Criterion of Periodicity of Solutions of a Certain Differential Equation with a Periodic Coefficient.

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Summary. - *In this paper the differential equation* (1) $y'' = q(t)y$ *is considered where* $q(t)$ *is a real continuous function with period* π. *There is proved a necessary and sufficient condition for the stability of the trivial solution of Equation* (1) *when the zeros of the characteristic equation* $\lambda^2 - A\lambda + 1 = 0$, *coincide. Moreover, there is shown the construction of all Equations* (1) *admitting only periodic or half-periodic solutions with period* π.

1. - Let us consider the differential equation

$$y'' = q(t)y, \tag{1}$$

where $q(t)$ is a real continuous periodic function with period π, defined on the entire interval $(-\infty, \infty)$. Denote by $C^n(i)$ the set of all continuous functions defined on the interval i and having, on the latter, continuous derivatives up to and including the order n. For brevity, set $C^n((-\infty, \infty)) = C^n$. By a solution of Equation (1) on the interval i we understand every function $y(t) \in C^2(i)$ complying, for all $t \in i$, with Equation (1). When speaking only about solutions of Equation (1) we mean solutions on the interval $(-\infty, \infty)$. The trivial identically zero solution is generally excluded from our considerations. A function $f(t)$ satisfying the relation $f(t + d) = -f(t)$, $d > 0$, is called half-periodic, with period d.

Let $u(t)$ and $v(t)$ be two linearly independent solutions of Equation (1), such that $u(0) = 0$, $u'(0) = 1$, $v(0) = 1$, $v'(0) = 0$. Their Wronskian $w = uv' - u'v$ is equal to -1. By Floquet's theory (see, e.g., [2] or [3]), we shall associate with Equation (1) the so called characteristic (fundamental) equation

$$\lambda^2 - A\lambda + 1 = 0, \tag{2}$$

where $A = u'(\pi) + v(\pi)$. If λ_1 and λ_2 are the zeros of the characteristic equation (2) ($\lambda_1\lambda_2 = 1 \neq 0$) and ρ_1 as well as ρ_2 comply with $\lambda_1 = e^{\pi\rho_1}$, $\lambda_2{}^{\pi\rho_2}$, then there exist two linearly independent solutions y_1 and y_2 of Equation (1), for which there holds either

$$y_1(t + \pi) = \lambda_1 y_1(t), \quad y_2(t + \pi) = \lambda_2 y_2(t), \tag{3}$$

$$y_1(t) = e^{t\rho_1} p_1(t), \qquad y_2(t) = e^{t\rho_2} p_2(t), \tag{3'}$$

or

$$y_1(t+\pi) = \lambda_1 y_1(t), \quad y_2(t+\pi) = y_1(t) + \lambda_1 y_2(t), \tag{4}$$

$$y_1(t) = e^{t\rho_1} p_1(t), \qquad y_2(t) = e^{t\rho_1}[p_2(t) + ctp_1(t)] \tag{4'}$$

for all the t, where $p_1 \in C^2$, $p_2 \in C^2$ are periodic, generally complex functions with period π. The forms (4) and (4') may (but need not) occur only for $\lambda_1 = \lambda_2$, hence for $|A| = 2$.

If we restrict our considerations to the real domain, then:

for $A > 2$, there exist two independent solutions of Equation (1) in the above forms even when ρ_1 and ρ_2 are required to be real numbers and p_1 and p_2 real functions;

for $A < -2$ there exist, in the above forms, two independent solutions of Equation (1), where $\rho_1 = \ln|\lambda_1|$, $\rho_2 = \ln|\lambda_2|$ and $p_1 \in C^2$, $p_2 \in C^2$ are real half-periodic functions with period π.

It is known, if $|A| > 2$, no trivial solution of Equation (1) is bounded (on the interval $(-\infty, \infty)$).

If $|A| < 2$, then every solution of Equation (1) is bounded, see, e.g., [1] p. 124. Moreover, noting by $e^{\pm a\pi i}$, $0 < a < 1$, the zeros of Equation (2), the general solution of Equation (1) is of the form

$$y(t; k_1, k_2) = k_1 \frac{\sin[P(t) + (2n+a)t + k_2]}{\sqrt{|P'(t) + 2n + a|}}, \tag{5}$$

where $P(t) \in C^3$ is a real function, n integer, $P'(t) + 2n + a \neq 0$. Furthermore, all Equations (1) having, for the zeros of their characteristic equation, the numbers $e^{\pm a\pi i}(0 < a < 1)$ or, in other words, all Equations (1) admitting only of bounded solutions which are neither periodic nor half-periodic with period π are expressed by

$$y'' = \left[-\frac{1}{2}\left(\frac{P''(t)}{P'(t) + 2n + a}\right)' + \frac{1}{4}\left(\frac{P''(t)}{P'(t) + 2n + a}\right)^2 - (P'(t) + 2n + a)^2\right] y,$$

where $P(t)$ and n, or $P(t)$, n and a satisfy the above requirements. For proof, see [4].

