On the global existence of solutions and Liapunov functions.

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Sunto. - *Per l'equazione differenziale ordinaria*

$$\text{(E)} \qquad x' = f(t, x)$$

le funzioni di Liapunov sono state costruite usando ipotesi diverse; ad esempio, (1) *la soluzione nulla è stabile su* $[0, \infty)$, (2) *la soluzione nulla é uniformemente stabile su* $[0, \infty)$, *e* (3) *tutte le soluzioni sono limitate nel futuro* (*cfr. ad es.* [A, H, Y]). *In questo lavoro costruiamo le funzioni di Liapunov partendo da ipotesi in certo senso minime, supponendo cioè soltanto l'esistenza globale delle soluzioni. Le funzioni di Liapunov costruite per l'esistenza sono poi usate per stabilire se le soluzioni hanno le proprietà addizionali* (1), (2) *e* (3).

Specificamente, nel teorema 1, *dimostriamo che l'esistenza come soluzione della* (E) *della funzione nulla si può caratterizzare in termini di funzioni di Liapunov; cioè, se la funzione nulla è una soluzione, allora esistono due funzioni di Liapunov, una delle quali si può poi usare per decidere circa la stabilità di quella soluzione, e l'altra per la stabilità uniforme. Queste stabilità si considerano su* $(-\infty, \infty)$ *anzichè su* $[0, \infty)$. *Nel teorema* 2 *troviamo una condizione necessaria e sufficiente sulle funzioni di Liapunov perchè tutte le soluzioni esistano su* $(-\infty, \infty)$. *Se tutte esistono su* $(-\infty, \infty)$, *allora la corrispondente funzione di Liapunov si può poi usare per decidere circa l'esistenza e stabilità della soluzione nulla. Nel teorema* 3 *si presentano alcuni risultati analoghi a quelli del teorema* 2 *ma «nel futuro.»*

1. - Introduction.

For the ordinary differential equation

$$\text{(E)} \qquad x' = f(t, x) \qquad \left(' = \frac{d}{dt}\right)$$

Liapunov functions have been constructed under various assumptions, including (1) the zero solution is stable on $[0, \infty)$, (2) the zero solution is uniformly stable on $[0, \infty)$, and (3) all solutions are bounded in the future (for details, see ANTOSIEWICZ [A], HAHN [H], and YOSHIZAWA [Y]). In this paper

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we construct Liapunov functions which correspond to a possibly more fundamental situation: the «global» existence of solutions. The Liapunov functions constructed for «global» existence are then used to determine whether or not the solutions have the additional properties (1), (2), and (3). Specifically, in Theorem 1 we show that the existence as a solution of (E) of the zero function can be characterized in terms of Liapunov functions. We prove that if the zero function is a solution, then there must exist two Liapunov functions, one of which can then be used as a test for the stability of the zero solution, and the other as a test for the uniform stability. These stabilities, are considered on $(-\infty, \infty)$ rather than on $[0, \infty)$. In Theorem 2 we find necessary and sufficient conditions in terms of Liapunov functions for all solutions to exist on $(-\infty, \infty)$. If they do exist on $(-\infty, \infty)$, then the Liapunov function that necessarily exists can then be used as a test for the boundedness of all solutions on $(-\infty, \infty)$ and for the existence and stability of the zero solution. Results similar to those in Theorem 2 but «in the future» are presented in Theorem 3.

2. - Definitions and results.

Let R^n denote Euclidean n-space. For simplicity, let R denote R^1. Let $|x|$ denote any norm of x in R^n. Let $A \subset R^r$ and let $g: A \to R^p$ for some p and r. Then g is C_0 at $y \in A$ if there exist a neighborhood N of y and a constant $k > 0$ such that

$$|g(y_1) - g(y_2)| \leq k |y_1 - y_2|$$

for all y_1 and y_2 in $N \cap A$. We say that g is C_0 (on A) if it is C_0 at every point of A.

Let Ω be an open subset of R^n containing the origin. We consider the differential equation (E) where

$$f: R \times \Omega \to R^n$$

is continuous and

(2.1) $$f(t, \cdot) \text{ is } C_0 \text{ on } \Omega \text{ for each real } t.$$

Hence for each (t_0, x_0) in $R \times \Omega$, there is precisely one solution $F(t)$ of (E) such that $F(t_0) = x_0$. This solution, which we often denote by $F(t; t_0, x_0)$, exists on some maximal interval (α, ω), $-\infty \leqslant \alpha < t_0 < \omega \leqslant +\infty$, where α and ω depend on (t_0, x_0). Furthermore (2.1) implies that

(2.2) $$F \text{ is } C_0 \text{ on the set } S,$$

where $S = \cup \{ (\alpha, \omega) \times \{(t_0, x_0)\} : t_0 \in R, x_0 \in \Omega \}$.

