Non-Linear Operators on Sets of Measures (*).

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Summary. – *If $M[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ is the collection of $U(\boldsymbol{C}, \boldsymbol{C})$-valued (non-linear) set functions defined on the Borel subsets $\mathfrak{B}$ of the compact Hausdorff space S, one may define operators on $M[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ which are «of the Hammerstein type». We initiate a study of a concept analogous to the second dual of a space of continuous functions by inquiring as to what representation theorems one may obtain for these operators. A «Lebesgue type» decomposition theorem for elements of $M[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ is obtained. A «density» theorem is also obtained for the space $M[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$.*

1. – Introduction.

Bounded linear transformations on spaces of continuous functions and their universal properties have been of special interest since A. Grothendieck's celebrated paper [6]. The early papers dealt with the case where the function spaces were spaces of real valued functions and where the functions themselves, were defined on a compact or locally compact Hausdorff space. Much was written on the representation (via integrals) of transformations (see [5]) on such spaces.

Later studies considered the underlying function spaces with functions having their values in some Banach space E. For example the representation of linear operators on the *space* $K(S, E)$ (with the usual supremum norm) *of E-valued functions with compact support* and defined on the locally compact Hausdorff space S, may be found in the compendium [4].

More recently has been the investigation of linear operators defined on the dual of such function spaces whose elements are E-valued (for example, see [1], [7], [9] and [10]). For example in [1], linear operators belonging to the second dual of $K(S, E)$ are represented on certain sets of measures in the dual of $K(S, E)$. It is shown that such an operator is in a certain sense *approximable by an integral* when computed over this subset of the dual.

Another direction of research has been to relax the condition of linearity. In [2], [11] and [12], the authors study operators Φ which are «*additive*». Essentially this replaces the condition of linearity with the condition that

$$\Phi(f_1 + f_2) = \Phi(f_1) + \Phi(f_2)$$

where f_1 and f_2 are functions in our function space which have disjoint supports.

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Let $L_E^p = L_E^p(\Omega, \Sigma, \mu)$ be the *Banach space of* (equivalence classes of) *Bochner μ-integrable E-valued functions defined on the measure space* (Ω, Σ, μ). In [12] E was, in addition, assumed to be separable. There a characterization is given of additive functions Φ from L_E^p into an arbitrary Banach space F which admit an integral representation of the form

$$\Phi(\varphi) = \int_\Omega \theta(\varphi(\xi), \xi(d\mu(\xi)$$

where θ is required to satisfy certain conditions related to those occurring in the theory of non-linear integral equations (see [8]). In the sequel these functions θ will be referred to as members of the *uniform Caratheodory class* $U - \mathrm{Car}(E, F)$ *relative to E on* $(\Omega \to F)$.

Let $C(S, E)$ be the *space of continuous E-valued functions* (with usual uniform supremum norm) defined on the compact Hausdorff space S. In [2] the « additivity » of the operator Φ from $C(S, E)$ into the Banach space F is replaced by the stronger *Hammerstein property* (¹). This is the algebraic property that

$$T(f + f_1 + f_2) = T(f + f_1) + T(f + f_2) - T(f)$$

where $f, f_1, f_2 \in C(S, E)$ and where f_1 and f_2 have disjoint supports. These non-linear transformations are represented as integrals with respect to additive «non-linear» set functions (which take their values in a linear space of operators from one Banach space into another which are uniformly continuous on bounded sets).

In this work we initiate a study of a concept analogous to the second dual of a space of continuous functions. More specifically if $M[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ is the *collection of $U(\boldsymbol{C}, \boldsymbol{C})$-valued set functions defined on the Borel subsets $\mathcal{B}$ of S and representing Hammerstein operators on* $C(S, E)$, one may define operators on $M[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ which would be « of the Hammerstein type ». We inquire as to what representation theorems one may obtain for these operators.

The elements of $M[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ are technically not measures. However, the subspace $M[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]_\alpha$ is a space of measures. Also, we may obtain a « Lebesgue type » decomposition theorem for elements of $M[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ (see Proposition 2): As will be discussed later the usual vector-valued decomposition theorem as in [4] is not applicable here. Our result yields a « density » type theorem (see Theorem 3) for elements of $M[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$. In essence, it shows that such an element can be approximated by an element of $M[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ which is « absolutely continuous ». with respect to some finite sum of elements from a maximal set $\mathcal{M}$ of mutually singular bounded non-negative Borel measures on $\mathcal{B}$.

Let $F(\mathbb{R}^+)$ be the *set of finite real-valued functions defined on the positive reals* $\mathbb{R}^+$. If $r \in \mathcal{M}$ and if $M_r[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ is that *subset of* $M[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ *whose elements are*

(¹) The class of Hammerstein operators satisfy this condition. It has sometimes been referred to as *strong additivity* in [3].

« *dominated* » (as defined below) *by* r, then certain additive operators Φ from $M_r[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ into $F(\mathbb{R}^+)$ are studied. For $\mu \in M[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$, we designate by μ_α the restriction of $\mu(B)$ for every $B \in \mathcal{B}$ to the ball $B(0, \alpha)$ of radius $\alpha > 0$ and center at 0. We consider certain « additive » Φ whose values at $\mu \in M_r[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ are determined by the restrictions μ_α, $\alpha > 0$, that is,

$$\Phi(\mu)(\alpha) = \Phi_\alpha(\mu_\alpha)$$

is an operator defined on this collection of restrictions.

The Φ under consideration will satisfy certain continuity conditions. Our main restult (Theorem 7) yields the interpretation that Φ_α may be considered as an operator on $L^1_{U_\alpha} = L^1_{U_\alpha(\boldsymbol{C}.\boldsymbol{C})}(S, \mathcal{B}, r)$. In fact,

$$\Phi_\alpha(\varphi) = \int \theta_{\alpha,r} \circ \varphi \, dr$$

for $\varphi \in L^1_{U_\alpha}$ where $\theta_{\alpha,r} \in U - \mathrm{Car}^1[U_\alpha, \mathbb{R}]$. A corollary to this yields a representation of the above operators Φ (see Theorem 8) in terms of slurs. In particular for $\mu \in M_{r_\beta}[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$,

$$\Phi(\mu)(\alpha) = \int \Psi_{\alpha,\beta} \, dr_\beta$$

where r_β is any element in $\mathcal{M}$ and $\Psi_{\alpha,\beta}$ is the slur $\{\psi_{1/n,\alpha}, \mathcal{F}_\beta\}$.

Using our « density » Theorem 3, we extend this representation theorem to yield representations of operators on $M_f[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ which is a space larger than $M_r[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ (see Theorem 10). In [2], $M_r[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ played an important role.

2. – Preliminary results.

A functional Φ from the space $C(S, E)$ into the scalar field $\boldsymbol{C}$ is said to have the *Hammerstein property* if

$$\Phi(f + f_1 + f_2) = \Phi(f + f_1) + \Phi(f + f_2) - \Phi(f)$$

for all f, f_1, and f_2 in $C(S, E)$ such that the supports of f_1 and f_2 are disjoint.

For the Banach spaces E and F let us denote by $U(E, F)$ the *linear space of all maps* ψ *from* E *into* F *with the following properties*:

(i) $\psi(0) = 0$.

(ii) If $B(0, \alpha)$ denotes the ball of radius α and center at 0, if ψ_α *denotes the restriction of* ψ *to* $B(0, \alpha)$ and if

$$D_\delta \psi_\alpha = \sup\{\|\psi(e) - \psi(e')\| : e, e' \in B(0, \alpha), \ \|e - e'\| < \delta\}$$

then $D_\delta \psi_\alpha$ converges to zero as δ converges to zero.

(iii) $\|\psi_\alpha\| = \sup\{\|\psi(e)\| : e \in B(0, \alpha)\} < \infty$, $\alpha > 0$.

Thus $U(E, F)$ is the set of maps from E into F that are bounded and are uniformly continuous on bounded subsets of E with the additional assumption that $\psi(0)=0$.

Let $U_\alpha(E, F)=\{\psi_\alpha: \psi\in U(E, F)\}$. The spaces $U_\alpha(E, F)$ are linear spaces and are considered to be normed by the norm $\|\ \|$ which takes each $\psi_\alpha\in U_\alpha(E, F)$ to $\|\psi_\alpha\|$ as defined above.

In the way of notation, we agree to always designate the restriction of an operator or function to the ball of radius α and center 0 (*α-ball*) by affixing the index α to the operator or function. When we are considering a space of set functions, brackets [,] will be used to enclose the domain and the superspace containing the range, whereas when point functions are under consideration, parentheses (,) will be used for these. Lower case Greek letters such as μ and ν will be used for vector-valued set functions and lower case Roman letters such as r and w will be used for scalar-valued set functions.