If $|A| = 2$, then there occur two cases: α) every solution of Equation (1) is bounded and then periodic or half-periodic with period π. This occur in case of the forms (3), (3') and when $A = 2$ or $A = -2$.

β) only the solution $y_1(t)$ and the solutions linearly dependent on it are bounded and therefore periodic or half-periodic with period π, whereas the other non-trivial solutions are unbounded. This occurs in case of the forms (4), (4′) and when $A = 2$ or $A = -2$.

In the present paper we have proved a necessary and sufficient condition indicating which of the mentioned cases, α) or β), occurs i.e. whether, in case that the zeros of the characteristic equation (2) coincide, every solution of Equation (1) is bounded or not. We have, moreover, shown the construction of all Equations (1) admitting only of periodic or half-periodic solutions with period π. This together with the results of paper [4], yields a survey of all Equations (1) which admit only of bounded solutions.

In what follows we shall only consider real numbers and real functions.

2. - Assume $A = 2$, $(A = -2)$, that is to say, the characteristic equation (2) of Equation (1) to have equal zeros: $\lambda_1 = \lambda_2 = 1$, (-1). Both in the case α), and in the case β) Equation (1) has at least one periodic (half-periodic) solution $y_1(t)$ with period π. Equation (1) has two independent solutions of the form (3) or (3′) (i.e. case α)) exactly when two arbitrary independent solutions of Equation (1) are periodic (half-periodic) functions with period π. But when at least one solution independent of $y_1(t)$ is not periodic (half-periodic) with period π, then every solution independent of $y_1(t)$ is no longer periodic (half-periodic) and is unbounded both on the interval $(-\infty, b)$ and (b, ∞). Let us now find out when there occurs α) and when β).

The zeros of the solution $y_1(t)$ may only be isolated points. Let $y_1(t_0) \neq 0$. Then there is on the interval $[t_0, t_0 + \pi]$ at most a finite number of zeros of the solution $y_1(t)$. Let us denote them by $a_1 < a_2 < \ldots < a_n$, $n \geq 0$ integer. Set, moreover, $a_0 = t_0$, $a_{n+1} = t_0 + \pi$. The zeros of $y_1(t)$ are simple. On every interval j on which one has $y_1(t) \neq 0$, the function

$$y_1(t) \int_{t^*}^{t} 1/y_1^2(\sigma) d\sigma, \quad t^* \in j,$$

is a solution of Equation (1) on the interval j; this solution is independent of $y_1(t)$. Therefore both this function and its 1^{st} and 2^{nd} derivatives have limits on the left (right) at the right (left) endpoint of the interval j. Let us choose $b_0 = a_0$, $b_n = a_{n+1}$, $b_i \in (a_i, a_{i+1})$ for $i = 1, 2, \ldots, n-1$. Set furthermore

$$\bar{y}_2(t) = y_1(t) \left[\int_{b_i}^{t} 1/y_1^2(\sigma) d\sigma + c_i \right], \quad \text{for} \quad t \in (a_i, a_{i+1}),$$

$i = 0, 1, \ldots, n$, where the numbers c_i are chosen in a way to enable us to define the value of the function $\bar{y}_2(t)$ for $t = a_i$ in such a manner that $\bar{y}_2(t)$

be continuous, even with its second derivative, on the interval $[t_0, t_0 + \pi]$. (That the numbers c_i may be chosen in such a way will be seen from the following considerations.) Thus $\bar{y}_2(t)$ coincides, on the interval $[t_0, t_0 + \pi]$ with some solution (let us say $y_2^*(t)$) of Equation (1). The solutions $y_1(t)$ and $y_2^*(t)$ are linearly independent and therefore we are to find out whether $y_2^*(t)$ is periodic (half-periodic) with period π or not. Since $q(t)$ is periodic with period π and $y_2^*(t)$ is a solution of Equation (1), even the function $y_2^*(t + \pi)$ is a solution of Equation (1). To ascertain whether $y_2^*(t)$ is periodic (half-periodic) with period π it is necessary and sufficient to find out whether $\bar{y}_2(t_0 + \pi) = \bar{y}_2(t_0)$, $\bar{y}'_2(t + \pi) = \bar{y}'_2(t_0)$ $(y_2(t_0 + \pi) = -y_2(t_0)$, $\bar{y}'_2(t_0 + \pi) = -\bar{y}_2(t_0))$.

