# The Ricci identity.

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**Summary.** - *The Ricci identities have been established by C.I. Ispas* [1] (1) *and H. Rund* [2] *in a Finsler space. Here we shall discuss some identities based on the principle of mathematical induction* [3], [4].

## Introduction.

**1.** - For a covariant vector $X^i(x, \dot{x})$ which is homogeneous of degree zero in it's directional argument $\dot{x}^i$, we have [2]

$$DX^i = \left\{ F \frac{\partial X^i}{\partial \dot{x}^h} + A^i_{kh}(x, \dot{x}) X^k \right\} Dl^h + X^i_{|h}\, dx^h \text{ (2), (3)}, \tag{1.1}$$

where

$$X^i_{|h} \overset{\text{def}}{=} \frac{\partial X^i}{\partial x^h} - \frac{\partial X^i}{\partial \dot{x}^k} \frac{\partial G^k}{\partial \dot{x}^h} + \Gamma^{*\,i}_{kh}(x, \dot{x}) X^k \tag{1.2}$$

and $l^i$ is a unit vector in the direction of the element of support and

$$A^i_{kh}(x, \dot{x}) \overset{\text{def}}{=} F(x, \dot{x})\, C^i_{kh}(x, \dot{x}), \tag{1.3a}$$

$$2G^i(x, \dot{x}) \overset{\text{def}}{=} Y^i_{kh}(x, \dot{x})\ \dot{x}^h \dot{x}^k. \tag{1.3b}$$

Furthermore, we put [2]

$$X^i|_h \overset{\text{def}}{=} F \frac{\partial X^i}{\partial \dot{x}^h} + A^i_{kh}(x, \dot{x}) X^k. \tag{1.4}$$

We get the commutation formulae corresponding to the repeated applications of the operation (1.4), which is given by

$$X^i|_{hk} - X^i|_{kh} = \{ F_{\dot{x}^k} X^i|_h - F_{\dot{x}^h} X^i|_k \} + S^i_{jkh}(x, \dot{x}) X^j, \tag{1.5}$$

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(1) Numbers in brackets refer to the references at the end of the paper.
(2) Greek indices run from *I* to *n*.
(3) Repeated indices always imply summation.

where

$$S^i_{jkh} \overset{\text{def}}{=} A^i_{k\gamma} A^\gamma_{jh} - A^i_{\gamma h} A^\gamma_{jk} \quad \text{and} \quad F_{\dot{x}^k} \overset{\text{def}}{=} \frac{\partial F}{\partial \dot{x}^k}. \tag{1.6}$$

This tensor is the first of Cartan's curvature tensors [2].

Secondly we get the commutation formulae involving both (1.2) and (1.4) processes

$$X^i|_{h|k} = F\left(\frac{\partial X^i}{\partial \dot{x}^h}\right)_{|k} + A^i_{h\gamma|k} X^\gamma + A^i_{h\gamma} X^\gamma_{|k} \tag{1.7}$$

and

$$X^i_{|k}|_h = F \frac{\partial}{\partial \dot{x}^h} (X^i_{|k}) + A^i_{h\gamma} X^\gamma_{|k} - A^\gamma_{hk} X^i_{|\gamma}. \tag{1.8}$$

With the help of the equations (1.7) and (1.8), we obtain

$$X^i|_{h|k} - X^i_{|k}|_h = - P^i_{jkh} X^j + X^i|_j A^j_{kh|\gamma} l^\gamma + X^i_{|j} A^j_{hk}, \tag{1.9}$$

where [2]

$$P^i_{jkh}(x, \dot{x}) \overset{\text{def}}{=} F \frac{\partial}{\partial \dot{x}^h} \Gamma^{*\,i}_{jk} + A^i_{jm} A^m_{hk|\gamma} l^\gamma - A^i_{jh|k}. \tag{1.10}$$

2. - THEOREM 2.1. - Let $A_{ij}(x, \dot{x})$ be a second order covariant tensor depending on the element of support. Then the RICCI identity for $A_{ij}(x, \dot{x})$ is given by

$$A_{ij}|_{hk} - A_{ij}|_{kh} = \{ F_{\dot{x}^k} A_{ij}|_h - F_{\dot{x}^h} A_{ij}|_k \} - A_{il} S^l_{jkh} - A_{lj} S^l_{ikh}, \tag{2.1}$$

where $S^i_{jkh}(x, \dot{x})$ is the first curvature tensor of CARTAN [2].