We denote by $HP(C(S, E), \mathbf{C})$ *the set of functionals in* $U(C(S, E), \mathbf{C})$ *with the Hammerstein property.*

Let μ be an additive set function from the σ-algebra $\mathfrak{B}$ of Borel subsets of S into $U(E, \mathbf{C})$. For every real number $\alpha>0$, we denote by μ_α the set function from $\mathfrak{B}$ to $U_\alpha(E, \mathbf{C})$ defined by restricting $\mu(B)$ for every $B\in\mathfrak{B}$ to the ball $B(0, \alpha)$.

The *semi-variation* of μ on S (see [2]) is defined to be

$$sv[\mu_\alpha, S]=\sup\{\|\Sigma\mu(B_j)(e_j)\|: e_j\in B(0,\alpha);\ B_j\in\mathfrak{B}'-a \text{ partition of } \mathfrak{B}\},$$

and the *variation* of μ on S is defined as

$$v[\mu_\alpha, S]=\sup\{\Sigma\|\mu_\alpha(B_j)\|: B_j\in\mathfrak{B}' \text{ a partition of } \mathfrak{B}\}.$$

Also for $\delta>0$, we define analogously the *δ-semi-variation* and *δ-variation*, respectively as,

$$sv_\delta[\mu_\alpha, S]=\sup\{\|\Sigma(\mu(B_j)e_j-\mu(B_j)e_j')\|: e_j, e_j'\in B(0,\alpha);\ \|e_j-e_j'\|\leqslant\delta;\ B_1\in\mathfrak{B}' \text{ a partition of } \mathfrak{B}\}$$

and

$$v_\delta[\mu_\alpha, S]=\sup\{\Sigma D_\delta\mu_\alpha(B_j): B_j\in\mathfrak{B}' \text{ a partition of } \mathfrak{B}\}.$$

Let us remark that these quantities may be defined on any subset $S'\subset S$ with the usual topological considerations. Later on we will make use of this.

We have

$$sv[\mu_\alpha, S]\leqslant v[\mu_\alpha, S]\leqslant 4sv[\mu_\alpha, S]$$

and

$$sv_\delta[\mu_\alpha, S]\leqslant v_\delta[\mu_\alpha, S]\leqslant 4sv_\delta[\mu_\alpha, S].$$

From [2] (see Theorem 1) the following theorem concerning the above will be needed.

THEOREM 1. – *There is an algebraic isomorphism between the space* $HP(C(S, E), \mathbf{C})$ *and the space of all additive non-linear set functions* μ *from* $\mathcal{B}$ *into* $U(E, \mathbf{C})$ *with the following properties*:

(1) $sv[\mu_\alpha, S] < \infty$ *and* $sv_\delta[\mu_\alpha, S]$ *converges to zero as* δ *converges to zero.*

(2) *Eeach* μ_α *from* $\mathcal{B}$ *into* $U_\alpha(E, \mathbf{C})$ (*and hence* $v(\mu_\alpha)$) *is regular* (*and therefore countably additive*) *for* $\alpha > 0$.

This correspondence is given by

$$\Phi(f) = \int f\, d\mu_\Phi$$

for $f \in C(S, E)$, $\Phi \in HP(C(S, E), \mathbf{C})$ *and* μ_Φ *its correspondent.*

For any algebra $\mathcal{A}$ of subsets of S, we define an *$\mathcal{A}$-partition* of S to be a finite system of pairwise disjoint sets from $\mathcal{A}$ whose union is S. Thus if μ is an additive set function from $\mathcal{A}$ into $U(E, F)$ then we may define an *$\mathcal{A}$-simple* function φ on S with values in E to be a function of the form

$$\varphi = \Sigma\{e_A \chi_A : A \in \mathcal{A}' - \text{an } \mathcal{A} \text{ partition of } S;\ e_A \in E\}$$

where χ_A represents the *characteristic function* of A.

Now the integral mentioned in the theorem is defined (in [2]) in the obvious way on simple functions. Then by a limit process, it is extended to $C(S, E)$ (in fact it is extended to the space $\mathcal{M}(\mathcal{B}, E)$ of all totally $\mathcal{B}$-measurable (¹) E-valued functions on S). The integral is linear in μ. With respect to f it has the following property. For all $f, f_1, f_2 \in \mathcal{M}(\mathcal{B}, E)$ such that the supports of f_1 and f_2 are disjoint one has

$$\int (f + f_1 + f_2)\, d\mu = \int (f + f_1)\, d\mu + \int (f + f_2)\, d\mu - \int f\, d\mu\,.$$

All integrals are over the whole space S.

We denote by $M[\mathcal{B}, U(\mathbf{C}, \mathbf{C})]$ *all those additive set functions from* $\mathcal{B}$ *into* $U(\mathbf{C}, \mathbf{C})$ *satisfying* (1) *and* (2) in the above Theorem 1. Thus $M[\mathcal{B}, U(\mathbf{C}, \mathbf{C})]$ represents $HP[C(S, \mathbf{C}), \mathbf{C}]$.

We now present a Lebesgue decomposition theorem for $U(\mathbf{C}, \mathbf{C})$ valued set functions. Since these are not technically measures, the usual vector valued Lebesgue decomposition is not valid.

(¹) These are the uniform limits of $\mathcal{B}$-simple functions S with values in E, where $\mathcal{M}(\mathcal{B}, E)$ is normed with the usual uniform norm.

PROPOSITION 2. – *Let $\mu \in M[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ and let r be a non-negative scalar measure on $\mathfrak{B}$. Then μ may be decomposed uniquely as a sum*

$$\mu = \mu_{ra} + \mu_{rs}$$

where μ_{ra} and μ_{rs} are elements in $M[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$, where μ_{ra} is absolutely continuous relative to r and where for every fixed $c \in \boldsymbol{C}$, the scalar-valued functon $\mu_{rs}(\ )(c)$ on $\mathfrak{B}$ is r-singular. Moreover for S' any subspace of S and for all $\alpha > 0$

(1) $v[(\mu_{ra})_\alpha, S'] \leqslant v[\mu_\alpha, S']$,
$v[(\mu_{rs})_\alpha, S'] \leqslant v[\mu_\alpha, S']$;

(2) $v_\delta[(\mu_{ra}), S'] \leqslant v_\delta[\mu_\alpha, S']$,
$v_\delta[(\mu_{rs}), S'] \leqslant v_\delta[\mu_\alpha, S']$.

PROOF. – If $c \in \boldsymbol{C}$, then $\mu(\ )(c)$ is a $\boldsymbol{C}$-valued finitely additive measure on $\mathfrak{B}$. The usual Lebesgue decomposition theorem yields a unique decomposition of $\mu(\ )(c)$ as

$$\mu(\ )(c) = \mu_{ra}(\ )(c) + \mu_{rs}(\ )(c)$$

where $\mu_{ra}(\ )(c)$ is absolutely continuous with respect to r and $\mu_{rs}(\ )(c)$ is r-singular. Since $0 = \mu(\ )(0)$, uniqueness implies that $\mu_{ra}(\ )(0) = 0 = \mu_{rs}(\ )(0)$.

For $\alpha > 0$, let $c \in \boldsymbol{C}$ and $B \in \mathfrak{B}$. Then

$$|\mu_{ra}(B)(c)| \leqslant v[\mu_\alpha, B] < \infty .$$

Consequently, if for $j = 1, \ldots, n$, $c_j \in C$, $|c_j| \leqslant \alpha$ and if B_j are pairwise disjoint subsets of $\mathfrak{B}$ (or of any collection $\mathfrak{B} \cap S'$ of sets of $\mathfrak{B}$ restricted to any subspace $S' \subset S$) we have

$$\Sigma |\mu_{ra}(B_j)(c_j)| \leqslant \Sigma v[\mu_\alpha, B_j] \leqslant v[\mu_\alpha, S] < \infty .$$

Similarly

$$\Sigma |\mu_{rs}(B_j)(c_j)| \leqslant v[\mu_\alpha, S] < \infty .$$

Suppose c and c' are in $\boldsymbol{C}$, $|c| \leqslant \alpha$, $|c'| \leqslant \alpha$ and $|c - c'| \leqslant \delta$ for $\delta > 0$. Then

$$\mu(B_j)(c) - \mu(B_1)(c') = \mu_{ra}(B_j)(c) - \mu_{ra}(B_j)(c') + \mu_{rs}(B_j)(c) - \mu_{rs}(B_j)(c')$$

implies that

$$|\mu_{ra}(B_1)(c) - \mu_{ra}(B_j)(c')| \leqslant v_\delta[\mu_\alpha, B_j] .$$

Consequently for c_j and c_j' in $\boldsymbol{C}$ with $|c_j|\leqslant\alpha$, $|c_j'|\leqslant\alpha$, $|c_j-c_j'|\leqslant\delta$ and for the pairwise disjoint subsets B_j

$$\Sigma|\mu_{ra}(B_j)(c_j)-\mu_{ra}(B_j)(c_j')|\leqslant\Sigma v_\delta[\mu_\alpha, B_j]\leqslant v_\delta[\mu_\alpha, S]$$

and therefore taking supremum over these collections we have

$$\sup\Sigma|\mu_{ra}(B_j)(c_j)-\mu_{ra}(B_j)(c_j')|\leqslant v_\delta[\mu_\alpha, S]\,.$$

Thus the left side of the last inequality converges to zero as δ converges to zero. A similar inequality will hold for μ_{ra} replaced by μ_{rs}.