Suppose $n \geq 1$. The function $\bar{y}_2(t)$ will be continuous, even with its second derivative, on the interval $[t_0, t_0 + \pi]$ exactly if, for every i, $1 \leq i \leq n$, there holds:

$$\lim_{t \to a_{i-}} \bar{y}_2(t) = \lim_{t \to a_{i+}} \bar{y}_2(t), \tag{6}$$

$$\lim_{t \to a_{i-}} \bar{y}'_2(t) = \lim_{t \to a_{i+}} \bar{y}'_2(t), \tag{7}$$

$$\lim_{t \to a_{i-}} \bar{y}''_2(t) = \lim_{t \to a_{i+}} \bar{y}''_2(t). \tag{8}$$

If (6) is true, then even (8) applies, since for $t \neq a_i$ one has $\bar{y}''_2(t) = q(t)\bar{y}_2(t)$ and the function $q(t)$ is continuous. Since $\bar{y}_2(t)$ is always, on the intervals (a_i, a_{i+1}), a solution of Equation (1), all the above limits exist. Let us consider the relation (6):

$$\lim_{t \to a_{i-}} y_1(t)\left[\int_{b_{i-1}}^{t} y_1^{-2}(\sigma)d\sigma + c_{i-1}\right] - \lim_{t \to a_{i+}} y_1(t)\left[\int_{b_i}^{t} y_1^{-2}(\sigma)d\sigma + c_i\right] =$$

$$= \lim_{t \to a_{i-}} y_1(t) \int_{b_{i-1}}^{t} y_1^{-2}(\sigma)d\sigma - \lim_{t \to a_{i+}} y_1(t) \int_{b_i}^{t} y_1^{-2}(\sigma)d\sigma =$$

$$= \lim_{t \to a_{i-}} \frac{\int_{b_{i-1}}^{t} y_1^{-2}(\sigma)d\sigma}{1/y_1(t)} - \lim_{t \to a_{i+}} \frac{\int_{b_i}^{t} y_1^{-2}(\sigma)d\sigma}{1/y_1(t)} = -\frac{1}{y'_1(a_i)} + \frac{1}{y'_1(a_i)} = 0,$$

since $y_1(a_i) = 0$ and therefore $y'_1(a_i) \neq 0$.

Let us now determine the numbers c_i and b_i in order that the relation (7) be true. Suppose $V_{\varepsilon_i}(a_i)$ is such a deleted (i.e. not containing the number a_i)

neighbourhood of the point a_i that in $V_{\varepsilon_i}(a_i)$ there holds $y_1(t) \neq 0$. Set $f_i^*(t) = y_1(t)/\sin(t - a_i)$, for $t \in V_{\varepsilon_i}(a_i) \equiv \{t;\ 0 < |t - a_i| < \varepsilon_i\}$. Calculate $\lim_{t\to a_i} f_i^*(t) = \lim_{t\to a_i} y'_1(t)/\cos(t - a_i) = y'_1(a_i) \neq 0$. Furthermore, one has

$$\lim_{t\to a_i} f_i^{*\prime}(t) = \lim_{t\to a_i} \frac{y'_1(t)\sin(t - a_i) - y_1(t)\cos(t - a_i)}{\sin^2(t - a_i)} =$$

$$= \lim_{t\to a_i} \frac{[q(t)y_1(t) + y_1(t)]\sin(t - a_i)}{2\sin(t - a_i)\cos(t - a_i)} = 0.$$

And for the 2^{nd} derivative we obtain:

$$\lim_{t\to a_i} f_i^{*\prime\prime}(t) = y'_1(a_i)[q(a_i) + 1]/3.$$

Hence, setting $f_i^*(a_i) = y_1'(a_i)$, one has $f_i^*(t) \in C^2((a_i - \varepsilon_i,\ a_i + \varepsilon_i))$, $f_i^{*\prime}(a_i) = 0$, $f_i^{*\prime\prime}(a_i) = 1/3\, y'_1(a_i)[q(a_i) + 1]$. Because $y_1(t) \neq 0$ for $t \in V_{\varepsilon_i}(a_i)$ and $y'_1(a_i) \neq 0$, there also holds $f_i^*(t) \neq 0$ in the entire interval $(a_i - \varepsilon_i,\ a_i + \varepsilon_i)$. The relation (7) may then be modified:

