A class of hypergeometric polynomials.

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Summary. - *In this paper we study the properties of the polynomials ${}_3F_2\left[\begin{matrix} -n, n+\gamma+1, \zeta; x \\ 1+\alpha, 1+\beta \end{matrix}\right]$.*

1. - Introduction.

BATEMAN [4, 5] studied the polynomial defined by the operational formula

$$(1.1) \quad F_n(D) \operatorname{sech} z = \operatorname{sech} z P_n(\tanh z),$$

where $P_n(z)$ is the LEGENDRE polynomial defined by

$$P_n(z) = {}_2F_1\left[\begin{matrix} -n, n+1; 1-z \\ 1 \end{matrix}\right],$$

and $D = d/dz$.

BATEMAN'S polynomials are hypergeometric and can be represented by

$$(1.2) \quad F_n(z) = {}_3F_2\left[\begin{matrix} -n, n+1, \frac{1}{2}(z+1); 1 \\ 1, 1 \end{matrix}\right],$$

$n = 0, 1, 2, \dots$. These polynomials are also orthogonal and BATEMAN [5] gave the relation

$$(1.3) \quad \int_{-\infty}^{+\infty} \operatorname{sech}^2\left(\frac{1}{2}\pi x\right) F_m(ix) F_n(ix) dx = \frac{4}{2n+1} \delta_{mn},$$

where $\delta_{mn} = 0$ if $n \neq m$ and $\delta_{nn} = 1$.

Later PASTERNAK [12] generalized (1.1) and studies the polynomials defined by

$$(1.4) \quad F_n^m(D) \operatorname{sech}^{m+1} z = \operatorname{sech}^{m+1} z P_n(\tanh z).$$

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They are also hypergeometric and have the representation

$$(1.5) \quad F_n^m(z) = {}_3F_2 \left[\begin{matrix} -n, n+1, \frac{1}{2}(z+n+1) \\ 1, m+1 \end{matrix}; 1 \right].$$

Although PASTERNAK gave the three-term recurrence relation

$$(1.6) \quad (n+1)(n+m+1)F_n^m(z) + (2n+1)zF_n^m(z) = n(n-m)F_{n-1}^m(z)$$

which suggests that they are also orthogonal, he gave no orthogonality relation. However CARLITZ proved in [7] that

$$(1.7) \quad \frac{\sin \pi \beta}{2i\beta} \int_{\alpha-i\alpha}^{\alpha+i\alpha} F_n^\beta(1-\beta+2z) F_m^\beta(1-\beta+2z) \frac{dz}{\sin \pi z \sin \pi(z-\beta)} = K_n^{(\beta)} \delta_{nm},$$

where

$$K_n^{(\beta)} = \frac{(-1)^m (2_m^m) (2_n^n) (n!)^2 (1-\beta)_n}{2^{n+m} (\beta-1)_m (2n+1) \{1.3.5. \dots (2n-1)\}^2},$$

$-1 < \alpha < 0$ and $0 < \beta < 1$.

Both the BATEMAN and the PASTERNAK polynomials were generalized further by RICE [15] who studied the polynomials

$$(1.8) \quad H_n(\zeta, p, v) = {}_3F_2 \left[\begin{matrix} -n, n+1, \zeta \\ 1, p \end{matrix}; v \right]$$

and gave many of their properties. However these polynomials are not orthogonal as we shall prove in §3 unless they degenerate to the PASTERNAK polynomials ($v=1$) if they are considered as polynomials in ζ or to the JACOBI polynomials ($\zeta=p$) if they are considered as polynomials in v .

In this paper we study a certain class of hypergeometric polynomials defined by

$$(1.9) \quad G_n(\zeta, x; \alpha, \beta, \gamma) = {}_3F_2 \left[\begin{matrix} -n, \gamma+n+1, \zeta \\ \alpha+1, \beta+1 \end{matrix}; x \right] (n=0, 1, 2, \dots).$$

These polynomials generalize the BATEMAN polynomials (1.2), the PASTERNAK polynomials (1.5) and the RICE polynomials (1.8). The case $x=1$ appeared in some investigations of HAHN [9] in connection with a study of polynomials satisfying q -difference equations.

We shall first give in §2 some preliminary results and notation. In §3 we determine when the $G_n(\zeta, x; \alpha, \beta, \gamma)$ are orthogonal and in fact we shall prove that the $G_n(\zeta, 1; \alpha, \beta, \gamma)$ are the only orthogonal polynomials of the form

$$\sum_{k=0}^n \frac{(-n)_k (\gamma+n+1)_k (\zeta)_k}{k!} H(k),$$

where $H(k)$ is an arbitrary function of k .