Thus permitting c to vary in $\boldsymbol{C}$, we have defined two finitely additive set functions μ_{ra} and μ_{rs} on $\mathcal{B}$ with values in $U(\boldsymbol{C},\boldsymbol{C})$. Since $v[\mu_\alpha, S]$ is regular, it follows that both $(\mu_{ra})_\alpha$ and $(\mu_{rs})_\alpha$ are both regular. Consequently the above discussion shows that (1) and (2) of Theorem 1 are satisfied, that is μ_{ra} and μ_{rs} are elements of $M[\mathcal{B}, U(\boldsymbol{C},\boldsymbol{C})]$.

The conditions (1) and (2) of the present proposition are given by the above computations. The uniqueness of μ_{ra} and μ_{rs} follows from the uniqueness of $\mu_{ra}(\ )(c)$ and $\mu_{rs}(\ )(c)$ for every $c\in\boldsymbol{C}$. The absolute continuity of μ_{ra} follows from that of $\mu_{ra}(\ )(c)$ with respect to r and the r-singularity of $\mu_{rs}(\ )(c)$ follows from the r-singularity of $\mu_{rs}(\ )(c)$ for each $c\in\boldsymbol{C}$. This completes the proof of the proposition.

In the next theorem we will let $\mathcal{M}$ denote a maximal set of non-negative (finite) Borel measures on S which are mutually singular (Zorn's Lemma). Under the finiteness assumption this is equivalent to the measures being concentrated on disjoint sets (see [4]). We will also assume that $\mathcal{M}$ can be well-ordered so that each proper initial segment of $\mathcal{M}$ is countable.

The following « density » theorem is similar to Theorem 1 of [1]. Our proof for the following also follows closely to that of [1].

THEOREM 3. - *Let* $\mu\in M[\mathcal{B}, U(\boldsymbol{C},\boldsymbol{C})]$, *let* $c_0\in\boldsymbol{C}$ *and let* $\varepsilon>0$. *There is a finite subset* I *of* $\mathcal{M}$ *and a* $\mu_\varepsilon\in M[\mathcal{B}, U(\boldsymbol{C},\boldsymbol{C})]$ *such that*

(1) $\mu_\varepsilon(\ )(c)$ *is absolutely continuous with respect to* $\Sigma\{r\colon r\in I\}$ *for every* $c\in\boldsymbol{C}$.

(2) $v[(\mu-\mu_\varepsilon)(\ )(c_0), S]<\varepsilon$.

PROOF. - For $\mu\in M[\mathcal{B}, U(\boldsymbol{C},\boldsymbol{C})]$, let $I=\{r_1,\dots,r_n\}$ be a finite subset of $\mathcal{M}$ and let $r=\Sigma\{r_i\colon r_i\in I\}$. Proposition 2 implies that μ may be written as

$$\mu=\mu_{ra}+\mu_{rs}$$

for μ_{ra} and μ_{rs} in $M[\mathcal{B}, U(\boldsymbol{C},\boldsymbol{C})]$, μ_s absolutely continuous with respect to r and $\mu_{rs}(\ )(c)$ singular with respect to r for every $c\in\boldsymbol{C}$.

For $c \in \boldsymbol{C}$, let

$$\mu^c(B) = \mu(B)(c) \; ; \quad \mu^c_{ra}(B) = \mu_{ra}(B)(c) \; ; \quad \mu^c_{rs}(B) = \mu_{rs}(B)(c)$$

for all $B \in \mathcal{B}$. Then μ^c, μ^c_{ra} and μ^c_{rs} are countably additive scalar valued measures on $\mathcal{B}$. For any bounded real measure m on $\mathcal{B}$, let $m = m^+ - m^-$ be the Hahn decomposition of m. Then

$$\mu^c = \mu_{1c} + i\mu_{2c} \; ; \quad \mu^c_{ra} = \mu^c_{1ra} + i\mu^c_{2ra} \; ; \quad \mu^c_{rs} = \mu^c_{1rs} + i\mu^c_{2rs} \, .$$

As shown in [1] one may show easily that

$$(\mu^c_i)^+ = (\mu^c_{ira})^+ + (\mu^c_{irs})^+$$
$$(\mu^c_i)^- = (\mu^c_{ira})^- + (\mu^c_{irs})^-$$

for $i = 1, 2$.

Let

$$\mathcal{M}_1^+(\mu, c) = \left\{(\mu^c_{1ra})^+ : r = \Sigma\{r_i : r_i \in I\}; \; I \text{ finite subset of } \mathcal{M}\right\}.$$

Then again as in [1], it is shown that if $B \in \mathcal{B}$ and if

$$(\bar{\mu}^c_1)^+(B) = \sup\{(\mu^c_{1ra})^+(B) : (\mu^c_{1ra})^+ \in \mathcal{M}_1^+(\mu, c)\}$$

then $(\mu^c_1)^+ = (\bar{\mu}^c_1)^+$.

For the next statements we will designate by $(\mu^c_{1Na})^+$ the function which would normally be designated as $(\mu^c_{1ra})^+$ where $r = \Sigma\{r_i : r_i \in N\}$ and N is a finite subset of I.

Thus there is a finite subset N_c of I such that

$$(\mu^c_1)^+(S) - \frac{1}{n} < (\mu^c_{1N_ca})^+(S) \leqslant (\mu^c_1)^+(S) \, .$$

Thus

$$v\left[\left((\mu^c_1)^+ - (\mu^c_{1N_ca})^+\right)(S)\right] < 1/n \, .$$

Similarly there is a finite subset K_c of I such that

$$v\left[\left((\mu^c_1)^- - (\mu^c_{1K_ca})^-\right)(S)\right] < 1/n$$

and there is a finite subset of M_c of I such that

$$v[(\mu^c_1 - \mu^c_{1M_ca})(S)] < 1/n \, .$$

A similar computation holds for μ_2^c. Thus we obtain a finite subset I of $\mathcal{M}$ such that

$$v[(\mu^c - \mu_{Ia}^c)(\ ),\ S] < 1/n\ ,$$

that is

$$v[(\mu - \mu_{Ia})_c(\ ),\ S] < 1/n\ .$$

By construction $\mu_{Ia}(\ )(c)$ is mutually singular with $r = \Sigma\{r_i : r_i \in I\}$ for every $c \in \boldsymbol{C}$. This completes the proof of our theorem.

Suppose I is a finite subset of $\mathcal{M}$ and we let $r_I = \Sigma\{r_i \in I\}$. By $M_I[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ we denote that subset of $M[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ consisting of those $\mu \in M[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ such that

$$v[\mu_\alpha, B] \leqslant L_\alpha(I)\, r_I(B)$$

where $L_\alpha(I)$ is a constant depending on α and the finite subset I of $\mathcal{M}$. By $M_D[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ *we designate that subset of* $M[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ *consisting of elements* μ *for which there is a finite subset* I_0 *of* $\mathcal{M}$ *such that for all* $I \subset I_0$, I *finite,*

$$v[(\mu_{ra})_\alpha, B] \leqslant K_\alpha \cdot r(B)$$

where $r = r\text{-}$, $B \in \mathfrak{B}$, and $(\mu_{ra})_\alpha$ is the continuous part of μ_α is its Lebesgue decomposition relative to r_I. Thus $M[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ may be considered as a set of elements in $M[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ whose absolutely continuous parts are eventually not too large.