$$0 = \lim_{t\to a_{i-}} \bar{y}'_2(t) - \lim_{t\to a_{i+}} \bar{y}'_2(t) = \lim_{t\to a_{i+}} \left[y'_1(t)\left[\int_{b_{i-1}}^{t} y_1^{-2}(\sigma)d\sigma + c_{i-1}\right] + \right.$$

$$\left. + y_1^{-1}(t)\right] - \lim_{t\to a_{i+}} \left[y'_1(t)\left[\int_{b_i}^{t} y_1^{-1}(\sigma)d\sigma + c_i\right] + y_1^{-2}(t)\right] =$$

$$= \lim_{t\to a_{i-}} \left[y'_1(t)\int_{b_{i-1}}^{t}\left(\frac{1}{y_1^2(\sigma)} - \frac{1}{y_1'^2(a_i)\sin^2(\sigma - a_i)}\right)d\sigma + c_{i-1}y'_1(t) + \right.$$

$$\left. + \left(\frac{1}{y_1(t)} - \frac{y'_1(t)\operatorname{cotg}(t - a_i)}{y_1'^2(a_i)}\right)\right] + \frac{1}{y'_1(a_i)}\operatorname{cotg}(b_{i-1} - a_i) -$$

$$- \lim_{t\to a_{i+}} \left[y'_1(t)\int_{b_i}^{t}\left(\frac{1}{y_1^2(\sigma)} - \frac{1}{y_1'^2(a_i)\sin^2(\sigma - a_i)}\right)d\sigma + c_i y'_1(t) + \right.$$

$$\left. + \left(\frac{1}{y_1(t)} - \frac{y'_1(t)\operatorname{cotg}(t - a_i)}{y_1'^2(a_i)}\right)\right] - \frac{1}{y'_1(a_i)}\operatorname{cotg}(b_i - a_i).$$

However

$$\lim_{t\to a_i}\left(\frac{1}{y_1(t)} - \frac{y'_1(t)\operatorname{cotg}(t - a_i)}{y_1'^2(a_i)}\right) =$$

$$= \lim_{t \to a_i} \frac{1/f_i^*(t) - y'_1(t)/y'^2_1(a_i) \cos(t - a_i)}{\sin(t - a_i)} =$$

$$= \lim_{t \to a_i} \frac{-f_i^{*\prime}(t)/f_i^{*2}(t) + \sin(t - a_i) y'_1(t)/y'^2_1(a_i) - \cos(t - a_i) q(t) y_1(t)/y'^2_1(a_i)}{\cos(t - a_i)} = 0.$$

Therefore the relation (7) may further be modified:

$$(9_i) \qquad 0 = y'_1(a_i)[c_{i-1} - c_i] + 1/y'_1(a_i)[\operatorname{cotg}(b_{i-1} - a_i) - \operatorname{cotg}(b_i - a_i)] +$$

$$+ y'_1(a_i) \int_{b_{i-1}}^{b_i} \left(\frac{1}{y_1^2(\sigma)} - \frac{1}{y'^2_1(a_i) \sin^2(\sigma - a_i)} \right) d\sigma.$$

Thus, if $b_1, \ldots, b_{n-1}$ are within the corresponding intervals and c_0 is an arbitrary but fixed number, then $c_1, \ldots, c_n$ are uniquely defined by the relations (9_i).

Let us now find out when $\bar{y}_2(t_0 + \pi) =_{(-)}^{+} \bar{y}_2(t_0)$ and $\bar{y}'_2(t_0 + \pi) =_{(-)}^{+} \bar{y}'_2(t_0)$. (In what follows, there always holds either $+$ or $-$.)

Modifying the first postulate, we obtain $0 = y_1(t_0 + \pi)[\int_{b_n}^{t_0+\pi} y_1^{-2}(\sigma)d\sigma + c_n] \mp y_1(t_0)[\int_{b_0}^{t_0} y_1^{-2}(\sigma)d\sigma + c_0] =_{(-)}^{+} y_1(t_0)[c_n - c_0]$, since $y_1(t + \pi) =_{(-)}^{+} y_1(t)$. Modifying the second postulate yields

$$0 = \left\{ y_1(t) \left[\int_{t_0+\pi}^{t} y_1^{-2}(\sigma)\,d\sigma + c_n \right] \right\}'_{t=t_0+\pi} \underset{(+)}{-} \left\{ y_1(t) \left[\int_{t_0}^{t} y_1^{-2}(\sigma)d\sigma + c_0 \right] \right\}'_{t=t_0} =$$

$$= y'_1(t_0 + \pi)\, c_n \underset{(+)}{-} y'_1(t_0)\, c_0 = \underset{(-)}{+} y'_1(t_0)[c_n - c_0].$$