PROPOSITION 1. — *There exists a C_0 function $\rho: \Omega \to [0, \infty)$ such that $\rho(x) \to \infty$ when either $|x| \to \infty$ or*

$$d(x, \partial\Omega) = \inf \{ |x - z| : z \in \partial\Omega \} \to 0,$$

where $\partial\Omega$ is the boundary of Ω. Furthermore,

$$\rho(x) = 0 \text{ iff } x = 0.$$

DEFINITIONS 1. — Let $\theta : R \to \{0\} \subset R^n$. If the zero function $\theta(t)$ is a solution of (E), we say that $\theta(t)$ is *stable* if for all $\varepsilon > 0$ and all real t_0, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $|x_0| < \delta$ and $t \geqslant t_0$ imply that

$$|F(t;\ t_0,\ x_0)| < \varepsilon. \tag{2.3}$$

We say that $\theta(t)$ is *uniformly stable* if $\delta = \delta(\varepsilon)$ is independent of t_0, see [H].

DEFINITIONS 2. — We say that $F(t;\ t_0,\ x_0)$ *exists forever* if it exists for all real t, and that $F(t;\ t_0,\ x_0)$ *exists in the future* if it exists for all $t \geqslant t_0$. All solutions (of (E)) are *equi-bounded* if for all $M > 0$ and all real t_0, there exists $\beta = \beta(t_0, M) > 0$ such that $\rho(x_0) \leqslant M$ implies that

$$\rho[F(t;\ t_0,\ x_0)] \leqslant \beta \tag{2.4}$$

for all real t. The solutions are *uniformly bounded in the future* if $\beta = \beta(M)$ is independent of t_0 and (2.4) holds merely for all $t \geqslant t_0$, see [Y].

DEFINITIONS 3. — Let $V: R \times \Omega \to [0, \infty)$. We say that V is *positive definite* if for all $\varepsilon > 0$, there exists $\mu = \mu(\varepsilon) > 0$ such that $|x| \geqslant \varepsilon$ and t real imply

$$V(t, x) \geqslant \mu.$$

V is *decrescent* if $V(t, x) \to 0$ as $|x| \to 0$ uniformly in t for t in R. V is *radially unbounded* if

$$V(t, x) \to \infty \text{ as } \rho(x) \to \infty \tag{2.5}$$

uniformly in t for t in R. V is *mildly unbounded* if (2.5) holds uniformly in t for t in any compact subset of R.

If $\Omega = R^n$, the first three definitions in Definitions 3 may be found in [H], the last in [S]. Definitions 2 and 3 are expressed in terms of the function ρ constructed in Proposition 1.

DEFINITION 4. — For $V: R \times \Omega \to [0, \infty)$, define the *generalized derivative* of V by

$$\dot{V}(t, x) = \limsup_{h \to 0^+} h^{-1} [V(t+h, x+hf(t, x)) - V(t, x)].$$

Proposition 2. *If V is C_0 on $R \times \Omega$, then*

$$\dot{V}(t, x) = \limsup_{h \to 0^+} h^{-1} [V(t+h, F(t+h; t, x)) - V(t, x)],$$

We now state our main results Below, iff means if and only if.

Theorem 1. — *The zero function $\theta(t)$ is a solution iff there exist two non-negative C_0 functions V_1 and V_2 such that on $R \times \Omega$ for $i = 1$ and 2,*

(a) $V_i(t, x) = 0$ *iff* $x = 0$ *and*

(b) $\dot{V}_i(t, x) \leq 0$.

Furthermore, V_1 is positive definite iff $\theta(t)$ is stable, while V_2 is positive definite and decrescent iff $\theta(t)$ is uniformly stable.

Theorem 2. — *All solutions exist forever iff there exists a non-negative C_0 function U such that on $R \times \Omega$*

(a′) *U is mildly unbounded, and*

(b′) $\dot{U}(t, x) \equiv 0$.

Furthermore, U is radially unbounded iff all solutions are equi-bounded. Finally, U satisfies (a) *iff $\theta(t)$ is a solution, while U is positive definite iff $\theta(t)$ is stable.*

Theorem 3. — *All solutions exist in the future iff there exists a non-negative C_0 function V such that on $R \times \Omega$*

(a′) *V is mildly unbounded, and*

(b) $\dot{V}(t, x) \leq 0$.

Furthermore, V is radially unbounded and bounded on $R \times K$ for every compact K in Ω iff all solutions are uniformly bounded in the future. Finally, V satisfies (a) *iff $\theta(t)$ is a solution, while V is positive definite and decrescent iff $\theta(t)$ is uniformly stable.*

Remark on Theorem 3. — Note that the function $V + V_1$ is C_0, non-negative, and satisfies (a′) and (b); while it is positive definite iff $\theta(t)$ is a stable solution. This generalizes Theorem 5.1 of [S], the proof of which was incorrect anyway, because the statement made there that $V_m = V_n$ on $D_n \cap D_m$ is not always true.