As a corollary to the theorem we now have

COROLLARY 4. – *For* $\mu \in M_D[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$, $c \in C$ *and* $\varepsilon > 0$ *there is a finite subset* N_c *of* $\mathcal{M}$ *and a* $\mu_{N_c} \in M_I[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ *such that*

$$v[(\mu - \mu_{N_c})(\ )(c),\ S] < \varepsilon\ .$$

PROOF. – From Theorem 3 there is a finite subset I of $\mathcal{M}$ and $\mu_\varepsilon \in M[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ such that

$$v[(\mu - \mu_\varepsilon)(\ )(c),\ S] < \varepsilon\ .$$

By the definition of $M_D[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ there is a finite subset I_0 of $\mathcal{M}$ such that for all $\alpha > 0$,

$$v[(\mu_{ra})_\alpha, B] \leqslant K_\alpha\, r(A)$$

where $r = r_I = \Sigma\{r_i : r_i \in I\}$ whenever I is a finite subset of $\mathcal{M}$, $I_0 \subset I$. Thus we may choose a finite subset N_c of $\mathcal{M}$ which satisfies both conditions simultaneously. Consequently $\mu_{N_c} \in M_I[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$. This completes the proof.

3. – Non-linear operators on set function spaces.

Suppose r is a fixed element of the maximal set $\mathcal{M}$ of finite measures. By $M_r[\mathcal{B}, U(\mathbf{C}, \mathbf{C})]$ *we designate those elements* μ *of* $M[\mathcal{B}, U(\mathbf{C}, \mathbf{C})]$ *with the property that for every* $\alpha > 0$,

$$v[\mu_\alpha, B] \leqslant L_\alpha(r)\, r(B)$$

where $B \in \mathcal{B}$ and $L_\alpha(\beta)$ denotes a constant depending on α and r. Thus $M_r[\mathcal{B}, U(\mathbf{C}, \mathbf{C})]$ consists of those elements μ of $M[\mathcal{B}, U(\mathbf{C}, \mathbf{C})]$ which are «dominated» by r.

We wish to obtain a representation theorem for operators on $M[\mathcal{B}, U(\mathbf{C}, \mathbf{C})]$. For the ensuing discussion the $r \in \mathcal{M}$ is fixed. First we need to make use of the following lemma (see [2], Lemma 10) for its proof.

Lemma 5. – *Let* $(\Omega, \mathcal{B}, w)$ *be a measure space with a bounded non-negative measure* w. *Then there exists an algebraic isomorphism between the functions* u *from* Ω *into* $U(\mathbf{C}, \mathbf{C})$ *such that* $u(\;)_\alpha \in L^\infty_{U_\alpha}(\Omega, \mathcal{B}, w)$ *where* $U_\alpha = U_\alpha(\mathbf{C}, \mathbf{C})$ *and the additive set functions* μ *from* $\mathcal{B}$ *into* $U(\mathbf{C}, \mathbf{C})$ *satisfying*

(1) μ_α *is countably additive,* $v[\mu_\alpha, \Omega] < \infty$, *and* $v_\delta[\mu_\alpha, \Omega]$ *converges to zero as* δ *converges to zero for every* $\alpha > 0$.

(2) $v[\mu_\alpha, B] \leqslant L_\alpha w(B)$ *for* $B \in \mathcal{B}$ *and where* L_α *is a constant depending on* α. *The correspondence is given by*

$$\mu(B)_\alpha = \int_B u(t)_\alpha\, dw(t) \qquad B \in \mathcal{B},\ \alpha > 0,\ t \in \Omega.$$

Also for corresponding μ *and* u *we have*

$$v[\mu_\alpha, B] = \int_B \|u(t)_\alpha\|\, dw(t) \qquad B \in \mathcal{B},\ \alpha > 0$$

$$v_\delta[\mu_\alpha, B] = \int_B D_\delta u(t)_\alpha\, dw(t) \qquad B \in \mathcal{B},\ \alpha > 0,\ \delta > 0$$

and

$$\int_\Omega g\, d\mu = \int_\Omega u(t)\, g(t)\, dw(t) \qquad t \in \Omega$$

for all g *which are totally measurable.*

Actually a more general version of this lemma is given in [2]. However the present form of it suffices for our purposes.

Suppose now that $\mu \in M_r[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$. Then for every $\alpha > 0$, $\mu(\ )_\alpha$ satisfies (1) and (2) of Lemma 5 where $\Omega = S$ and $w = r$. Thus to each $\mu(\ )$ there corresponds a function u_μ mapping S into $U(\boldsymbol{C}, \boldsymbol{C})$ such that the mapping $u_\mu(\ )_\alpha$ from S into $U_\alpha = U_\alpha(\boldsymbol{C}, \boldsymbol{C})$ is an element of $L^\infty_{U_\alpha}(S, \mathfrak{B}, r)$ and for which

$$\mu(B)_\alpha = \int_B u_\mu(t)_\alpha \, dr(t) .$$

This correspondence is in fact an isometry for each $\alpha > 0$ where one considers $L^\infty_{U_\alpha}(S, \mathfrak{B}, r)$ as a subspace of $L^1_{U_\alpha}(S, \mathfrak{B}, r)$ (which is true since r is finite). For we have by Lemma 5

$$\|\mu(\ )_\alpha\| = v[\mu(\ )_\alpha, S] = \int_S \|\mu_\mu(t)_\alpha\| \, dr(t) .$$

This is the $L^1_{\mathbb{R}}$-norm of $u_\mu(\ )_\alpha$, where $\mathbb{R}$ is the reals.

Let $F(\mathbb{R}^+)$ *be the set of finite real valued functions defined on the positive reals* $\mathbb{R}^+$. We wish to consider operators Φ from $M_r(S)$ into $F(\mathbb{R}^+)$ which satisfy a natural additivity condition. To do this we need to consider an orthogonality relation on $M_r[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$.

For μ_1 and μ_2 in $M_r[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ we shall say that μ_1 *is orthogonal to* μ_2 if for every $\alpha > 0$, $(\mu_1)_\alpha$ is mutually singular with $(\mu_2)_\alpha$. We may interpret this in terms of the functions u_μ discussed above in the following manner. Let S_i be the support of $(\mu_i)_\alpha$ for $i = 1, 2$. If u_i is the function from the previous disussion corresponding to μ_i, $i = 1, 2$ then

$$\int_{B \cap S_1} \|\mu_2(\ )_\alpha\| \, dr = 0 = \int_{A \cap S_2} \|u_1(\ )_\alpha\| \, dr$$

for $B \in \mathfrak{B}$. Thus μ_1 is orthogonal to μ_2 if and only if the intersection of the supports of $u_2(\ )_\alpha$ and $u_1(\ )_\alpha$ is r-null for every $\alpha > 0$.

Thus we define an operator Φ from $M_r[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ into $F(\mathbb{R})^+$ to be *additive* if

$$\Phi(\mu_1 + \mu_2) = \Phi(\mu_1) + \Phi(\mu_2)$$

whenever μ_1 is orthogonal to μ_2, for μ_1 and μ_2 in $M_r[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$.

In the proof of Theorm 7, we shall use the characterization of orthogonality of μ_1 and μ_2 in terms of the correspondents u_1 and u_2. Specifically we shall assume that the operator Φ is of the form

$$\Phi(\mu)(\alpha) = \Phi_\alpha(\mu_\alpha)$$

whe reΦ_α is a function on the set $M_r[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]_\alpha$ of *restrictions* μ_α *of measures in* $M_r[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ for each $\alpha > 0$. But by Lemma 5, each μ corresponds to a u_μ (which also depends on r but complication of notation refrains us from inserting it) such that for each $\alpha > 0$, μ_α and $u_\mu(\ )_\alpha$ correspond. Thus we will consider the Φ_α as defined either on μ_α or on $u_\mu(\ )_\alpha$. Consequently Φ_α may be considered as a mapping from a subset of $L^1_{U_\alpha}(S, \mathcal{B}, r)$ into the reals.

With this understood, it is clear that

$$\Phi(\mu_1 + \mu_2) = \Phi(\mu_1) + \Phi(\mu_2)$$

is equivalent to

$$\Phi_\alpha[u_{\mu_1}(\ )_\alpha + u_{\mu_2}(\ )_\alpha] = \Phi_\alpha[u_{\mu_1}(\ )_\alpha] + \Phi_\alpha[u_{\mu_2}(\ )_\alpha]$$

for each $\alpha > 0$ assuming throughout that μ_1 is orthogonal to μ_2.