In order that both postulates be satisfied it is necessary and sufficient that $c_n = c_0$. Since $y'_1(a_i) \neq 0$ for $1, \ldots, n$, let us divide the relations (9_i) by $y'_1(a_i)$ and sum for $i = 1, \ldots, n$. We obtain

$$0 = c_0 - c_n + \sum_{i=1}^{n} \left\{ \int_{b_{i-1}}^{b_i} \left[\frac{1}{y_1^2(\sigma)} - \frac{1}{y'^2_1(a_i) \sin^2(\sigma - a_i)} \right] d\sigma + \right.$$

$$\left. + \frac{1}{y'^2_1(a_i)} [\operatorname{cotg}(b_{i-1} - a_i) - \operatorname{cotg}(b_i - a_i)] \right\}.$$

Hence $c_0 = c_n$ occurs exactly when

$$(10) \qquad 0 = \sum_{i=1}^{n} \left\{ \int_{b_{i-1}}^{b_i} \left[\frac{1}{y_1^2(\sigma)} - \frac{1}{y_1'^2(a_i) \sin^2(\sigma - a_i)} \right] d\sigma + \right.$$

$$\left. + \frac{1}{y_1'^2(a_i)} [\operatorname{cotg}(b_{i-1} - a_i) - \operatorname{cotg}(b_i - a_i)] \right\}.$$

Let us arrange the last relation into a clearer form. There holds:

$$- \int_{b_{i-1}}^{b_i} \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{y_1'^2(a_j) \sin^2(\sigma - a_j)} \right] d\sigma = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{y_1'^2(a_j)} [\operatorname{cotg}(b_i - a_j) - \operatorname{cotg}(b_{i-1} - a_j)].$$

Let us therefore set

$$(11) \qquad r(t) = \sum_{j=1}^{n} \frac{1}{y_1'^2(a_i) \sin^2(t - a_i)}$$

(in the case when $y_1(t)$ oscillates, and $r(t) \equiv 0$ when $y_1(t)$ is a non-oscillatory solution; we do so in order to simplify, formally, some of our following statements). Then

$$\sum_{i=1}^{n} \left\{ \int_{b_{i-1}}^{b_i} \left[\frac{1}{y_1^2(\sigma)} - \frac{1}{y_1'^2(a_i) \sin^2(\sigma - a_i)} \right] d\sigma + \right.$$

$$\left. + \frac{1}{y_1'^2(a_i)} [\operatorname{cotg}(b_{i-1} - a_i) - \operatorname{cotg}(b_i - a_i)] \right\} -$$

$$- \sum_{i=1}^{n} \int_{b_{i-1}}^{b_i} \left[\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{y_1'^2(a_j) \sin^2(\sigma - a_j)} \right] d\sigma -$$

$$- \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{y_1'^2(a_j)} \operatorname{cotg}(b_i - a_j) - \operatorname{cotg}(b_{i-1} - a_j)] =$$

$$= \sum_{i=1}^{n} \left\{ \int_{b_{i-1}}^{b_i} \left[\frac{1}{y_1^2(\sigma)} - r(\sigma) \right] d\sigma - \right.$$

$$-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{y_1'^2(a_j)}[\operatorname{cotg}(b_i-a_j)-\operatorname{cotg}(b_{i-1}-a_j)]=$$

$$=\int_{t_0}^{t_0+\pi}\left[\frac{1}{y_1^2(\sigma)}-r(\sigma)\right]d\sigma+R,$$

where

$$R=-\sum_{j=1}^{n}\frac{1}{y_1'^2(a_j)}\sum_{i=1}^{n}[\operatorname{cotg}(b_i-a_j)-\operatorname{cotg}(b_{i-1}-a_j)]=$$

$$=-\sum_{j=1}^{n}\frac{1}{y_1'^2(a_j)}[\operatorname{cotg}(b_n-a_j)-\operatorname{cotg}(b_0-a_j)]=$$

$$=-\sum_{j=1}^{n}\frac{1}{y_1'^2(a_j)}[\operatorname{cotg}(b_0-a_j+\pi)-\operatorname{cotg}(b_0-a_j)]=0.$$

We see that the relation (10) is equivalent to the relation

$$\int_{t_0}^{t_0+\pi}\left(\frac{1}{y_1^2(\sigma)}-r(\sigma)\right)d\sigma=0. \tag{12}$$