PROOF. - Let $X^i(x, \dot{x})$ be an arbitrary contravariant vector. The inner product of $X^i(x, \dot{x})$ and $A_{ij}(x, \dot{x})$ is a covariant vector, given by

$$T_i \overset{\text{def}}{=} A_{ij}(x, \dot{x}) X^j. \tag{2.2}$$

We know that [2]

$$T_i|_{hk} - T_i|_{kh} = \{ F_{\dot{x}^k} T_i|_h - F_{\dot{x}^h} T_i|_k \} - T_j S^j_{ikh}. \tag{2.3}$$

With the help of (2.2), the equation (2.3) reduces to the form

$$X^j [A_{ij|hk} - A_{ij|kh} - \{ F_{\dot{x}}^{\cdot k} A_{ij|h} - F_{\dot{x}}^{\cdot h} A_{ij|k} \} + A_{il} S^l_{jkh} + A_{lj} S^l_{ikh}] = 0. \tag{2.4}$$

Since $X^i(x, \dot{x})$ is an arbitrary vector, hence we have the Theorem.

THEOREM 2.2. - The RICCI identity for a covariant tensor $A_{i_1, \ldots, i_p}(x, \dot{x})$ of order $p$ is given by

$$\begin{aligned} A_{i_1, i_2, \ldots, i_p|hk} - A_{i_1, i_2, \ldots, i_p|kh} &= \{ F_{\dot{x}}^{\cdot k} A_{i_1, i_2, \ldots, i_p|h} - \\ &- F_{\dot{x}}^{\cdot h} A_{i_1, \ldots, i_p|k} \} - \sum_{\alpha=1}^{p} A_{i_1, \ldots, i_{\alpha-1}, l, i_{\alpha+1}, \ldots, i_p} S^l_{i_\alpha kh}. \end{aligned} \tag{2.5}$$

PROOF. - Let us assume that the identity is true for a covariant tensor of order $m$. Let $A_{i_1, \ldots, i_m, j}(x, \dot{x})$ be a $(m+1)$ order covariant tensor and $X^i(x, \dot{x})$ as before be an ordinary contravariant vector. The inner product of $X^i(x, \dot{x})$ and $A_{i_1, \ldots, i_m, j}(x, \dot{x})$ is given by

$$T_{i_1, \ldots, i_m} \stackrel{\text{def}}{=} A_{i_1, \ldots, i_m, j}(x, \dot{x}) X^j. \tag{2.6}$$

We have [2]

$$\begin{aligned} T_{i_1, \ldots, i_m|hk} - T_{i_1, \ldots, i_m|kh} &= \{ F_{\dot{x}}^{\cdot k} T_{i_1, \ldots, i_m|h} - \\ &- F_{\dot{x}}^{\cdot h} T_{i_1, \ldots, i_m|k} \} - \sum_{\beta=1}^{m} T_{i_1, \ldots, i_{\beta-1}, l, i_{\beta+1}, i_m} S^l_{i_\beta kh}. \end{aligned} \tag{2.7}$$

Substituting the value of $T_{i_1, \ldots, i_m}(x, \dot{x})$ from (2.6) and using the equations (1.5) and (1.6), we obtain

$$\begin{aligned} & X^j [A_{i_1, \ldots, i_m, j|hk} - A_{i_1, \ldots, i_m, j|kh} - \\ & - \{ F_{\dot{x}}^{\cdot k} A_{i_1, \ldots, i_m, j|h} - F_{\dot{x}}^{\cdot h} A_{i_1, \ldots, i_m, j|k} \} + \\ & + A_{i_1, \ldots, i_m, l} S^l_{jkh} + \sum_{\beta=1}^{m} A_{i_1, \ldots, i_{\beta-1}, l, i_{\beta+1}, i_m, j} S^l_{i_\beta kh}] = 0. \end{aligned} \tag{2.8}$$

Since $X^i(x, \dot{x})$ is an arbitrary vector, then we get from (2.8) by replacing the index $j$ by $i_{m+1}$

$$\begin{aligned} A_{i_1, \ldots, i_{m+1}|hk} - A_{i_1, \ldots, i_{m+1}|kh} &= \{ F_{\dot{x}}^{\cdot k} A_{i_1, \ldots, i_{m+1}|h} - \\ &- F_{\dot{x}}^{\cdot h} A_{i_1, \ldots, i_{m+1}|k} \} - \sum_{\alpha=1}^{m+1} A_{i_1, \ldots, i_{\alpha-1}, l, i_{\alpha+1}, \ldots, i_{m+1}} S^l_{i_\alpha hk}. \end{aligned} \tag{2.9}$$

The equation (2.9) shows that it is true for a covariant tensor of order $(m+1)$ also. Since the identity is true for a covariant vector of order two, hence it must be true for all the indices that is for order $p$ also.