The operator Φ defined on the subset $M_r[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ of $U(\boldsymbol{C}, \boldsymbol{C})$ also gives rise to a real valued set function defined on $\mathcal{B}$. For any $\psi \in U(\boldsymbol{C}, \boldsymbol{C})$, $B \in \mathcal{B}$ and $\alpha > 0$, using the characteristic function χ_B of B, we may define the function $(\chi_B \psi)_\alpha = \chi_B \psi_\alpha$ from S into $U_\alpha = U_\alpha(\boldsymbol{C}, \boldsymbol{C})$ by

$$(\chi_A \psi_\alpha)(s) = \chi_A(s)\, \psi_\alpha \qquad s \in S.$$

Furthermore, $\chi_A \psi_\alpha \in L^\infty_{U_\alpha}(S, \mathcal{B}, r) \subset L^1_{U_\alpha}(S, \mathcal{B}, r)$. Thus by Lemma 5 for $\alpha > 0$ and $B \in \mathcal{B}$

$$\mu_{\psi,B}(B')_\alpha = \int_{B'} \chi_B(\ )\, \psi_\alpha\, dr \qquad B' \in \mathcal{B}$$

defines an element $\mu_{\psi,B}$ of $M_r[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$. Let us notice that the step functions in $L^1_{U_\alpha}(S, \mathcal{B}, r)$ are finite sums of functions of the type $\chi_A \psi_\alpha$.

Let us now define the real valued set function r_Φ for which conciseness of notation refrains us from writting the fact that it also depends on $\psi \in U(\boldsymbol{C}, \boldsymbol{C})$ and $\alpha > 0$. It is defined for $B \in \mathcal{B}$ by

$$r_\Phi(B) = \Phi[\mu_{\psi,B}](\alpha)\,.$$

It will be of interest when this set function r_Φ has locally almost compact average range. We define this for the more general situation that (Ω, Σ, w) is a measure space and that v is an additive set function from Σ into the Banach space E. We define the *average range* of v on the measurable set $B \in \mathcal{B}$, $0 < w(B) < \infty$, to be

$$A(v, B) = \left\{\frac{v(B')}{w(B')} \cdot B' \in \Sigma,\ B' \subset B,\ 0 < \mu(B')\right\}.$$

Then v is said to have *locally almost compact average range* if whenever $B \in \Sigma$, $0 < w(B) < \infty$, and $\varepsilon > 0$ there exists $B' \in \Sigma$, $B' \subset B$ such that $w(B \setminus B') < \varepsilon$ and $A(v, B')$ is a precompact subset of E (see [13]).

Some results from [11] and [12] will also be necessary. Suppose the measure space (Ω, Σ, μ) is assumed to be also finite and complete and that E is also separable. Let $B(\Omega, E)$ be the vector space of E-valued Bochner measurable functions on Ω. A function Γ from $B(\Omega, E)$ into another Banach space F is said to be *additive* if

$$\Gamma(\varphi+\eta)=\Gamma(\varphi)+\Gamma(\eta)$$

whenever φ and η are functions in $B(\Omega, E)$ with (almost everywhere) disjoint supports. More specifically, concern is required for such additive F-valued functions Γ defined on the associated space $L_E^p=L_E^p(\Omega, \Sigma, \mu)$ for $1\leqslant p\leqslant\infty$ of (equivalence classes of) Bochner μ integrable E-valued functions. If Γ is such an additive function then for every $e\in E$ we may define the set function Γ_e (2) from Σ into F by

$$\Gamma_e(B)=\Gamma(e\cdot\chi_B) \qquad B\in\Sigma.$$

If $d>0$ and $\delta>0$ then we may define

$$V_d(\delta,\Gamma)=\sup\left\{\sum_i v[\Gamma_{e_i}-\Gamma_{f_i}](E_i)\colon e_i, f_i\in E_i\right\},$$

$$V_d(\delta,\Gamma)=\sup\Big\{\sum_i v[\Gamma_{e_i}-\Gamma_{f_i}](E_i)\colon e_i, f_i\in E_i;\ \|e_i\|\leqslant d,\ \|f_i\|\leqslant d;$$
$$\|e_i-f_i\|\leqslant\delta;\ 1\leqslant i\leqslant n;\ \{E_i\}\text{ pairwise disjoint subsets of }\mathcal{B}\Big\}.$$

The family $\{\Gamma_e\}_{e\in E}$ of set functions is *locally uniformly continuous in variation* provided the

$$\lim_{\delta\to 0^+} V_d(\delta,\Gamma)=0$$

for every $d>0$.

Let us designate the *variation of* Γ on a set $B\in\Sigma$ by $V(\Gamma)(B)$.

A function θ from $E\times\Omega$ into F is said to be in the *uniform Caratheodory class relative to* E *on* $(X\to F)$, in brief,

$$\theta\in U\text{-Car}(E,F)$$

if $\theta(e,\ )$ is a F-valued Bochner measurable function for each vector $e\in E$ and $\theta(\cdot,\xi)$ is uniformly continuous on bounded subsets of E for all $\xi\in\Omega$ outside a μ-null set.

Given a p, $1\leqq p\leqq\infty$, $\theta\in U$-Car(E,F) is said to be in U-Car-(E,F) if the composition operator $x\to\theta\circ x$, where $\theta\ x(\xi)=\theta(x(\xi),\xi)$, maps L_E^p into L_F^1.

(2) In [11] and [12] the space (Ω, Σ, μ) is assumed to be σ-finite and complete. Then Γ would be defined on sets in Σ of finite measure.

The following theorem will be used in our representation theorem (see Theorem 5 of [12] for its proof).

THEOREM 6. – *Let (Ω, Σ, μ) be as above, let E be a separable Banach space and let F be an arbitrary Banach space. Let Γ be an additive function mapping $L_E^p(1 \leqslant p \leqslant \infty)$ into F satisfying the following conditions:*

(1) *For each vector $e \in E$ the set function Γ_e from Σ into F has locally almost compact average range.*

(2) *For each $e \in E$, if $B \in \Sigma$, $\mu(B) < \infty$ then $V(\Gamma_e)(B) < \infty$.*

(3) *On each set $B \in \Sigma$, the family of set functions $\{\Gamma_e\}_{e \in E}$ is locally uniformly continuous in variation.*

(4) *The function Γ is continuous relative to the L_E^p norm, if $p < \infty$, and is continuous with respect to bounded* a.e. *convergence of $p = \infty$.*

Then there exists a function $\theta \in U\text{-}\mathrm{Car}^p(E, F)$ such that

$$\Gamma(\varphi) = \int_\Omega \theta \circ \varphi \, d\mu \qquad \varphi \in L_E^p .$$

Moreover θ can be taken to satisfy

$$\theta(0, \ ) = 0 \qquad \text{a.e.}$$

and is then unique up to sets of the form $E \times N$ with N a null set in Ω (3).

At last our representation theorem may now be formulated.

THEOREM 7. – *Let Φ be an operator from $M_r[\mathfrak{B}, U(\mathbf{C}, \mathbf{C})]$ into $F(\mathbb{R}^+)$ of the form*

$$\Phi(\mu)(\alpha) = \Phi_\alpha(\mu_\alpha)$$

where Φ_α is a transformation defined on the space of restrictions μ_α of elements $\mu \in M_r[\mathfrak{B}, U(\mathbf{C}, \mathbf{C})]$. Assume

(1) *Φ is additive (in the sense defined above).*

(2) *Φ is uniformly continuous on bounded subsets of $M_r[\mathfrak{B}, U(\mathbf{C}, \mathbf{C})]_\alpha$ for every α, that is for every $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$v[\mu_\alpha - v_\alpha, S] = \|\mu_\alpha - v_\alpha\| < \delta$$

implies that

$$|\Phi(\mu)(\alpha) - \Phi(v)(\alpha)| = |\Phi_\alpha(\mu_\alpha) - \Phi_\alpha(v_\alpha)| < \varepsilon$$

$$\mu_\alpha, v_\alpha \in M_r[\mathfrak{B}, U(\mathbf{C}, \mathbf{C})]_\alpha .$$

(3) A converse is given in [12] but is not needed here.

(3) *The set function* r_Φ *defined for* $B \in \mathfrak{B}$ *by* $r_\Phi(B) = \Phi[\mu_{\psi,B}](\alpha)$ *has locally almost compact average range for all* $\alpha > 0$ *and all* $\psi \in U(\mathbf{C}, \mathbf{C})$.

(4) *The* $\sup\left\{\sum_{B\in\mathfrak{B}'} |\Phi[\mu_{\psi,B}](\alpha)|: \mathfrak{B}' \text{ partition of } \mathfrak{B}\right\}$ *is finite for all* $\alpha > 0$ *and* $\psi \in U(\mathbf{C}, \mathbf{C})$.

