Oscillation of Nonlinear Vector Differential Equations (*).

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Summary. — *Oscillation criteria are obtained for vector partial differential equations of the type $\Delta v + b(x, v)v = 0$, $x \in G$, $v \in E^m$, where G is an exterior domain in E^n , and b is a continuous nonnegative valued function in $G \times E^m$. A solution $v: G \rightarrow E^m$ is called h -oscillatory in G whenever the scalar product $[v(x), h]$ ($|h| = 1$) has zeros x in G with $|x|$ arbitrarily large. It is shown that the spherical mean of $[v(x), h]$ over a hypersphere of radius r in E^n satisfies a nonlinear ordinary differential inequality. As a consequence, the main theorems give sufficient conditions on $b(x, t)$, depending upon the dimension n , for all solutions v to be h -oscillatory in G .*

Let G be an exterior domain in n -dimensional Euclidean space E^n , i.e. suppose that G contains the complement of some n -ball in E^n . Points in E^n will be denoted by $x = (x_1, \dots, x_n)$ and the Euclidean length of x by $|x|$. We shall consider m -vector functions $v: G \rightarrow E^m$ whose components satisfy a system of nonlinear partial differential equations in G . The notation $[v(x), w(x)]$ will be used for the scalar product of $v(x), w(x) \in E^m$ and $|v(x)|$ for the Euclidean length of $v(x)$. Our objective is to obtain oscillation criteria for the vector differential operator defined by

$$(1) \quad Lv \equiv \Delta v + b(x, v)v = 0, \quad x \in G$$

where

$$\Delta = \sum_{i=1}^n D_i D_i, \quad D_i = \partial / \partial x_i, \quad i = 1, \dots, n.$$

The following assumptions on the function b will be retained in the sequel.

(2) ASSUMPTIONS

$b: G \times E^m \rightarrow E^1$ is a continuous nonnegative valued function such that

$$b(x, v) \geq \lambda(|x|) \varphi(|v|)$$

for all $x \in G$, $v \in E^m$, where

- (i) λ is a nonnegative continuous function in $[0, \infty)$;
- (ii) $\varphi \in C^1 [0, \infty)$ with $\varphi(t) > 0$ for $t \neq 0$ and $\varphi'(t) \geq 0$ for all t ; and
- (iii) The function φ defined by $\varphi(t) = t\psi(t)$ is convex in $[0, \infty)$.

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The domain $\mathfrak{D}_L$ of L is defined to be the set of all m -vector functions $v: G \rightarrow E^m$ with real components $v_k \in C^2(G)$, $k = 1, \dots, m$.

The following notation will be used:

$$S_a = \{x \in E^n: |x| > a\};$$

$$G_a = G \cap S_a;$$

$$\Omega_a = \{x \in E^n: |x| = a\}, \quad a > 0.$$

Our assumption that G is an exterior domain means that $S_a \subset G$ for some $a > 0$.

The definition of oscillation of L in a direction h (called h -oscillation) given below has been used by the authors [2] and by Ju. I. DOMŠLAK [1] in special cases.

DEFINITION. — A function $v: G \rightarrow E^m$ is called h -osc in $G \subset E^n$ iff $[v(x), h]$ has a zero in G_a for all $a > 0$, where h is a constant unit vector in E^m . The differential operator L is called h -osc in G whenever every solution $v \in \mathfrak{D}_L$ of $Lv = 0$ is h -osc in G .

This definition means that the projection of an h -osc function v on h is oscillatory in G in the classical sense, i.e. this projection has zeros $x \in G$ with $|x|$ arbitrarily large. Examples are given in [2] to show that the h -osc property of an operator L might depend upon h , e.g. it might be satisfied for $h^* = (1, 0)$ but not for $h^* = (0, 1)$.

An example of a function $b: G \times E^m \rightarrow E^1$ for which assumptions (2) are satisfied is

$$b(x, v) = g^*(v)A(x)g(v), \quad x \in G,$$

where $A(x)$ is a symmetric, uniformly positive definite, continuous matrix function, and g is a continuous m -vector valued function such that

$$|g(v)|^2 \geq \psi(|v|)$$

with ψ satisfying assumptions (2ii) and (2iii). Then $b(x, v) \geq \lambda(r)\psi(|v|)$ for all $x \in G$, $v \in E^m$, where $\lambda(r)$ denotes the minimum of the smallest eigenvalue of $A(x)$ on the hypersphere Ω_r . Clearly this example provides a fairly wide class of nonlinear systems (1) for which all the assumptions (2) are satisfied. An Emden-Fowler prototype is the case $g(v) = |v|^\alpha v$, $\alpha \geq -1$. A slightly different example is

$$b(x, v) = [v^*A(x)v]^\gamma, \quad x \in G$$

with $A(x)$ as before and $\gamma \geq 0$.