Both $y_1^2(t)$ and $r(t)$ are, except the points $a_i+k\pi$, periodic functions with period π and therefore, for an arbitrary b, there holds

$$\int_{t_0}^{t_0+\pi}\left[\frac{1}{y_1^2(t)}-r(t)\right]dt=\int_{b}^{b+\pi}\left[\frac{1}{y_1^2(t)}-r(t)\right]dt=\int_{0}^{\pi}\left[\frac{1}{y_1^2(t)}-r(t)\right]dt.$$

When the solution $y_1(t)$ has no zeros on the interval $[t_0, t_0+\pi]$ (i.e. $n=0$, which, of course, cannot occur for the half-periodic solution y_1), then $\bar{y}_2(t)=$ $=y_1(t)\int_{t_0}^{t}y_1^{-2}(\sigma)d\sigma$ is a solution of Equation (1), independent of $y_1(t)$ and thus defined on the entire interval $(-\infty,\infty)$. Then it is necessary, for Equation (1) to have two independent solutions of the form (3) or (3') that $0=\bar{y}_2(t_0+\pi)-$ $-\bar{y}_2(t_0)=y_1(t_0+\pi)\int_{t_0}^{t_0+\pi}y_1^{-2}(\sigma)d\sigma$. Of course, this cannot occur, since $y_1(t)\neq 0$.

Hence we come to the conclusion:

There exist two independent solutions of Equation (1) *that may be written in the form*

$$y_1=p_1(t),\ y_2=p_2(t)$$

or in the form $y_1=p_1(t)$, $y_2=p_2(t)+ctp_1(t)$, $c\neq 0$ *real*, p_1, p_2 *real periodic or* *half-periodic functions with period* π, $p_1 \in C^2$, $p_2 \in C^2$, *exactly when Equation* (1) *has a non-trivial solution* $y_1(t)$ *periodic or half-periodic with period* π. *The solution* $y_2(t)$ *is in the first of the mentioned forms exactly when* $y_1(t)$ *is oscillatory and there holds*

$$\int_0^{\pi} \left(\frac{1}{y_1^2(t)} - r(t)\right) dt = 0, \tag{13}$$

where $r(t) \sum_{i=1}^{n} = \frac{1}{y_1'^2(a_i)\sin^2(t-a_i)}$; $a_1, a_2, \ldots, a_n$, $n \geq 1$ *are all the zeros of the solution* $y_1(t)$ *on the interval* $[0, \pi)$ ($r(t) \equiv 0$ *for* $y_1(t)$ *non-oscillatory*).

Every solution of Equation (1) for $A = 2$ or $A = -2$ is bounded on the interval $(-\infty, \infty)$ and therefore periodic or half-periodic with period π exactly when (13) applies. When (13) does not apply, then there exists a non-trivial solution of Equation (1), bounded on the interval $(-\infty, \infty)$ and every solution independent of this solution is already unbounded both on $(-\infty, b)$ and (b, ∞).

If it is required that the Wronskian of the pair y_1 and y_2 be 1, then the constant c in the expression of the solution y_2 is already uniquely determined: $y_2(t+\pi) = p_2(t+\pi) + ctp_1(t+\pi) + c\pi p_1(t+\pi) = \underset{(-)}{+} p_2(t) \underset{(-)}{+} ctp_1(t) \underset{(-)}{+} c\pi p_1(t) = \underset{(-)}{+} y_2(t) \underset{(-)}{+} c\pi y_1(t)$. Thus $c = [\underset{(-)}{+} y_2(t+\pi) - y_2(t)]/(\pi y_1(t)) = [\underset{(-)}{+} y_2(t_0+\pi) - y_2(t_0)]/(\pi y_1(t_0))$. When y_1 is oscillatory, then $\underset{(-)}{+} y_2(t_0+\pi) - y_2(t_0) = \underset{(-)}{+} y_1(t_0+\pi)\left[\int_{b_n}^{t_0+\pi} y_1^{-2}(\sigma)d\sigma + c_n\right] - y_1(t_0)\left[\int_{b_0}^{t_0} y_1^{-2}(\sigma)d\sigma + c_0\right] = y_1(t_0)[c_n - c_0]$. Therefore $c = [c_n - c_0]/\pi = \frac{1}{\pi}\int_0^{\pi}[y_1^{-2}(t) - r(t)]\,dt$. When y_1 is not oscillatory, then $y_1(t) \neq 0$ and $y_2(t) = y_1(t)\int_{t_0}^{t} y_1^{-2}(\sigma)\,d\sigma + k$. Then $y_2(t_0+\pi) - y_2(t_0) = y_1(t_0+\pi)\int_{t_0}^{t_0+\pi} y_1^{-2}(\sigma)\,d\sigma$ or $c = \frac{1}{\pi}\int_0^{\pi} y_1^{-2}(t)\,dt$.