(5) *The* $\limsup\left\{\Sigma v\left[\Phi[\mu_{\psi_i^1,B_i}](\alpha) - \Phi[\mu_{\psi_i^2,B_i}](\alpha)\right]: \|\psi_i^j(\alpha)\| \leq d,\ j = 1, 2;\ \|\psi_i^1(\alpha) - \psi_i^2(\alpha)\| \leq \delta,\ \{B_i\} \text{ a partition of } \mathfrak{B}\right\} = 0.$

Then Φ_α *may be considered as a transformation on* $L^1_{U_\alpha}(S, \mathfrak{B}, r)$ *into the reals (where* $U_\alpha = U_\alpha(\mathbf{C}, \mathbf{C})$*) and there exists a* $\theta_{\alpha,r} \in U\text{-Car}^1[U_\alpha, \mathbb{R}]$ *such that*

(6) $\Phi_\alpha(\varphi) = \int_S (\theta_{\alpha,r} \circ \varphi)\, dr$ *for* $\varphi \in L^1_{U_\alpha}(S, \mathfrak{B}, r)$

and

(7) $\theta_{\alpha,r}(0, \ ) = 0$ (r a.e.).

In particular

$$\Phi(\mu)(\alpha) = \Phi_\alpha(\mu_\alpha) = \Phi_\alpha(\varphi_{r,\alpha}) = \int_S (\theta_{\alpha,r} \circ \varphi_{r,\alpha})\, dr$$

where $\varphi_{r,\alpha} = (\varphi_r)_\alpha \in L^1_{U_\alpha}(S, \mathfrak{B}, r)$ for $\bar{\mu}_r$ the correspondent of μ in Lemma 5.

PROOF. – Firstly the function $\theta_{\alpha,r} \circ \varphi$ is the function from S into $\mathbb{R}$ defined by

$$(\theta_{\alpha,r} \circ \varphi)(s) = \theta_{\alpha,r}(\varphi(s), s) \qquad s \in S.$$

Let us abbreviate by setting $L^1_{U_\alpha} = L^1_{U_\alpha}(S, \mathfrak{B}, r)$ and $L^\infty_{U_\alpha} = L^\infty_{U_\alpha}(S, \mathfrak{B}, r)$. We show that Φ may be extended to the step functions in $L^1_{U_\alpha}$ and is additive on that class.

If $\Phi = \sum_i \chi_{B_i} \psi_{\alpha,i}$ where $B_1, \dots, B_n$ are pairwise disjoint elements of Σ then define

$$\Phi_\alpha(u) = \sum_i \Phi_\alpha(\chi_{B_i} u)\,.$$

Let $\varphi \in L^1_{U_\alpha}$ and let $\{\varphi_n\}$ be a sequence of step functions in $L^1_{U_\alpha}$ for which $\{\varphi_n - \bar{\mu}\}$ converges to zero in the $L^1_{U_\alpha}$ norm. The sequence $\{\Phi_\alpha(\varphi_n)\}$ of real numbers is a Cauchy sequence. Consequently by assumption (2) if $\varepsilon > 0$ there is a $\delta > 0$ such that $|\Phi_\alpha(\varphi_1) - \Phi_\alpha(\varphi_2)| < \varepsilon$ whenever $\|\varphi_1 - \varphi_2\| < \delta$. Since $\{\varphi_n\}$ is a $L^1_{U_\alpha}$ Cauchy sequence there is an integer N such that in the $L^1_{U_\alpha}$ norm φ_m and φ_n are less than δ whenever $m, n > N$. Thus $|\Phi_\alpha(\varphi_m) - \Phi_\alpha(\varphi_n)| < \varepsilon$ whenever $m, n > N$. As usual we may then define $\Phi_\alpha(\varphi) = \lim_n \Phi_\alpha(\varphi_n)$ and this limit is independent of the particular Cauchy sequence chosen.

This extension of Φ_α is additive on $L^1_{U_\alpha}$. In particular if φ_a and φ_b are any two step functions in $L^1_{U_\alpha}$ with disjoint supports then we may write

$$\Phi_\alpha(\varphi^a + \varphi^b) = \Phi_\alpha(\varphi^a) + \Phi_\alpha(\varphi^b) .$$

Consequently for any two elements φ^a and φ^b of $L^1_{U_\alpha}$ with disjoint supports we may select corresponding sequences $\{\varphi^a_n\}$ and $\{\varphi^b_n\}$ of step functions in $L^1_{U_\alpha}$ converging to them and such that for each n, φ^α_n and φ^b_n have disjoint supports. A pass to the limit yields the result.

The Φ_α is uniformly continuous on bounded subsets of $L^1_{U_\alpha}$. Again by assumption (2), Φ_α is uniformly continuous on bounded subsets of $L^\infty_{U_\alpha}$. For $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|\Phi_\alpha(\varphi_1) - \Phi_\alpha(\varphi_2)| < \delta/3$$

where φ_1 and φ_2 are in $L^\infty_{U_\alpha}$ with ther difference less than δ in the $L^1_{U_\alpha}$ norm. Let φ and η be in $L^1_{U_\alpha}$ satisfying the condition that their difference is less than $\delta/3$ in the $L^1_{U_\alpha}$-norm. Choose φ_1 and η_1 to be functions in $L^\infty_{U_\alpha}$ such that in the $L^1_{U_\alpha}$ norm the difference of φ and φ_1 and the difference of η and η_1 are both less than $\delta/3$ and

$$|\Phi_\alpha(\varphi) - \Phi_\alpha(\varphi_1)| < \varepsilon/3; \qquad |\Phi_\alpha(\eta) - \Phi_\alpha(\eta_1)| < \varepsilon/3.$$

Thus the difference of φ_1 and η_1 is less than δ in the $L^1_{U_\alpha}$ norm and hence

$$|\Phi_\alpha(\varphi) - \Phi_\alpha(\eta)| < \varepsilon .$$

Now define the following real valued set functions on $\mathfrak{B}$. For every $\psi \in U_\alpha(\boldsymbol{C}, \boldsymbol{C})$, let Φ_ψ be defined for $B \in \mathfrak{B}$ by

$$\Phi_\psi(B) = \Phi_\alpha(\chi_B \psi) = \Phi_\alpha[\mu_{\psi,B}](\alpha) .$$

Assumption (3) says that Φ_ψ has locally almost compact average range for each $\psi \in U(\boldsymbol{C}, \boldsymbol{C})$. Also for $\mathfrak{B}'$ a finite family of pairwise disjoint subsets from the family $\mathfrak{B}$ we have

$$\Sigma\{|\Phi[\mu_{\psi,B}](\alpha)|: B \in \mathfrak{B}'\} = \Sigma\{|\Phi_\psi(B)|: B \in \mathfrak{B}'\}$$

and

$$\Sigma v[\Phi[\mu_{\psi^1,B_i}](\alpha) - \Phi[\mu_{\psi^2_i,B_i}](\alpha)] = \Sigma v[\Phi_{\psi^1} - \Phi_\psi](B_i).$$

Thus assumption (5) translates as the family $\{\Phi_\psi\}_{\psi \in U_\alpha(\boldsymbol{C},\boldsymbol{C})}$ being locally uniformly continuous in variation. Assumption (4) means just that each Φ_ψ has finite variation.

All of the hypotheses of Theorem 6 are satisfied with $p=1$, $E=U_\alpha(\mathbf{C},\mathbf{C})$, $F=\mathbb{R}$, $\Gamma=\Phi$, and $\{\Gamma_e\}_{e\in E}=\{\Phi_\varphi\}_{\varphi\in U_\alpha(\mathbf{C},\mathbf{C})}$. Consequently that theorem yields the existence of a function $\theta_{\alpha,r}\in U\text{-Car}^1\,U_\alpha(\mathbf{C},\mathbf{C})$ such that

$$\Phi_\alpha(\varphi)=\int_S(\theta_{\alpha,r}\circ\varphi)\,dr$$

for all $\varphi\in L^1_{U_\alpha}$. The function $\theta_{\alpha,r}$ also satisfies

$$\theta_{\alpha,r}(0,\ )=0 \qquad (r \text{ a.e.}).$$

This completes the proof of our theorem.

4. – Representation via approximate integration.

This representation just presented may be viewed in a different manner. To do this we utilize the concept of a slur (as found in [14]) and the technique of approximate integration as developed by ALÒ and DE KORVIN in [1].

Let r_β be a typical element of the maximal subset $\mathcal{M}$ where β and γ will represent typical elements in a well-ordered set used to index the elements of $\mathcal{M}$. In accordance with our assumption on $\mathcal{M}$, we assume that each proper initial segment of this index set is countable.