3. - Proofs.

PROOF OF THEOREM 1. — The existence of a non-negative C_0 function satisfying (a) and (b) shows that $\theta(t)$ is a solution. Namely, by (a), $V(t, 0) = 0$. Thus for t_0 real and $t \geqslant t_0$,

$$0 = V(t_0, F(t_0; t_0, 0)) \geqslant V(t, F(t; t_0, 0)) \geqslant 0,$$

using (b). Hence $V(t, F(t; t_0, 0)) = 0$, and (a) now implies that $F(t; t_0, 0) \equiv \theta(t)$ for all $t \geqslant t_0$.

Conversely, let $\theta(t)$ be a solution of (E) on R. We shall first construct V_1 and show that V_1 is as desired. Then we will construct V_2 and show that V_2 satisfies its desired conditions.

Define a non-increasing, C_0 function $\gamma : [0, \infty) \to (0, \infty)$ so that $|x_0| < \gamma(t)$ implies $F(\tau; 0, x_0)$ exists on $[0, t]$ for $t \geqslant 0$.

Next, define a C_0 function $\Phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ as follows: $\Phi(t, r) > 0$ for $r > 0$, $\Phi(\cdot, r)$ is a non-increasing function on $[0, \infty)$ for each fixed $r > 0$, $\Phi(0, \cdot)$ is non-decreasing, and

$$\Phi(t, r) = \begin{cases} 0 \text{ if } r = 0, \\ \gamma(t) \text{ if } r \geqslant \gamma(t)/2. \end{cases}$$

Finally, define V_1: $R \times \Omega \to [0, \infty)$ by

$$V_1(t, x) = \begin{cases} \Phi(t, |F(0; t, x)|), \text{ if } t \geqslant 0 \text{ and } |F(0; t, x)| < \gamma(t), \\ \gamma(t), \text{ if } t \geqslant 0 \text{ and either } |F(0; t, x)| \geqslant \gamma(t) \text{ or} \\ \qquad F(\tau; t, x) \text{ does not exist for } 0 \leqslant \tau \leqslant t, \\ \sup\limits_{t \leq \tau \leq 0} \Phi(0, |F(\tau; t, x)|), \text{ if } t < 0. \end{cases}$$

Here, as elsewhere, we understand that the supremum is evaluated over that part of $[t, 0]$ on which $F(\tau; t, x)$ is defined; i.e., $[t, 0] \cap (\alpha, \omega)$.
Clearly, V_1 satisfies (a). If $t < 0$, then $V_1(t, F(t))$ is non-increasing. If $t \geqslant 0$ and $h \geqslant 0$, then $\gamma(t + h) \leqslant \gamma(t)$ and

$$\Phi(t + h, |F(0; t + h, F(t + h))|) = \Phi(t + h, |F(0; t, F(t))|) \leqslant \\ \leqslant \Phi(t, |F(0; t, F(t))|),$$

hence $V_1(t, F(t))$ is again non-increasing, proving (b).

We now prove V_1 is C_0. Let $t > 0$. If (t, x) is such that $|F(0; t, x)| < \gamma(t)$, then for (t_1, x_1) near (t, x), we have $|F(0; t_1, x_1)| < \gamma(t_1)$, and the result follows by the smoothness of γ, Φ, and by (2.2). If (t, x) is such that $|F(0; t, x)| > \gamma(t)/2$, then for (t_1, x_1) near (t, x), we have $|F(0; t_1, x_1)| > \gamma(t_1)/2$, and the result again follows easily. Let (t, x) be such that $F(\tau; t, x)$ does not exist on $0 \leqslant \tau \leqslant t$. We claim that there exists a neighborhood N of (t, x) in which $V_1(t_1, x_1) = \gamma(t_1)$. If so, V_1 is C_0 at (t, x). Therefore, suppose the claim is false. Choose a sequence $(t_n, x_n) \to (t, x)$ such that $V_1(t_n, x_n) < \gamma(t_n)$. Thus $|F(0; t_n, x_n)| < \gamma(t_n)/2$. Assume without loss that

$$y_n \equiv F(0; t_n, x_n) \to y_0 \text{ as } n \to \infty.$$

Then $|y_0| \leqslant \gamma(t)/2 < \gamma(t)$, hence $F(\tau; 0, y_0)$ exists for $0 \leqslant \tau \leqslant t$. But

$$|x - F(t; 0, y_0)| \leqslant |x - F(t_n; 0, y_n)| + \\ + |F(t_n; 0, y_n) - F(t; 0, y_0)| \to 0$$

as $n \to \infty$ by continuous dependence, since $x_n = F(t_n; 0, y_n)$. Thus $F(t; 0, y_0) = x$; that is, the solution through (t, x) exists on $[0, t]$, a contradiction. Thus the result is proved for $t > 0$.