The h -oscillation theorems given by the authors in [2] do not apply to the above nonlinear examples since the majorization hypothesis given there is not fulfilled. On the other hand, we do not consider the case that b is matrix-valued in the present

paper. Unlike the theorems in [2], our results are primarily nonlinear. For completeness, however, we state Theorem 3 in such a way that it is applicable to both the linear and nonlinear cases.

The spherical mean-projection of an L^1 m -vector function $v: G \rightarrow E^m$ over Ω_r , $r > a$, is defined as

$$(3) \quad f(r) = f_h(r; v) = \frac{1}{s(\Omega_r)} \int_{\Omega_r} [v(x), h] ds,$$

where s denotes the measure on Ω_r .

LEMMA. — Under assumptions (2), let $v \in \mathfrak{D}_L$ be a solution of (1) such that $[v(x), h]$ has constant sign on Ω_r , where h is a constant unit m -vector. Then the spherical mean-projection (3) of v over Ω_r satisfies the ordinary differential inequality

$$(4) \quad [r^{n-1}|f(r)|']' + r^{n-1}\lambda(r)\varphi[|f(r)|] \leq 0$$

for $a < r < \infty$.

PROOF. — First consider the case that $[v(x), h] > 0$ on Ω_r . Since $[v(x), h] \leq |v(x)| |h| = |v(x)|$ and φ is a nondecreasing function by (2ii), we find from (1) that

$$(5) \quad \begin{aligned} \Delta[v, h] &= -b(x, v)[v, h] \\ &\leq -\lambda(r)\varphi(|v|)[v, h] \\ &\leq -\lambda(r)\varphi([v, h])[v, h], \end{aligned}$$

where $r = |x|$. For $[v(x), h] < 0$ we have similarly

$$(6) \quad \Delta[v, h] \geq -\lambda(r)\varphi(|[v, h]|)[v, h].$$

It follows from (5) and (6) that $u = |[v, h]|$ satisfies the differential inequality

$$(7) \quad \Delta u \leq -\lambda(r)u\varphi(u) = -\lambda(r)\varphi(u).$$

The definition (3) reduces to

$$|f(r)| = \frac{r^{1-n}}{\omega_n} \int_{\Omega_r} u(x) ds,$$

where ω_n denotes the measure of the unit $(n-1)$ -sphere in E^n . Thus

$$(8) \quad |f(r)| = \frac{r^{1-n}}{\omega_n} \int_{\Omega_r} u(x) \sum_{i=1}^n \frac{x_i^2}{r^2} ds, \quad r = |x|.$$

A formula which can be used to differentiate such an integral with respect to r has been given by the authors [3]:

$$(9) \quad \frac{d}{dr} \int_{\Omega_r} w(x) \frac{x_i}{r} ds = \int_{\Omega_r} D_i w(x) ds, \\ D_i = \partial/\partial x_i, \quad i = 1, \dots, n.$$

We apply (9) to (8) to obtain

$$\begin{aligned} \omega_n r^{n-1} |f(r)|' &= \int_{\Omega_r} \sum_i D_i \left(\frac{u(x)x_i}{r} \right) ds + \frac{1-n}{r} \int_{\Omega_r} u(x) ds \\ &= \int_{\Omega_r} \left(\sum_i D_i u \right) \frac{x_i}{r} ds \end{aligned}$$

since $\sum D_i(x_i/r) = (n-1)/r$. Applying (9) again we find that

$$(10) \quad \begin{aligned} \omega_n [r^{n-1} |f(r)|']' &= \int_{\Omega_r} \sum D_i^2 u ds \\ &= \int_{\Omega_r} \Delta u ds \\ &\leq -\lambda(r) \int_{\Omega_r} \varphi(u(x)) ds \end{aligned}$$

on account of (7). From Jensen's inequality for convex functions φ [4, p. 160],

$$\varphi \left[\frac{1}{s(\Omega_r)} \int_{\Omega_r} u(x) ds \right] \leq \frac{1}{s(\Omega_r)} \int_{\Omega_r} \varphi[u(x)] ds.$$

Then (10) implies the differential inequality

$$\begin{aligned} [r^{n-1} |f(r)|']' &\leq -\frac{\lambda(r)}{\omega_n} s(\Omega_r) \varphi[|f(r)|] \\ &= -r^{n-1} \lambda(r) \varphi[|f(r)|]. \end{aligned}$$

This completes the proof of the Lemma.