Hence we have the Theorem.

THEOREM 2.3. - The RICCI identity for a mixed tensor of contravariant order $p$ and covariant order $q$ is given by

$$A^{i_1,\dots,i_p}_{j_1,\dots,j_q}|_{hk} - A^{i_1,\dots,i_p}_{j_1,\dots,j_q}|_{kh} = \{F^{\cdot k}_{\dot{x}} A^{i_1,\dots,i_p}_{j_1,\dots,j_q}|_h - F^{\cdot h}_{\dot{x}} A^{i_1,\dots,i_p}_{j_1,\dots,j_q}|_k\} + \sum_{\alpha=1}^{p} A^{i_1,\dots,i_{\alpha-1},l,i_{\alpha+1},\dots,i_p}_{j_1,\dots,j_q} S^{i_\alpha}_{lkh} - \sum_{\beta=1}^{q} A^{i_1,\dots,i_p}_{j_1,\dots,j_{\beta-1},l,j_{\beta+1},\dots,j_q} S^{l}_{j_\beta kh}. \tag{2.10}$$

PROOF. - The proof of the Theorem (2.3) follows the pattern of the proofs of Theorems (2.1) and (2.2).

3. - THEOREM 3.1. - Let $A_{ij}(x, \dot{x})$ be a second order covariant tensor depending on the element of support. The RICCI identity for this tensor is given by

$$A_{ij}|_{h|k} - A_{ij|k}|_h = A_{il} P^l_{jkh} + A_{lj} P^l_{ikh} + A_{ij}|_l A^l_{hk|\gamma} t^\gamma + A_{ij|l} A^l_{hk} \tag{3.1}$$

where $P^l_{jkh}(x, \dot{x})$ is a second curvature tensor of CARTAN [2], $t^\gamma$ is a unit vector in the direction of the element of support.

PROOF. - Let $X^i(x, \dot{x})$ be an arbitrary contravariant vector. The inner product of $X^i(x, \dot{x})$ and $A_{ij}(x, \dot{x})$ is given by (2.2). We know [2]

$$T_i|_{h|k} - T_{i|k}|_h = T_j P^j_{ikh} + T_i|_j A^j_{hk|\gamma} t^\gamma + T_{i|j} A^j_{hk}. \tag{3.2}$$

Substituting the value of $T_i(x, \dot{x})$ from (2.2) we get

$$X^j(A_{ij}|_{h|k} - A_{ij|k}|_h) + A_{ij}(X^j|_{h|k} - X^j{}_{|k}|_h) = A_{lj} P^l_{ikh} X^j + A^j_{hk}(A_{il|j} X^l + A_{il} X^l{}_{|j}) + (A_{il}|_j X^l + A_{il} X^l|_j) A^j_{hk|\gamma} t^\gamma. \tag{3.3}$$

By arranging (3.3) and using (1.9) the above equation reduces to the form

$$X^j [A_{ij}|_{h|k} - A_{ij|k}|_h - A_{lj} P^l_{ikh} - A_{il} P^l_{jkh} - A_{ij}|_l A^{l}_{hk|\gamma} t^\gamma - A_{ij|l} A^l_{hk}] = 0. \tag{3.4}$$

Since $X^j(x, \dot{x})$ is an arbitrary vector, then we get the result. Hence we have the Theorem.