Consequently, we may state:

Equation (1) *with* $A = 2$ *or* $A = -2$ *has two independent solutions whose Wronskian is equal to* 1 *and which are of the form* $y_1 = p_1(t)$, $y_2 = p_2(t) + \frac{1}{\pi}\int_0^{\pi}[y_1^{-2}(\sigma) - r(\sigma)]\,d\sigma .\, t.p_1(t)$, *where* p_1, p_2 *are periodic or half-periodic functions with period* π, $p_1 \in C^2$, $p_2 \in C^2$ *and* $r(t)$ *is given by* (11). *Even in the form of* $y_3 = p_3(t) + \frac{1}{\pi}\int_0^{\pi}[y_1^{-2}(\sigma) - r(\sigma)]\,d\sigma .\, t.p_1(t)$ ($p_3 \in C^2$ *is periodic or half-periodic with period* π) *one may write any solution* $y_3(t)$ *of Equation* (1) *independent of* y_1 *and such that the Wronskian of* y_1 *and* y_3 *is* 1.

We can therefore deduce that, for $\int_0^\pi [y_1^{-2}(t) - r(t)]\,dt \neq 0$, the character of the behaviour of the solutions y_3 and y_4, independent of y_1, and having the same Wronskian with the latter, is, in a certain sense, the same. More precisely:

$$\lim_{m\to\infty} \frac{y_3(t+m\pi)}{m} = \lim_{m\to\infty} \frac{y_4(t+m\pi)}{m} = \left(wy_1(t)\int_0^\pi [y_1^{-2}(\sigma) - r(\sigma)]\,d\sigma\right.$$

where w is the Wronskian of the pair y_1, y_3 or y_1, y_4.

3. - In the simplest case the half-periodic solution $y_1(t)$ has exactly one zero a_1 in the interval $[t_0, t_0+\pi)$. Then the function $f_1^*(t)$ is defined on the entire interval $(-\infty, \infty)$, $f_1^* \in C^2$, $f_1^*(t) \neq 0$, $f_1^*(t+\pi) = f_1^*(t)$, $f_1^*(a_1) = y'_1(a)$, $f_1^{*\prime}(a_1) = 0$. Set $f(t) = \ln[f_1^*(t)/y'_1(a_1)]$. Then $f(t) \in C^2$, $f(t+\pi) = f(t)$, $f(a_1) = f'(a_1) = 0$. We may then write $y_1(t) = y'_1(a_1)e^{f(t)}\cdot\sin(t-a_1)$ for $t \in (-\infty, \infty)$. At the same time $r(t) = \dfrac{1}{y'^2_1(a_1)\sin^2(t-a_1)}$. Therefore:

Let Equation (1) *have a half-periodic solution* $y_1(t)$ *with period* π *and exactly one zero* a_1 *on the interval* $[0, \pi)$. *Then* $y_1(t) = y'_1(a_1)e^{f(t)}\cdot\sin(t-a_1)$, *where* $f \in C^2$, $f(t+\pi) = f(t)$, $f(a_1) = f'(a_1) = 0$. *In that case every solution of this equation is half-periodic with period* π *exactly when*

$$(14)\qquad 0 = \int_0^\pi \frac{e^{-2f(t)} - 1}{\sin^2(t-a_1)}\,dt \left(= y'^2_1(a_1)\int_0^\pi \left[\frac{1}{y_1^2(t)} - \frac{1}{y'^2_1(a_1)\sin^2(t-a_1)}\right]dt\right).$$

From the theorem on the separation of zeros it also follows that if (14) is true, every solution has, on every interval $[b, b+\pi)$, exactly one zero.

4. - The above considerations enable us directly to form all the differential equations of the mentioned type. Since, for all these types, the construction is similar, we shall carry out, in detail, only the construction of all the differential equations whose solutions are half-periodic functions with period π (i.e. such that -1 is a 2^{nd} order zero of their characteristic equation, each solution being bounded).