On the basis of this Lemma, h -oscillation theorems for (1) will follow from appropriate oscillation criteria for the ordinary differential inequality (4). Theorems A and B below apply to the differential inequality

$$(11) \quad [a(r)y'(r)]' + b(r)\varphi[y(r)] \leq 0$$

under the following hypotheses

$$\begin{aligned}
 (12) \quad & (A) \ a \in C^1[0, \infty) \text{ with } a(r) > 0 \text{ in } [0, \infty); \\
 & (B) \ b \in C[0, \infty) \text{ with } b(r) \geq 0 \text{ in } [0, \infty); \\
 & (C) \ \varphi \in C^1[0, \infty), \varphi(t) > 0 \text{ for } t \neq 0, \\
 & \text{and } \varphi'(t) > 0 \text{ in } (0, \infty).
 \end{aligned}$$

Evidently (4) is the special case of (11) for which

$$a(r) = r^{n-1}, \quad b(r) = r^{n-1} \lambda(r),$$

and hypotheses (12) are all fulfilled on account of (2).

THEOREM A. – Inequality (11) has no solution y which is positive throughout $[r_0, \infty)$ for any $r_0 > 0$ if a, b , and φ satisfy hypotheses (12) and the following conditions

$$\begin{aligned}
 (i) \quad & \int_c^\infty \frac{dr}{a(r)} = \infty; \\
 (ii) \quad & \int_c^\infty \left(\int_c^r \frac{ds}{a(s)} \right) b(r) dr = \infty; \text{ and} \\
 (iii) \quad & \int_c^\infty \frac{dt}{\varphi(t)} < \infty
 \end{aligned}$$

for some $c > 0$.

In the event that condition (i) fails, we need the additional hypothesis that the function φ in (11) satisfies

$$(13) \quad \varphi(st) \geq \varphi_0(s) \varphi_0(t)$$

for all $s > 0, t > 0$, where φ_0 is a function satisfying assumption (12C).

THEOREM B. – Inequality (11) has no solution y which is positive throughout $[r_0, \infty)$ for any $r_0 > 0$ if a, b , and φ satisfy (12), (13), and the following conditions

$$(i) \quad \int_c^\infty \frac{dr}{a(r)} < \infty;$$

$$(ii) \int_c^\infty \varphi_0 \left(\int_r^\infty \frac{ds}{a(s)} \right) b(r) dr = \infty; \text{ and}$$

$$(iii) \int_c^\infty \frac{dt}{\varphi_0(t)} < \infty$$

for some $c > 0$.

Proofs of Theorems A and B are given in the appendix.

THEOREM 1. — The differential operator L given by (1) is h -osc in an exterior domain G of E^2 if assumptions (2) hold and

$$(i) \int_r^\infty r \log r \lambda(r) dr = \infty;$$

$$(ii) \int \frac{du}{\varphi(u)} < \infty.$$

PROOF. — Suppose to the contrary that v is a solution of (1) such that $[v(x), h]$ has a fixed sign in G_r for all $r > R$. If $[v(x), h] > 0$ in G_r for $r > R$, then $[v(x), h] > 0$ on Ω_r for $r > r_0$, where $r_0 = \max(a, R)$. Hence $f(r) > 0$ for $r > r_0$ (see (3)) and $f(r)$ satisfies (4) by the Lemma. This contradicts Theorem A in the case

$$a(r) = r, \quad b(r) = r\lambda(r)$$

according to hypotheses (i) and (ii). If $[v(x), h] < 0$ in G_r for all $r > R$, then $|f(r)|$ satisfies (4), and a similar contradiction is obtained.

THEOREM 2. — The differential operator L is h -osc in an exterior domain G of E^n ($n \geq 3$) if (2) and (13) hold, and

$$(i) \int \varphi_0 \left(\frac{r^{3-n}}{n-2} \right) r^{n-1} \lambda(r) dr = \infty;$$

$$(ii) \int \frac{du}{\varphi_0(u)} < \infty.$$

This follows from Theorem B in the case

$$a(r) = r^{n-1}, \quad b(r) = r^{n-1} \lambda(r), \quad n \geq 3$$

similarly to Theorem 1.