THEOREM 3.2. - The RICCI identity for a covariant tensor of order $p$ is given by

$$A_{i_1, \ldots, i_p}|_{h|k} - A_{i_1, \ldots, i_p}|_{k|h} = A_{i_1, \ldots, i_p|l} A^l{}_{hk} + \sum_{\alpha=1}^{p} A_{i_1, \ldots, i_{\alpha-1}, l, i_{\alpha+1}, \ldots, i_p} P^l{}_{i_\alpha kh} + A_{i_1, \ldots, i_p}|_l A^l{}_{hk|\gamma} t^\gamma . \tag{3.5}$$

PROOF. - Let us assume that the identity is true for a covariant tensor of order $m$. Let $A_{i_1, \ldots, i_m, j}(x, \dot{x})$ be a $(m+1)$ order covariant tensor and $X^i(x, \dot{x})$ be an arbitrary contravariant vector. The inner product of $X^i(x, \dot{x})$ and $A_{i_1, \ldots, i_m, j}(x, \dot{x})$ is given by (2.6).
With the help of (1.9) we get

$$T_{i_1, \ldots, i_m}|_{h|k} - T_{i_1, \ldots, i_m}|_{k|h} = T_{i_1, \ldots, i_m|l} A^l{}_{hk} + T_{i_1, \ldots, i_m}|_l A^l{}_{hk|\gamma} t^\gamma + \sum_{\alpha=1}^{m} T_{i_1, \ldots, i_{\alpha-1}, l, i_{\alpha+1}, i_m} P^l{}_{i_\alpha kh}. \tag{3.6}$$

In view of the equations (2.6) and (3.6) we obtain

$$\begin{aligned} X^j(A_{i_1, \ldots, i_m, j}|_{h|k} - A_{i_1, \ldots, i_m, j|k|h}) + A_{i_1, \ldots, i_m, j}(X^j|_{h|k} - X^j{}_{|k|h}) = \\ = X^j(A_{i_1, \ldots, i_m, j|l} A^l{}_{hk} + A_{i_1, \ldots, i_m, j}|_l A^l{}_{hk|\gamma} t^\gamma) + \\ + A_{i_1, \ldots, i_m, j}(X^j{}_{|l} A^l{}_{hk} + X^j|_l A^l{}_{hk|\gamma} t^\gamma) + \\ + \sum_{\alpha=1}^{m} A_{i_1, i_{\alpha-1}, l, i_{\alpha+1}, \ldots, i_m, j} P^l{}_{i_\alpha kh} X^j, \end{aligned} \tag{3.7}$$

which yields the form

$$\begin{aligned} X^j[A_{i_1, \ldots, i_m, j}|_{h|k} - A_{i_1, \ldots, i_m, j|k}|_h - A_{i_1, \ldots, i_m, j}|_l A^l{}_{hk|\gamma} t^\gamma - \\ - A_{i_1, \ldots, i_m, j|l} A^l{}_{hk} - \sum_{\alpha=1}^{m} A_{i_1, \ldots, i_{\alpha-1}, l, i_{\alpha+1}, \ldots, i_m, j} P^l{}_{i_\alpha kh} - \\ - A_{i_1, \ldots, i_m, l} P^l{}_{jkh}] = 0. \end{aligned} \tag{3.8}$$

Since $X^i(x, \dot{x})$ is an arbitrary vector, we get by replacing the index $j$ by $i_{m+1}$ in the equation (3.8)

$$A_{i_1, \ldots, i_{m+1}}|_{h|k} - A_{i_1, \ldots, i_{m+1}|k|h} = A_{i_1, \ldots, i_{m+1}|l} A^l{}_{hk} + \sum_{\alpha=1}^{m+1} A_{i_1, \ldots, i_{\alpha-1}, l, i_{\alpha+1}, \ldots, i_{m+1}} P^l{}_{i_\alpha kh} + A_{i_1, \ldots, i_{m+1}}|_l A^l{}_{hk|\gamma} t^\gamma . \tag{3.9}$$

Equation (3.9) shows that the identity is true for a covariant tensor of order $(m+1)$ also. Since the identity is true for the covariant tensors of orders 2 and 3, hence it is true for the tensors of all the orders.

Hence we have the Theorem.

