boltzmann equation in the kinetic theory of gases*, Mat. Sbornik, **58** (1962), pp. 65-86; translated in A.M.S. Translation, Series 2, **47** (1965), pp. 193-216.
-