Theorems 1 and 2 exclude the linear case $\varphi(u) = u$ because of the hypothesis (ii) of either theorem. In order to obtain similar (but slightly weaker) results applicable to both linear and nonlinear operators L , we employ the following replacement for condition (13): For arbitrary $\delta > 0$ there exists a corresponding $K(\delta) > 0$ and a number $\gamma \geq 1$ such that

$$(14) \quad \frac{u^\gamma}{\varphi(u)} \leq K(\delta)$$

for all $u \geq \delta$. Condition (14) holds, for example, when φ has the form

$$\varphi(u) = u^\gamma [\log(|u| + 1)]^\beta, \quad \gamma \geq 1, \beta \geq 0$$

including the linear case ($\gamma = 1, \beta = 0$) and the superlinear case ($\gamma > 1, \beta = 0$).

The following analogues of Theorems A and B are valid under hypothesis (14). The proofs following [3] are outlined in the appendix.

THEOREM C. — Inequality (11) has no solution y which eventually becomes positive as $r \rightarrow \infty$ if a, b , and φ satisfy (12), (14), and the following:

$$(i) \quad \int_c^\infty \frac{dr}{a(r)} = \infty; \text{ and}$$

$$(ii) \quad \int_c^\infty \left(\int_c^r \frac{ds}{a(s)} \right)^\varrho b(r) dr = \infty$$

for some $\varrho < 1$ and some $c \geq 0$.

THEOREM D. — Inequality (11) has no solution y which eventually becomes positive as $r \rightarrow \infty$ if a, b , and φ satisfy (12), (14), and the following:

$$(i) \quad \int_c^\infty \frac{dr}{a(r)} < \infty;$$

$$(ii) \quad \int_c^\infty \left(\int_r^\infty \frac{ds}{a(s)} \right)^{\mu+\gamma} b(r) dr = \infty$$

for some numbers $\mu > 0$ and $c \geq 0$.

THEOREM 3. — The differential operator L is h -osc in an exterior domain G of E^n if (2) and (14) hold, and

$$(15) \quad \int^\infty v(r) \lambda(r) dr = \infty,$$

where

$$\begin{aligned} r(r) &= r(\log r)^{\varrho} & \text{if } n = 2 \\ &= r^{\sigma(n)} & \text{if } n \geq 3, \\ \sigma(n) &= n - 1 - (\gamma + \mu)(n - 2) \end{aligned}$$

for some numbers $\varrho < 1, \mu > 0$, respectively.

PROOF. — For $n = 2$ we apply Theorem C in the case that $a(r) = r$ and $b(r) = r\lambda(r)$ in (11). Hypothesis (ii) of Theorem C is equivalent to (15) in this case. The proof is then completed in the same way as that of Theorem 1.

For $n \geq 3$ we apply Theorem D in the case that $a(r) = r^{n-1}$, $b(r) = r^{n-1}\lambda(r)$ in (11), and again use the same argument.

We remark in conclusion that it is a very simple matter to write analogues of Theorems 1-3 when (1) is replaced by

$$Lv = \Delta v + B(x, v, \nabla v)v = 0, \quad x \in G$$

where B is an $m \times m$ real symmetric matrix function with entries of class $C(G \times E^m \times E^m)$ such that

$$[B(x, v, \nabla v)v(x), h] \geq \lambda_h(|x|)\varphi([v(x), h])$$

whenever $[v(x), h] > 0$, $|h| = 1$, under assumptions similar to (2). There are other obvious generalizations, but it is not clear what to do when the operator Δ is replaced by a more general elliptic operator, or when the exterior domain G is replaced by a more general unbounded domain. In fact, if G is contained in a cone in E^2 with vertex angle $\alpha < 2\pi$, easy counterexamples similar to those in [3] show that the criterion (15), for instance, is not strong enough for all solutions of (1) to be h -osc in G .

Appendix.