Let $t < 0$ and $x \in \Omega$. If $|F(\tau; t, x)| > \gamma(0)/2$ for some τ in $[t, 0]$, then for (t_1, x_1) sufficiently near (t, x) we have also that $|F(\tau; t_1, x_1)| > \gamma(0)/2$, hence $V_1(t, x) \equiv \gamma(0)$ in some neighborhood of (t, x). Thus suppose that $|F(\tau; t, x)| \leqslant \gamma(0)/2$ for all τ in $[t, 0]$. Thus for (t_1, x_1) sufficiently near (t, x), we have $|F(\tau; t_1, x_1)| < \gamma(0)$ for all τ in $[t, 0]$, and hence for (t_1, x_1) and (t_2, x_2) near (t, x),

$$\begin{aligned} V_1(t_i, x_i) &= \sup_{t_i \leq \tau \leq 0} \Phi(0, |F(\tau; t_i, x_i)|) \\ &= \Phi(0, \max_{t_i \leq \tau \leq 0} |F(\tau; t_i, x_i)|) \\ &= \Phi(0, |F(\tau_i; t_i, x_i)|), \end{aligned}$$

where τ_i depends on (t_i, x_i), $i = 1, 2$. Because Φ is C_0, we need only estimate

$$\Big| |F(\tau_1; t_1, x_1)| - |F(\tau_2; t_2, x_2)| \Big| .$$

Assume $t_1 \leqslant t_2 \leqslant 0$, the argument for $t_2 \leqslant t_1 \leqslant 0$ is similar. Then $t_1 \leqslant \tau_2 \leqslant 0$, hence

$$\begin{aligned} |F(\tau_2; t_2, x_2)| - |F(\tau_1; t_1, x_1)| &\leqslant |F(\tau_2; t_2, x_2)| - |F(\tau_2; t_1, x_1)| \\ &\leqslant |F(\tau_2; t_2, x_2) - F(\tau_2; t_1, x_1)| . \end{aligned}$$

If $t_2 \leqslant \tau_1 \leqslant 0$, then

$$|F(\tau_1;\ t_1,\ x_1)| - |F(\tau_2;\ t_2,\ x_2)| \leqslant |F(\tau_1;\ t_1,\ x_1)| - |F(\tau_1;\ t_2,\ x_2)| \leqslant |F(\tau_1;\ t_1,\ x_1) - F(\tau_1;\ t_2,\ x_2)|,$$

while if $t_1 \leqslant \tau_1 < t_2$, then

$$\begin{aligned} |F(\tau_1;\ t_1,\ x_1)| - |F(\tau_2;\ t_2,\ x_2)| &\leqslant |F(\tau_1;\ t_1,\ x_1)| - |F(t_2;\ t_2,\ x_2)| \\ &= |F(\tau_1;\ t_1,\ x_1)| - |x_2| \\ &\leqslant |F(\tau_1;\ t_1,\ x_1) - x_2| \\ &\leqslant |F(\tau_1;\ t_1,\ x_1) - x_1| + |x_1 - x_2|. \end{aligned}$$

Thus V_1 is C_0 at $(t,\ x)$.

If $t = 0$, parts of the above analysis give the desired result. Thus V_1 is C_0 on $R \times \Omega$.

If, in addition, V_1 is positive definite, then $\theta(t)$ is stable by Liapunov's original stability theorem [H]. Conversely, if $\theta(t)$ is stable, then we may choose $\gamma(t) = \gamma(0)$ to be constant for $t \geqslant 0$, and $\Phi(t,\ r) = \Phi(0,\ r)$ to be independent of t for $t \geqslant 0$. Let $\varepsilon > 0$. Choose $\delta = \delta(\varepsilon,\ 0) > 0$ so that $|x_0| < \delta$ implies $|F(t;\ 0,\ x_0)| < \varepsilon$ for $t \geqslant 0$ by stability. Let $|x| \geqslant \varepsilon$. Then if $V_1(t,\ x) < \Phi(0,\ \delta)$ for some $t \geqslant 0$, we have $\Phi(0,\ |F(0;\ t,\ x)|) < \Phi(0,\ \delta)$ which implies $|F(0;\ t,\ x)| < \delta$ so that

$$|F(\tau;\ 0,\ F(0;\ t,\ x))| < \varepsilon$$

by stability for $\tau \geqslant 0$, a contradiction at $\tau = t$. Therefore, $V_1(t,\ x) \geqslant \Phi(0,\ \delta)$ for all $t \geqslant 0$, proving that V_1 is positive definite on $[0,\ \infty) \times \Omega$. But on $(-\infty,\ 0] \times \Omega$,

$$V_1(t,\ x) \geqslant \Phi(0,\ |x|),$$

so that V_1 is always positive definite on $(-\infty,\ 0] \times \Omega$, regardless of the stability of $\theta(t)$. Thus V_1 is positive definite on $R \times \Omega$, and the proof for V_1 is complete.