Let $y(t) \in C^2$ and suppose $y(t+\pi) = -y(t)$ applies. Let $a_1 < a_2 < \ldots < a_n$ be all the zeros of the function $y(t)$ on the interval $[0, \pi)$. If $y(t)$ is to be a solution of Equation (1), then, according to what has been said above, we may define the functions $f_i^*(t)$. Let us, furthermore, define $p(t) = \dfrac{y(t)}{y'(a_n)\prod_{i=1}^n \sin(t-a_i)}$, for $t \neq a_i + k\pi$, and elsewhere so that $p(t)$ be continuous. Since one has, in a suitable neighborhood of the point a_i, $y(t)/\sin(t - a_i) = f_i^*(t)$ and because $f_i^*(a_i) = y'(a_i)$, there holds:

$$p(a_n) = 1/\prod_{i=1}^{n-1} \sin(a_n - a_i) > 0;$$

moreover: $p(a_{n-1}) = \frac{y'(a_{n-1})}{y'(a_n)} / \prod_{i=1}^{n}{}' \sin(a_{n-1} - a_i) > 0$, since $\operatorname{sign} y'(a_{n-1}) = - \operatorname{sign} y'(a_n)$; the sign ' in the symbol Π denotes that the zero factor has been omitted. Analogously, $p(a_i) > 0$. Since both the numerator and the denominator in the expression for $p(t)$ are, except the points a_i, non-zero, one has $p(t) > 0$ for $t \in [0, \pi)$. The function $p(t)$ is periodic with period π and therefore $p(t) > 0$ for all the t. Because $f_i^*(t) \in C^2(V_{\varepsilon_i}(a_i + k\pi))$ and $y(t) \in C^2$, one also has $p(t) \in C^2$. Let us, therefore, set $f(t) = \ln p(t)$.

Obviously

$$y(t) = y'(a_n) \cdot \sin(t - a_1) \cdot \ldots \cdot \sin(t - a_n) \cdot e^{f(t)}. \tag{15}$$

The condition for all the solutions of Equation (1) to be bounded is

$$0 = \int_0^\pi \frac{e^{-2f(t)}/y'^2(a_n) - \sum_{i=1}^{n} 1/y'^2(a_i) \prod_{j=1}^{n,\, j\neq i} \sin^2(t - a_j)}{\prod_{i=1}^{n} \sin^2(t - a_i)} \, dt \tag{16}$$

where $\prod_{j=1}^{1,\, j\neq i} = 1$.

The function $q(t)$ in Equation (1) is then given by

$$q(t) = f''(t) + f'^2(t) + 2f'(t) \sum_{i=1}^{n} \operatorname{cotg}(t - a_i) + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq 1}}^{n} \operatorname{cotg}(t - a_i) \operatorname{cotg}(t - a_j) - n, \tag{17}$$

where $f(t) \in C^2$, $f(t + \pi) = f(t)$ and, moreover, $f(t)$ satisfies the relation (16), where $y'(a_i)$ are determined by the relation (15); the function $q(t)$ given by the formula (17) may be supplemented so that $q(t) \in C^0$. Conversely, Equation (1) with the function $q(t) \in C^0$ given by the formula (17), where $f(t)$ has the above properties, has all the solutions half-periodic with period π as well.

Particularly, for $n = 1$, one obtains $p(t) = f_1^*(t)/y'(a_1)$, or $f(t) = \ln[f_1^*(t)/y'(a_1)]$. Thus $f(a_1) = \ln p(a_1) = 0$ and $f'(a_1) = 0$. Then $y(t) = y'(a_1) e^{f(t)} \cdot \sin(t - a_1)$. Consequently:

All the differential equations whose solutions are half-periodic with period π and such that every solution has, on every interval $[b, b+\pi)$, exactly one zero, are given by the formula

$$y'' = [f''(t) + f'^2(t) + 2f'(t) \cdot \operatorname{cotg}(t - a_1) - 1] \cdot y, \tag{18}$$

where $f \in C^2$, $f(t+\pi) = f(t)$, $f(a_1) = f'(a_1) = 0$,

$$\int_0^\pi \frac{e^{-2f(t)} - 1}{\sin^2(t - a_1)} dt = 0.$$

Setting, for example,

$f(t) \equiv 0$, one obtains the differential equation $y'' = -y$;

$f(t) \equiv -1/2 \ln[1 - k \cdot \sin 2(t-c) \cdot \sin^2(t-c)]$, where $|k| < 1$, c arbitrary, we get the differential equation

$$y'' = \left\{ 3k \frac{\sin 4(t-c) + k \cdot \sin^4(t-c)}{[1 - k \cdot \sin 2(t-c) \sin^2(t-c)]^2} - 1 \right\} y. \tag{19}$$

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