PROOF OF THEOREM A. — Suppose to the contrary that there exists a solution y of (11) with $y(r) > 0$ throughout $[r_0, \infty)$ for some r_0 . We reduce (11) to the standard form

$$h''(s) + H(s)\varphi[h(s)] \leq 0, \quad s \geq 0$$

by the transformations

$$\begin{aligned} \alpha(r) &= \int_{r_0}^r \frac{ds}{a(s)}, & h(s) &= y(\beta(s)), \\ H(s) &= a(\beta(s))b(\beta(s)), \end{aligned}$$

where β denotes the uniquely determined inverse of the strictly increasing function α . Since $h''(s) \leq 0$ and $h(s) > 0$ for $s \geq 0$, it follows routinely that $h'(s) > 0$ for sufficiently large s , say for $s > s_1$. Define

$$(16) \quad g(s) = \frac{sh'(s)}{\varphi[h(s)]}, \quad s \geq s_1.$$

Then

$$g'(s) \leq -sH(s) + \frac{h'(s)}{\varphi[h(s)]}$$

and integration yields

$$(17) \quad g(s) - g(s_1) \leq - \int_{s_1}^s tH(t)dt + \int_{h(s_1)}^{h(s)} \frac{dh}{\varphi(h)}.$$

The transformation $t = \alpha(r)$, $r = \beta(t)$ reduces the first integral on the right side to

$$(18) \quad \int_{s_1}^s tH(t)dt = \int_{\beta(s_1)}^{\beta(s)} \alpha(r)b(r)dr,$$

which diverges as $s \rightarrow \infty$ by hypothesis (ii). Hence $g(s) < 0$ for sufficiently large s by hypothesis (iii), and consequently $h'(s)$ eventually becomes negative. This contradiction completes the proof of Theorem A.

PROOF OF THEOREM B. - The inequality (11) is reduced to the form

$$h''(s) + H(s)\varphi_0[h(s)] \leq 0, \quad s \geq 0$$

by the transformations

$$\alpha(r) = \left(\int_r^\infty \frac{ds}{a(s)} \right)^{-1}, \quad h(s) = sy(\beta(s)),$$

$$H(s) = \frac{1}{s^3} \varphi_0 \left(\frac{1}{s} \right) a(\beta(s)) b(\beta(s)),$$

where β denotes the inverse function of α . The proof is completed in the same way as that of Theorem A when (18) is replaced by

$$\int_{s_1}^s tH(t)dt = \int_{\beta(s_1)}^{\beta(s)} \varphi_0 \left(\frac{1}{\alpha(r)} \right) b(r)dr,$$

which diverges by hypothesis (ii) as $s \rightarrow \infty$.

PROOF OF THEOREM C. — The proof is virtually the same as that of Theorem A, with (16), (17), and (18) replaced by

$$(16') \quad g(s) = \frac{s^\varrho h'(s)}{\varphi[h(s)]},$$

$$(17') \quad g(s) - g(s_1) \leq - \int_{s_1}^s t^\varrho H(t) dt + K \int_{s_1}^s t^{\varrho-2} dt,$$

$$(18') \quad \int_{s_1}^s t^\varrho H(t) dt = \int_{\beta(s_1)}^{\beta(s)} \alpha^\varrho(r) b(r) dr,$$

respectively, where K is a constant (depending on γ and ϱ). The contradiction $h'(s) < 0$ for sufficiently large s is then obtained as in the proof of Theorem A.

PROOF OF THEOREM D. — As before, we suppose to the contrary that y is a solution of (11) with $y(r) > 0$ throughout $[r_0, \infty)$ for some number r_0 . With α, β , and h defined the same way as in Theorem B, it follows that there exist numbers δ and s_1 such that $y(\beta(s)) = s^{-1}h(s) > \delta$ for all $s \geq s_1$. Assumption (14) then implies that

$$\varphi\left(\frac{h(s)}{s}\right) \geq K \left(\frac{h(s)}{s}\right)^\gamma, \quad s \geq s_1$$

for some constants $K > 0$ and $\gamma \geq 1$. Since y satisfies (11), an easy calculation shows that $h(s) = sy(\beta(s))$ satisfies the differential inequality

$$h''(s) + H(s)[h(s)]^\gamma \leq 0 \quad s \geq s_1$$

where

$$H(s) = Ks^{-3-\gamma}a(\beta(s))b(\beta(s)).$$

The proof is then completed as before with (16'), (17'), and (18') replaced respectively by

$$(16'') \quad g(s) = \frac{s^{1-\mu}h'(s)}{[h(s)]^\gamma},$$

$$(17'') \quad g(s) - g(s_1) \leq - \int_{s_1}^s t^{1-\mu} H(t) dt + K \int_{s_1}^s t^{-1-\mu} dt,$$

$$(18'') \quad \int_{s_1}^s t^{1-\mu} H(t) dt = \int_{\beta(s_1)}^{\beta(s)} \left(\frac{1}{\alpha(r)}\right)^{\mu+\gamma} b(r) dr.$$

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