Let

$$(\varphi_\beta)_\alpha=\varphi_{\beta,\alpha}\in L^\infty_{U_\alpha}(S,\mathcal{B},r_\beta)$$

for every $\alpha>0$. Let $\{\varphi^n_{\beta,\alpha}\}_n$ be a sequence of simple functions converging r_β a.e. to $\varphi_{\beta,\alpha}$. These may be so chosen so that in the $L^1_{U_\alpha}$ norms $\varphi_{\beta,\alpha}$ is finite and is not less than the $\varphi^n_{\beta,\alpha}$ (see Theorem 2, page 99 of [4]). The dominated convergence theorem says that the sequence $\{\varphi^n_{\alpha,\beta}\}_n$ converges in the $L^1_{U_\alpha}$ norm to $\varphi_{\beta,\alpha}$. If $\theta_{\alpha\beta}=\theta_{\alpha,r_\beta}$ is the function from Theorem 7, let us define

$$\bar\theta_{\alpha,\beta}(\ ,\ )=|\theta_{\alpha,\beta}(\ ,\ )|\,.$$

Then

$$\bar\theta_{\alpha,\beta}\circ\varphi^n_{\beta,\alpha}=|\theta_{\alpha,\beta}\circ\varphi^n_{\beta,\alpha}|\,.$$

Since the sequence above does converge in the $L^1_{U_\alpha}$ norm, a modification of the proof of Theorem 2 in [11] will imply that $\{\bar\theta_{\beta,\alpha}\circ\varphi^n_{\beta,\alpha}\}_n$ converges to $\bar\theta\circ\varphi_{\beta,\alpha}$ also in the $L^1_{U_\alpha}$ norm. Thus the set functions

$$\sigma_n(B)=\int_B|\theta_{\alpha,\beta}\circ\varphi^n_{\beta,\alpha}|\,dr_\beta \qquad B\in\mathcal{B},\ n=1,2,\dots$$

form a uniformly absolutely continuous family with respect to r_β. Furthermore $\{\theta_{\alpha,\beta}\circ\varphi^n_{\beta,\alpha}\}_n$ converges to $\theta_{\alpha,\beta}\circ\varphi_{\beta,\alpha}$ (r_β a.e.). Thus the sequence

$$\left\{\int_S |\theta_{\alpha,\beta}\circ\varphi_{\beta,\alpha} - \theta_{\alpha,\beta}\circ\varphi^n_{\beta,\alpha}|\, dr_\beta\right\}_{n\in\mathrm{N}}$$

converges to zero (see Theorem 6, page 122 of [5]).

Now our argument follows closely that used in [1] and [10]. For $\gamma<\beta$, let B^β_γ be a set in $\mathfrak{B}$ such that $r_\gamma(B^\beta_\gamma)=0$ and $r_\beta(\complement B^\beta_\gamma)=0$ ($\complement$ refers to set theoretical complementation). Let $B^\beta=\cap\{B^\beta_\gamma\colon \gamma<\beta\}$. Since this is the intersection of a countable number of sets it follows that $B^\beta\in\mathfrak{B}$.

If $B\in\mathfrak{B}$ then there is at most one cardinal β such that $B\subset B^\beta$ and $r_\beta(B)>0$ (the proof of this is exactly as in [1]).

For $\varepsilon>0$, $\alpha>0$ and $B\in\mathfrak{B}$ define

$$\tilde\theta_{\varepsilon,\alpha}(B)=\begin{cases}\theta_{\alpha,1}\circ\varphi^{n(\alpha,1)}_{1,\alpha} & \text{if } r_1(B)>0\\ \theta_{\alpha,\beta}\circ\varphi^{n(\alpha,\beta)}_{\beta,\alpha} & \text{if } B\subset B_\beta \text{ and } r_\beta(B)>0\\ 0 & \text{otherwise.}\end{cases}$$

Here 1 indicates the first element in the indexing set for $\mathcal{M}$. The integer $n(\alpha,\beta)$ is so chosen so that for $m\geqslant n(\alpha,\beta)$,

$$\int_S|\theta_{\alpha,\beta}\circ\varphi_{\beta,\alpha}-\theta_{\alpha,\beta}\circ\varphi^m_{\beta,\alpha}|\,dr_\beta<\varepsilon\,.$$

Thus for every $B\in\mathfrak{B}$, $\tilde\theta_{\varepsilon,\alpha}(B)\in L^1_{\mathrm{R}}(S,\mathfrak{B},r_\beta)$ where β depends on B.

We may now give the representation of the operator Φ in Theorem 7 in terms of slurs.

Let w be a real valued set function on $\mathfrak{B}$. A *slur* is a sequence $\Psi=\{\psi_n,\mathcal{T}_n\}$ where ψ_n is a set function from $\mathfrak{B}$ into $L^1(S,\mathfrak{B},w)$ and where $\mathcal{T}_n$ is a partition of S by sets in $\mathfrak{B}$ for each $n=1,2,3,\ldots$.

If there is a number L such that for very $\varepsilon>0$ there is a positive integer N such that for $n\geqslant N$, and $\mathcal{T}'_n$ a refinement of $\mathcal{T}_n$,

$$\left|\Sigma\left\{\int\psi_n(B)\,dw\colon B\in\mathcal{T}'_n\right\}-L\right|<\varepsilon\,,$$

then L is denoted by

$$\int\{\psi_n,\mathcal{T}_n\}\,dw=\int\Psi\,dw\,.$$

Now let $\mathfrak{T}_\beta$ be a finite partition refining $\{B^\beta, \complement B^\beta\}$. Then

$$\Sigma\left\{\int_B \tilde{\theta}_{\varepsilon,\alpha}(B)\,dr_\beta\colon B\in\mathfrak{T}_\beta\right\} = \Sigma\left\{\int_B \tilde{\theta}_{\varepsilon,\alpha}(B)\,dr_\beta\colon B\in\mathfrak{T}_\beta,\, r_\beta(B)>0\,,\, B\subset B^\beta\right\}$$
$$= \Sigma\left\{\int_B \theta_{\alpha,\beta}\circ\varphi_{\alpha,\beta}^{n(\alpha,\beta)}\,dr_\beta\colon B\in\mathfrak{T}_\beta,\, r_\beta(B)>0,\, B\subset B^\beta\right\}$$
$$= \int_{B^\beta} \theta_{\alpha,\beta}\circ\varphi_{\alpha,\beta}^{n(\alpha,\beta)}\,dr_\beta$$
$$= \int_B \theta_{\alpha,\beta}\circ\varphi_{\alpha,\beta}^{n(\alpha,\beta)}\,dr_\beta\,.$$

Now let Φ be as in Theorem 7 and let $\mu\in M_r[\mathfrak{B},\ U(\boldsymbol{C},\boldsymbol{C})]$. Then

$$\Phi(\mu)(\alpha) = \Phi_\alpha(\mu_\alpha) = \int_S \theta_{\alpha,\beta}\circ\varphi_{\beta,\alpha}\,dr_\beta\,.$$

If $\varepsilon>0$ then

$$\left|\Phi(\mu)(\alpha) - \Sigma\left\{\int_B \tilde{\theta}_{\varepsilon,\alpha}(B)\,dr\colon B\in\mathfrak{T}_\beta\right\}\right| = \left|\int_S (\theta_{\alpha,\beta}\circ\varphi_{\beta,\alpha} - \theta_{\alpha,\beta}\circ\varphi_{\alpha,\beta}^{n(\alpha,\beta)})\,dr\right| < \varepsilon\,.$$

For each pair (α,β) with $\alpha>0$, let us define the slur $\Psi_{\alpha,\beta}=\{\tilde{\theta}_{1/n,\alpha},\mathfrak{T}_\beta\}_n$ where $\mathfrak{T}_\beta$ is a fixed partition refining $\{B^\beta, \complement B^\beta\}$. Then

$$\left|\Phi(\mu)(\alpha) - \Sigma\left\{\int \psi_{n,\alpha}(B)\,dr_\beta\colon B\in\mathfrak{T}_\beta\right\}\right| < 1/n\,.$$

Thus

$$\Phi(\mu)(\alpha) = \int \Psi_{\alpha,\beta}\,dr_\beta$$

for $\mu\in M_{r_\beta}[\mathfrak{B},\ U(\boldsymbol{C},\boldsymbol{C})]$. We have thus proved

THEOREM 8. – *If Φ is as in Theorem 7, then*

$$\Phi(\mu)(\alpha) = \int \Psi_{\alpha,\beta}\,dr_\beta$$

in the integral notation established above for the slur $\Psi_{\alpha,\beta}=\{\tilde{\theta}_{1/n,\alpha},\mathfrak{T}_\beta\}$ where for each n, $\mathfrak{T}_\beta$ is the fixed partition given above, $\mu\in M_{r_\beta}[\mathfrak{B},\ U(\boldsymbol{C},\boldsymbol{C})]$.

With this representation we can now give a representation for operators on $M_I[\mathfrak{B},\ U(\boldsymbol{C},\boldsymbol{C})]$ which is a larger class than each $M_{r_\beta}[\mathfrak{B},\ U(\boldsymbol{C},\boldsymbol{C})]$.