With $\gamma(t)$ and $\Phi(t,\ r)$ defined as before, we put

$$V_2(t,\ x) = \begin{cases} \inf\limits_{0 \leq \tau \leq t} \Phi(0,\ |F(\tau;\ t,\ x)|), & \text{if } t \geqslant 0, \\ \sup\limits_{t \leq \tau \leq 0} \Phi(0,\ |F(\tau;\ t,\ x)|), & \text{if } t < 0. \end{cases}$$

Repetitions of the arguments made for V_1 yield that V_2 is C_0, non-negative, and satisfies (a) and (b). If V_2 is positive definite and decrescent, $\theta(t)$ is uniformly stable [H]. Finally, let $\theta(t)$ be uniformly stable. For $t \geqslant 0$, $V_2(t, x) \leqslant \Phi(0, |x|)$, hence V_2 is decrescent on $[0, \infty) \times \Omega$. Let $\varepsilon > 0$. Choose $\delta = \delta(\varepsilon) > 0$ by uniform stability. Let $|x| \geqslant \varepsilon$. If $V_2(t, x) < \Phi(0, \delta)$ for some $t \geqslant 0$, then

$$|F(\tau;\ t,\ x)| < \delta$$

for some $0 \leqslant \tau \leqslant t$, hence

$$|F(s;\ \tau,\ F(\tau;\ t,\ x))| < \varepsilon$$

for all $s \geqslant \tau$ by uniform stability, a contradiction at $s = t$. Thus $V_2(t, x) \geqslant \Phi(0, \delta)$ for $t \geqslant 0$ proving V_2 is positive definite on $[0, \infty) \times \Omega$. For $t < 0$, V_2 is positive definite as was V_1. If $\varepsilon > 0$, choose $\delta = \delta(\varepsilon) < \gamma(0)/2$ by uniform stability so that if $|x| < \delta$, we have $V_2(t, x) \leqslant \Phi(0, \varepsilon)$, proving V_2 is decrescent on $(-\infty, 0] \times \Omega$, and completing the proof.

PROOF OF THEOREM 2. — Suppose there exists a non-negative C_0 function $U(t, x)$ satisfying (a′) and (b′). Assume that for some (t_0, x_0) in $R \times \Omega$, $F(t;\ t_0,\ x_0)$ fails to exist forever. Then either $\omega < +\infty$ and

$$\rho[F(t;\ t_0,\ x_0)] \rightarrow \infty \text{ as } t \rightarrow \omega^-, \tag{3.1}$$

or $\alpha > -\infty$ and

$$\rho[F(t;\ t_0,\ x_0)] \rightarrow \infty \text{ as } t \rightarrow \alpha^+, \tag{3.2}$$

Suppose (3.1) holds. Then $U(s,\ F(t;\ t_0,\ x_0)) \rightarrow \infty$ as $t \rightarrow \omega^-$ uniformly for s in $[t_0, \omega]$, hence by (b′)

$$U(t_0,\ x_0) \equiv U(t,\ F(t;\ t_0,\ x_0)) \rightarrow \infty \text{ as } t \rightarrow \omega^-,$$

a contradiction. A similar contradiction results if (3.2) holds.

Conversely, suppose all solutions exist forever. Define

$$U(t,\ x) = \rho[F(0;\ t,\ x)]$$

for (t, x) in $R \times \Omega$. Clearly, U is non-negative and C_0. If h is real, $F(0;\ t + h,\ F(t + h)) = F(0;\ t,\ F(t))$, hence U satisfies (b′). If U does not satisfy (a′), then there exist $T > 0$, $M > 0$, and a sequence $\{(t_n, x_n)\}$ in $R \times \Omega$ such that $-T \leqslant t_n \leqslant T$, $\rho(x_n) \rightarrow \infty$, and $U(t_n, x_n) \leqslant M$. Assume without loss of generality that $t_n \rightarrow t_0$ and $F(0;\ t_n,\ x_n) \rightarrow y_0$. Then $y_0 \in \Omega$. Therefore, $F(t;\ 0,\ y_0)$ exists for all real t and by continuous dependence

$$F(t;\ 0,\ F(0;\ t_n,\ x_n)) \rightarrow F(t;\ 0,\ y_0)$$

as $n \to \infty$ uniformly in t for $-T \leqslant t \leqslant T$.

Thus

$$|x_n - F(t_0;\ 0,\ y_0)| \leqslant |F(t_n;\ 0,\ F(0;\ t_n,\ x_n)) - F(t_n;\ 0,\ y_0))| + \\ + |F(t_n;\ 0,\ y_0) - F(t_0;\ 0,\ y_0)| \to 0$$

as $n \to \infty$, contradicting $\rho(x_n) \to \infty$. Thus U satisfies (a').