THEOREM 3.3. - The RICCI identity for a mixed tensor of contravariant order $p$ and covariant order $q$ is given by

$$A^{i_1,\dots,i_p}_{j_1,\dots,j_q}|_{h|k} - A^{i_1,\dots,i_p}_{j_1,\dots,j_q}|_{k|h} = A^{i_1,\dots,i_q}_{j_1,\dots,j_q}|_l A^l{}_{hk|\gamma} t^\gamma + \tag{3.10}$$
$$+ A^{i_1,\dots,i_p}_{j_1,\dots,j_q}|_l A^l{}_{hk} - \sum_{\alpha=1}^{p} A^{i_1,\dots,i_{\alpha-1},l,i_{\alpha+1},\dots,i_p}_{j_1,\dots,j_q} P^{i_\alpha}_{lkh} +$$
$$+ \sum_{\beta=1}^{q} A^{i_1,\dots,i_p}_{j_1,\dots,j_{\beta-1},l,j_{\beta+1},\dots,j_q} P^l{}_{j_\beta kh}.$$

PROOF. - The proof the Theorem (3.3) follows the pattern of the proofs of Theorems (3.1) and (3 2). Hence we have the Theorem.

4. - THEOREM 4.1. - The RICCI identity for a covariant tensor $A_{ij}(x, \dot{x})$ is

$$A_{ij|hk} - A_{ij|kh} = -A_{lj} R^l{}_{ihk} - A_{il} R^l{}_{jhk} - A_{ij}|_l K^l{}_{\gamma hk} t^\gamma, \tag{4.1}$$

where $t^\gamma$ is a unit vector in the direction of the element of support and $K^l{}_{\gamma hk}(x, \dot{x})$, $R^l{}_{\gamma hk}(x, \dot{x})$ are the Cartan's curvature tensors [2].

PROOF. - Let $X^i(x, \dot{x})$ be an arbitrary contravariant vector. The inner product of $X^i(x, \dot{x})$ and $A_{ij}(x, \dot{x})$ is given by (2.2). We know [2]

$$T_{i|hk} - T_{i|kh} = -T_j R^j{}_{ihk} - T_i|_j K^j{}_{\gamma hk} t^\gamma. \tag{4.2}$$

With the help of the equation (2.2) the equation (4.2) reduces to the form

$$X^j(A_{ij|hk} - A_{ij|kh}) + A_{ij}(X^j{}_{|hk} - X^j{}_{|kh}) = -A_{jl} X^l R^j{}_{ihk} - \tag{4.3}$$
$$- A_{il} X^l|_j K^j{}_{\gamma hk} t^\gamma - X^l A_{il}|_j K^j{}_{\gamma hk} t^\gamma.$$

Which yields

$$X^j[A_{ij|hk} - A_{ij|kh} + A_{lj} R^l{}_{ihk} + A_{il} R^l{}_{jhk} + A_{ij}|_l K^l{}_{\gamma hk} t^\gamma] = 0. \tag{4.4}$$

Since $X^j(x, \dot{x})$ is an arbitrary vector, then we get the Theorem.

THEOREM 4.2. - The RICCI identity for a covariant tensor of order $p$ is given by

$$A_{i_1, \dots, i_p|hk} - A_{i_1, \dots, i_p|kh} = - \tag{4.5}$$
$$- \sum_{\alpha=1}^{p} A_{i_1, \dots, i_{\alpha-1}, l, i_{\alpha+1}, \dots, i_p} R^l{}_{i_\alpha hk} - A_{i_1, \dots, i_p|l} K^l{}_{\gamma hk} t^\gamma .$$

where $R^l{}_{ijk}(x, \dot{x})$ is Cartan's third curvature tensor [2] and $t^\gamma$ is a unit vector in the direction of the line of support.

THEOREM 4.3. - The RICCI identity for a mixed tensor of contravariant order $p$ and covariant order $q$ is given by

$$A^{i_1, \dots, i_p}_{j_1, \dots, j_q|hk} - A^{i_1, \dots, i_p}_{j_1, \dots, j_q|kh} = \sum_{\alpha=1}^{p} A^{i_1, \dots, i_{\alpha-1}, l, i_{\alpha+1}, \dots, i_p}_{j_1, \dots, j_q} R^{i_\alpha}_{lhk} - \tag{4.6}$$
$$- \sum_{\beta=1}^{q} A^{i_1, \dots, i_p}_{j_1, \dots, j_{\beta-1}, l, j_{\beta+1}, \dots, j_q} R^l{}_{j_\beta hk} - A^{i_1, \dots, i_p}_{j_1, \dots, j_q|l} K^l{}_{\gamma hk} t^\gamma .$$

PROOF. - The proofs of Theorems (4.2) and (4.3) follow the pattern of the proofs of Theorems (3.2) and (3.3).

## REFERENCES

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