COROLLARY 9. – *Let Φ be an additive function from $M_I[\mathfrak{B},\ U(\boldsymbol{C},\boldsymbol{C})]$ to $F(\mathbb{R}^+)$. If $\mu\in M_I[\mathfrak{B},\ U(\boldsymbol{C},\boldsymbol{C})]$ then there is a finite set $\{r_i\}_{i\in I}$ in $\mathcal{M}$ such that for $B\in\mathfrak{B}$*

$$v[\mu_\alpha, B] \leqslant L_\alpha(I)\cdot\Sigma r_i(B)\,.$$

Assume Φ restricted to each $M_{r_i}[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ satisfies the conditions of Theorem 7. Then

$$\Phi(\mu)(\alpha) = \Sigma\left\{\int \Psi_{\alpha,i}\, dr_i : i \in I\right\}$$

where each $\Psi_{\alpha,i}$ is a slur as in Theorem 8.

PROOF. – By Lemma 5 there is a φ from S into $U(\boldsymbol{C}, \boldsymbol{C})$ such that

$$\mu(B)_\alpha = \int_B \varphi(t)_\alpha\, d\big(\Sigma\{r_i : i \in I\}\big) = \Sigma\left\{\int_B \varphi(t)_\alpha\, dr_i : i \in I\right\}$$

for every $\alpha > 0$. Let μ_i from $\mathfrak{B}$ into $U(\boldsymbol{C}, \boldsymbol{C})$ be defined by

$$\mu_i(B)_\alpha = \int_B \varphi(t)_\alpha\, dr_i$$

for every $\alpha > 0$. Note that $\varphi(\ )_\alpha \in L^\infty_{U_\alpha}(S, \mathfrak{B}, r_i)$ since $\varphi(\ )_\alpha \in L^\infty_{U_\alpha}\Big(S, \mathfrak{B}, \sum_{i \in I} r_i\Big)$. Then $\mu_i \in M_{r_i}[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$. Thus we have $\mu = \Sigma\{\mu_i : i \in I\}$. But recall that the μ_i's are mutually singular. Thus for each $\alpha > 0$, the $\mu_i(\ )_\alpha$, $i \in I$, are concentrated on mutually disjoint sets. Thus

$$\Phi(\mu) = \Phi\big(\Sigma\{\mu_i : i \in I\}\big) = \Sigma\{I(\mu_i) : i \in I\}\ .$$

By Theorem 8, for each i and $\alpha > 0$ there is a slur $\Psi_{\alpha,i}$ such that

$$\Phi(\mu_i)(\alpha) = \int \Psi_{\alpha,i}\, dr_i\ .$$

Thus,

$$\Phi(\mu)(\alpha) = \Sigma\left\{\int \Psi_{\alpha,i}\, dr_i : i \in I\right\}.$$

Using Corollaries 4 and 9 we may now approximate certain additive operators on $M[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$.

Suppose $\mu \in M_D[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$, $c \in \boldsymbol{C}$ and $\delta > 0$. By Corollary 4, there is a finite set $I \subset \mathcal{M}$ and a $\mu_I \in M_I[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ such that $v[(\mu - \mu)_I(c), S] < \delta$.

Let us now define a particular subset $M_A[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ of $M[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ over which we will define our integral. A set function $\mu \in M[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ is said to be *simple* if there are finite collections $\mu_1, \ldots, \mu_n$ of set functions in $M_D[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$, functions $\psi^1, \ldots, \psi^n$ in $U(\boldsymbol{C}, \boldsymbol{C})$ and points $c_1, \ldots, c_n$ in $\boldsymbol{C}$ such that for $B \in \mathfrak{B}$,

$$\mu(B) = \Sigma \mu_i(B)(c_i)\, \psi^i\ .$$

The measure μ is *α-simple*, $\alpha > 0$, if it is simple and if the functions $\psi^1, \ldots, \psi^n$ are the functions $\psi^1_\alpha, \ldots, \psi^n_\alpha$ from $U_\alpha(\boldsymbol{C}, \boldsymbol{C})$. Let $M_A[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ be the collection of all set functions $\mu \in M[\mathfrak{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ such that for every $\alpha > 0$ and $\delta > 0$ there is an α-simple set function v^α such that the variation $v[\mu_\alpha - v^\alpha, S] < \delta$.

THEOREM 10. – *Let Φ be an operator from $M[\mathfrak{B}, U(\mathbf{C}, \mathbf{C})]$ into $F(\mathbb{R}^+)$ of the form $\Phi(\mu)(\alpha) = \Phi_\alpha(\mu_\alpha)$, $\mu \in M[\mathfrak{B}, U(\mathbf{C}, \mathbf{C})]$ and $\alpha > 0$. Assume also that Φ_α is continuous in the variation norm on $M_A[\mathfrak{B}, U(\mathbf{C}, \mathbf{C})]_\alpha$ and that Φ restricted to each $M_{r_\beta}[\mathfrak{B}, U(\mathbf{C}, \mathbf{C})]$, $r_\beta \in \mathcal{M}$ satisfies the conditions of Theorem 7. If $\varepsilon > 0$, $\alpha > 0$ and $\mu \in M_A[\mathfrak{B}, U(\mathbf{C}, \mathbf{C},)]$, then there exists a set of slurs Ψ_{α,β_i}, $i = 1, \dots, p$ of the type described in Theorem 8 such that*

$$\left|\Phi(\mu) - \Sigma \int \Psi_{\alpha,\beta_i} dr_{\beta_i}\right| < \varepsilon .$$

PROOF. – If $\mu \in M_A[\mathfrak{B}, U(\mathbf{C}, \mathbf{C})]$ then for $\delta > 0$ there is an α-simple set function $v^\alpha = \sum_{i=1}^{k} \mu_i(\ )(c_i) \cdot \psi_\alpha^i$ such that

$$v[\mu - v^\alpha, S] < \delta .$$

Choose $v_1, \dots, v_k \in M_I[\mathfrak{B}, U(\mathbf{C}, \mathbf{C})]$ such that

$$v[\mu_i(\ )(c_i) - v_i(\ )(c_i), S] < \frac{\delta}{k \cdot K}$$

where $K = \max\{\|\psi_\alpha^i\| : i = 1, \dots, k\}$. Thus

$$\Phi\Big(\sum_{i=1}^{k} \mu_i(\ )(c_i)\psi_\alpha^i - \sum_{i=1}^{k} v_i(\ )(c_i)\psi_\alpha^i\Big) \leqslant \delta .$$

For sufficiently small δ, the continuity of Φ_α on $M_A[\mathfrak{B}, U(\mathbf{C}, \mathbf{C})]_\alpha$ implies that

$$|\Phi(\mu)(\alpha) - \Phi_\alpha\Big(\sum_{i=1}^{k} \mu_i(\ )(c_i)\psi_\alpha^i\Big)| < \varepsilon .$$

Since $\sum_{i=1}^{k} \mu_i(\ )(c_i)\psi_\alpha^i \in M_I[\mathfrak{B}, U(\mathbf{C}, \mathbf{C})]$, Corollory 9 yields slurs Ψ_{α,β_i}, $i = 1, \dots, p$ such that

$$\Phi_\alpha\Big(\sum_{i=1}^{k} \mu_i(\ )(c_i)\psi_\alpha^i\Big) = \sum_{i=1}^{p} \int \Psi_{\alpha,\beta_i} dr_{\beta_i} .$$

This yields

$$\left|\Phi(\mu) - \sum_{i=1}^{p} \int \Psi_{\alpha,\beta_i} dr_{\beta_i}\right| < \varepsilon .$$

5. – Conclusion.

For operators Φ from $C(S, E)$ into a Banach space F it is clear that the condition that Φ be additive is definitely weaker than the condition that Φ satisfy the Hammerstein property. For example, for each $f \in C(S, E)$ where $S = E = [0, 1]$, let

$$\Phi(f) = \inf\{|f(s)| : s \in S\} .$$

Such an additive functional may not be represented as an integral with respect to an additive non-linear set function as Theorem 1 would indicate.

We proceeded to show that certain additive operators on the subset $M_r[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ of $M[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ may be represented in terms of certain uniform Caratheodory functions (see Theorem 7) as discussed in [12]. However this representation may also be given (see Theorem 8) in terms of slurs and the technique of approximate integration as developed in [1]. The latter leads to the representation of additive operators on the larger subset $M_I[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ of $M[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ (see Corollary 9). From here we are led to (Theorem 10) the representation of operators on $M[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ itself. The representation of such operators is given through approximation over the subset $M_A[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ of $M[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$. Let us recall that the Hammerstein condition was used in the representation of $M[\mathcal{B}, U(\boldsymbol{C}, \boldsymbol{C})]$ as the space $HP[C(S, \boldsymbol{C}), \boldsymbol{C}]$.

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