To prove the boundedness statement, suppose first that U is radially unbounded. If the solutions are not equi-bounded, then there exist $M > 0$, t_0 real, and sequences $\{t_n\}$ and $\{x_n\}$, where t_n is real and $\rho(x_n) \leqslant M$, for which

$$\rho[F(t_n;\ t_0,\ x_n)] \to \infty \text{ as } n \to \infty.$$

Thus $U(t,\ F(t_n;\ t_0,\ x_n)) \to \infty$ as $n \to \infty$, uniformly in t for t in R. Hence

$$U(t_0,\ x_n) \equiv U(t_n;\ F(t_n;\ t_0,\ x_n)) \to \infty \text{ as } n \to \infty,$$

a contradiction because U is bounded on the set $\{t_0\} \times \{x\colon \rho(x) \leqslant M\}$.

Conversely, suppose that all solutions are equi-bounded. If U is not radially unbounded, thére exist $P > 0$ and sequences $\{t_n\}$ and $\{x_n\}$, where t_n is real and $\rho(x_n) \to \infty$, for which

$$U(t_n,\ x_n) \leqslant P.$$

Then

$$\rho[F(t;\ 0,\ F(0;\ t_n,\ x_n))] \leqslant \beta(0,\ P)$$

for all t in R by (2.4), a contradiction at $t = t_n$ for large enough n, proving the boundedness result.

The proof of the existence of $\theta(t)$ is trivial and the proof of the stability of $\theta(t)$ is nearly identical to its analog in the proof of Theorem 1. Theorem 2 is now proved.

PROOF OF THEOREM 3. — Suppose there exists a non-negative C_0 function V satisfying (a') and (b). If, for some $(t_0,\ x_0)$, $F(t;\ t_0,\ x_0)$ does not exist for all $t \geqslant t_0$, then $\omega < +\infty$ and $\rho[F(t;\ t_0,\ x_0)] \to \infty$ as $t \to \omega^-$, hence as before,

$$V(t_0,\ x_0) \geqslant V(t,\ F(t;\ t_0,\ x_0)) \to \infty$$

as $t \to \omega^-$, a contradiction.

Conversely, suppose all solutions exist in the future. Define

$$V(t, x) = \begin{cases} \inf\limits_{0 \leq \tau \leq t} \rho[F(\tau;\ t,\ x)] & \text{if } t > 0, \\ \sup\limits_{t \leq \tau \leq 0} \rho[F(\tau;\ t,\ x)] & \text{if } t \leqslant 0. \end{cases}$$

Clearly, V is non-negative and satisfies (b). Suppose (a′) does not hold. Then there exist $T > 0$, $M > 0$, and sequences $\{t_n\}$ and $\{x_n\}$, where $-T \leqslant t_n \leqslant T$ and $\rho(x_n) \to \infty$, for which $V(t_n, x_n) \leqslant M$. Since $t \leqslant 0$ implies $V(t, x) \geqslant \rho(x)$, we must have $t_n > 0$ for all sufficiently large n. Thus there is another sequence $\{\tau_n\}$ with $0 \leqslant \tau_n \leqslant t_n$ such that

$$\rho[F(\tau_n;\ t_n,\ x_n)] \leqslant 2M.$$

Assume without loss that $t_n \to t_0 \geqslant 0$, $\tau_n \to \tau_0$, and $F(\tau_n;\ t_n,\ x_n) \to y_0$. Then $0 \leqslant \tau_0 \leqslant t_0$ and $y_0 \in \Omega$. By hypothesis, $F(t;\ \tau_0,\ y_0)$ exists on $[\tau_0, \infty)$. Also, there exists $\eta > 0$ such that $F(t;\ \tau_0,\ y_0)$ exists on $[\tau_0 - \eta, \infty)$. Therefore,

$$F(s;\ \tau_n,\ F(\tau_n;\ t_n,\ x_n)) \to F(s;\ \tau_0,\ y_0)$$

uniformly in s for s in $[\tau_0 - \eta, 2T]$. For large n, $\tau_0 - \eta < t_n$, hence

$$\begin{aligned} |x_n - F(t_0;\ \tau_0,\ y_0)| \leqslant |F(t_n;\ \tau_n,\ F(\tau_n;\ t_n,\ x_n)) - F(t_n;\ \tau_0,\ y_0)| + \\ + |F(t_n;\ \tau_0,\ y_0) - F(t_0;\ \tau_0,\ y_0)| \to 0 \end{aligned}$$

as $n \to \infty$, contradicting $\rho(x_n) \to \infty$, proving (a′).

To establish that V is C_0, we first note that V is C_0 on $(-\infty, 0) \times \Omega$ by the same argument as in the proof of Theorem 1. If $t \geqslant 0$, $x \in \Omega$, and $F(\tau;\ t,\ x)$ exists on $0 \leqslant \tau \leqslant t$, then a simple modification of the argument used in the proof of Theorem 1 gives the result. Thus let $t > 0$ and assume that $F(\tau;\ t,\ x)$ does not exist on $0 \leqslant \tau \leqslant t$. Then $\alpha \geqslant 0$ and

$$\rho[F(\tau;\ t,\ x)] \to \infty \text{ as } t \to \alpha^+.$$

Choose σ so that $\alpha < \sigma < t$ and

$$\rho[F(\tau;\ t,\ x)] > \rho(x)$$

for all $\alpha < \tau \leqslant \sigma$. We claim that there is a neighborhood N of (t, x) such that

$$V(t', x') = \min_{\sigma \leq \tau \leq t+1} \rho[F(\tau;\ t',\ x')]$$

for every (t', x') in N. (This is not immediate because $F(\tau;\ t',\ x')$ might exist on $[0, t']$.) Suppose not. Then there exists a sequence $(t_n, x_n) \to (t, x)$ such that

$$V(t_n, x_n) = \rho[F(\tau_n; t_n, x_n)] \leqslant \rho(x_n),$$

where $\tau_n < \sigma$. Assume without loss that $F(\tau_n; t_n, x_n) \to y_0$ and $\tau_n \to \tau_0 \leqslant \sigma$. Then $y_0 \in \Omega$ because $\rho(y_0) \leqslant \rho(x)$. By hypothesis, $F(\tau; \tau_0, y_0)$ exists on $[\tau_0, \infty)$, hence

$$F(t; \tau_n, F(\tau_n; t_n, x_n)) \to F(t; \tau_0, y_0),$$

where t was chosen previously. In addition

$$F(t; \tau_n, F(\tau_n, t_n, x_n)) = F(t, t_n, x_n) \to F(t, t, x) = x$$

by continuous dependence. Thus $F(t; \tau_0, y_0) = x$, implying $F(\tau; \tau_0, y_0) \equiv F(\tau; t, x)$ for $\tau_0 \leqslant \tau \leqslant t$. But

$$\rho[F(\tau_0; \tau_0, y_0)] = \rho(y_0) \leqslant \rho(x),$$

while also $\rho[F(\tau_0; t, x)] > \rho(x)$ because $\tau_0 \leqslant \sigma$, a contradiction. Thus our claim is proved. In view of the claim, the C_0 property of V can again be proved using the arguments of the proof of Theorem 1.

The arguments relating to the existence and stability of $\theta(t)$ are the same as those used to prove the analogous results in Theorem 1. It remains only to prove the statement on boundedness.

Let V be radially unbounded and bounded on $R \times K$ for each compact subset K of Ω. If the solutions are not uniformly bounded in the future, there exist $M > 0$ and sequences $\{t_n\}$, $\{\tau_n\}$, and $\{x_n\}$ such that t_n is real, $t_n \leqslant \tau_n$, $\rho(x_n) \leqslant M$, and $\rho[F(\tau_n; t_n, x_n)] \to \infty$. Thus $V(t, F(\tau_n; t_n, x_n)) \to \infty$ uniformly in t for t in R, hence

$$V(t_n, x_n) \geqslant V(\tau_n, F(\tau_n; t_n, x_n)) \to \infty,$$

a contradiction to the boundedness of V on the set $R \times \{x: \rho(x) \leqslant M\}$.

Conversely, let the solutions be uniformly bounded in the future. First let $t \leqslant 0$. Then $V(t, x) \geqslant \rho(x)$ so that V is radially unbounded on $(-\infty, 0] \times \Omega$. If $M > 0$, choose $\beta(M) > 0$ by (2.4), hence for $\rho(x) \leqslant M$, we have

$$V(t, x) = \sup_{t \leq \tau \leq 0} \rho[F(\tau; t, x)] \leqslant \beta(M),$$

hence V is bounded on $(-\infty, 0] \times \{x: \rho(x) \leqslant M\}$.

Now let $t > 0$. Since $V(t, x) \leqslant \rho(x)$, the boundedness property holds. If V is not radially unbounded on $[0, \infty) \times \Omega$, then there exist $P > 0$ and sequences $\{t_n\}$, $\{\tau_n\}$, and $\{x_n\}$ where $0 \leqslant \tau_n \leqslant t_n$ and $\rho(x_n) \to \infty$ such that

$$\rho[F(\tau_n;\ t_n,\ x_n)] \leqslant P.$$

Choose $\beta(P) > 0$ by (2.4), then

$$\rho[F(s;\ \tau,\ F(\tau_n;\ t_n,\ x_n))] \leqslant \beta(P)$$

for all $s \geqslant \tau \geqslant 0$. If we choose $\tau = \tau_n$ and $s = t_n$, then the above becomes $\rho(x_n) \leqslant \beta(P)$, a contradiction. This completes the proof of Theorem 3.

Proof of Proposition 1. — If $\Omega = R^n$ so that $\partial\Omega$ is empty, choose $\rho(x) = |x|$. Clearly, ρ has all the desired properties.

If $\Omega \neq R^n$, choose

$$\rho(x) = |x| \left[1 + \frac{1}{d(x,\ \partial\Omega)}\right].$$

Then $\rho(x) \geqslant 0$ for all $x \in \Omega$, and $\rho(x) \to \infty$ when either $|x| \to \infty$ or $d(x,\ \partial\Omega) \to 0$. Clearly, $\rho(x) = 0$ iff $x = 0$. To prove that ρ is C_0 on Ω, we first note that if x and y belong to Ω, then for every z in $\partial\Omega$,

$$d(x,\ \partial\Omega) \leqslant |x - z| \leqslant |x - y| + |y - z|,$$

hence

$$d(x,\ \partial\Omega) - d(y,\ \partial\Omega) \leqslant |x - y|.$$

Reversing the roles of x and y, we have

$$|d(x,\ \partial\Omega) - d(y,\ \partial\Omega)| \leqslant |x - y|.$$

Now let $x \in \Omega$ and choose a compact neighborhood N of x so that $N \subset \Omega$. Thus there exist $P > 0$ and $Q > 0$ such that $y \in N$ imply $|y| \leqslant P$ and $Q \leqslant d(y,\ \partial\Omega) \leqslant P$. Thus

$$|\rho(x) - \rho(y)| \leqslant ||x| - |y|| + \frac{||x|\,d(y,\ \partial\Omega) - |y|\,d(x,\ \partial\Omega)|}{d(x,\ \partial\Omega) \cdot d(y,\ \partial\Omega)}$$

$$\leqslant (1 + 2PQ^{-2})\,|x - y|,$$

proving Proposition 1.

Proposition 2 is proved in [Y, p. 3].

4. - Examples.

Consider the scalar equation

$$x' = \begin{cases} 0 \text{ if } x \geqslant 1 \text{ or } x \leqslant 0, \\ x(x-1) \text{ if } 0 < x < 1. \end{cases} \tag{4.1}$$

Then the zero solntion $\theta(t)$ is uniformly stable for (4.1). By the proof of Theorem 1, we may choose $\Phi(t, r) = \Phi(r)$ to be independent of t. For each x, $0 < x < 1$, $|F(0; t, x)| \to 1$ as $t \to \infty$. Thus

$$V_1(t, x) = \Phi(|F(0; t, x)|) \to \Phi(1)$$

as $t \to \infty$, hence V_1 is not decrescent. This show that V_1 cannot be used as a test for the uniform stability of $\theta(t)$.

Consider the scalar equation

$$x' = \frac{g'(t)}{g(t)} x, \tag{4.2}$$

where g is C^1, $g(2n+1) = (2n+1)^{-1}$ and $g(2n) = 1$ for $n = 0, 1, 2, \ldots$, $g(t) = 1$ for $t \leqslant 0$, and g is monotone between any two consecutive integers. The solution of (4.2) is

$$F(t; t_0, x_0) = g(t) x_0 [g(t_0)]^{-1},$$

and the zero solution $\theta(t)$ is stable for (4.2) because (4.2) is linear and g is bounded. (However, $\theta(t)$ is not uniformly stable because $g(2n)[g(2n-1)]^{-1} = 2n-1$). Again, we may choose Φ independent of t. Then

$$V_2(2n, x) = \inf_{0 \leq \tau \leq 2n} \Phi(|x| g(\tau) [g(2n)]^{-1})$$
$$= \Phi\left(\frac{|x|}{2n-1}\right) \to 0$$

as $n \to \infty$, hence V_2 is not positive definite. This show that V_2 cannot be used as a test for the stability of $\theta(t)$.

Indeed, we do not know whether there exists *one* Liapunov function satisfying (a) and (b) in Theorem 1 which will serve as a test for *both* the stability and uniform stability of $\theta(t)$.

We remarked in Section 2 that if all solutions exist in the future, then V characterizes the uniform stability of $\theta(t)$, while $V + V_1$ characterizes the stability of $\theta(t)$. However, we do not know whether it is possible to characterize the uniform stability of $\theta(t)$ with a function constructed from the existence forever of all solutions. Certainly, U is not such a function because all solutions of (4.1) exist forever and $\rho(x) = |x|$, yet for $0 < x < 1$

$$U(t, x) = |F(0; t, x)| \to 1$$

as $t \to \infty$, so that U fails to be decrescent. This, then, remains an open problem.

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