The three-term recurrence relation satisfied by $G_n(\zeta, 1; \alpha, \beta, \gamma)$ is also given in this section.

§4 contains various contiguous relations which are satisfied by $G_n(\zeta, x; \alpha, \beta, \gamma)$, $G_n(\zeta, 1; \alpha, \beta, \gamma)$ and the general ${}_3F_2(\alpha, b, c; d, e)$.

§5 contains some operational formulas and a second proof of the three-term recurrence relations. We give in §6 various generating relations for $G_n(\zeta, x; \alpha, \beta, \gamma)$ and some of its special cases.

§7 contains some integral representations. In §8 we shall give various expansion formulas. Finally, in §9, we shall give some miscellaneous results.

2. - Preliminaries.

Let a be an arbitrary number (real or complex) and k a non-negative integer. We define

$$(2.1) \quad (a)_k = a(a+1)(a+2) \dots (a+k-1) \quad (k \geq 1), \quad (a)_0 = 1.$$

We shall make use of the JACOBI polynomial defined by

$$(2.2) \quad P_n^{(\alpha, \beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1 \left[\begin{matrix} -n, \alpha+\beta+n+1; \\ \alpha+1; \end{matrix} \frac{1}{2}(1-x) \right];$$

this reduces to the LEGENDRE polynomial when $\alpha = \beta = 0$,

$$(2.3) \quad P_n(x) = {}_2F_1 \left[\begin{matrix} -n, n+1; \\ 1; \end{matrix} \frac{1}{2}(1-x) \right].$$

The JACOBI function $Q_n^{(\alpha, \beta)}(x)$ is defined by

$$(2.4) \quad Q_n^{(\alpha, \beta)}(x) = 2^{\alpha+\beta+n} \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+2n+2)} (x-1)^{-\alpha-n-1} \\ \cdot (x-1)^{-\beta} {}_2F_1 \left[\begin{matrix} \alpha+n+1, n+1; \\ \alpha+\beta+2n+2; \end{matrix} \frac{1}{1-x} \right].$$

The LEGENDRE function $Q_n(x) = Q_n^{(0,0)}(x)$.

The associated LEGENDRE function is

$$(2.5) \quad P_n^{-m}(x) = \frac{1}{(m+1)} \left(\frac{1+x}{1-x} \right)^{-\frac{1}{2}m} {}_2F_1 \left[\begin{matrix} -n, n+1; \\ m+1; \end{matrix} \frac{1}{2}(1-x) \right] \quad (-1 < x < 1).$$

Two of the Appell's Functions of two variables that we shall employ are defined by means of

$$(2.6) \quad F_2(a; b, b'; c, c'; x, y) = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k} (b)_k (b')_n x^k y^n}{k! n! (c)_k (c')_n},$$

and

$$(2.7) \quad F_4(a, b; c, c'; x, y) = \sum_{n,k=0}^{\infty} \frac{(a)_{n+k} (b)_{n+k} x^k y^n}{k! n! (c)_k (c')_n}$$

We shall use $B(p, q)$ for the beta function defined by

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad (\operatorname{Re}(p) > 0, \operatorname{Re}(q) > 0).$$

We shall need also the following result.

THEOREM. - (SHOHAT [17]). *A necessary and sufficient condition that the sequence of polynomials*

$$\begin{aligned} \Phi_0(x) &= 1, \quad \Phi_1(x) = x + b_0 \\ \Phi_n(x) &= \sum_{k=0}^n a_k x^k, \quad a_n = 1 \quad (n \geq 2) \end{aligned}$$

forms a sequence of orthogonal polynomials in the generalized sense is that they satisfy a difference equation of the form

$$(2.8) \quad \Phi_{n+1}(x) = (x + b_n)\Phi_n(x) + c_n\Phi_{n-1}(x) \quad (n \geq 1)$$

where c_n, b_n are constant and $c_n \neq 0$.

DEFINITION. - A system of polynomials $\{\Phi_n(x)\}$ is orthogonal in the general sense if there exist at least one function $\Psi(x)$ of bounded variation over (a, b) such that

$$\int_a^b \Phi_m(x)\Phi_n(x)d\Psi(x) = 0 \quad (n \neq m).$$

REMARK. - There is no loss of generality in assuming that the coefficient of x^n is 1. If it is $K_n (K_n \neq 0)$ then (2.8) takes the form

$$(2.9) \quad \Phi_{n+1}(x) = (A_n x + B_n)\Phi_n(x) + C_n\Phi_{n-1}(x).$$

With SHOHAT [17], $\Phi_n(x)$ is orthogonal provided (2.9) is satisfied and $A_n C_n \neq 0$.

We shall often use the shorter notation $G_n(\zeta, x)$ when it is not necessary to mention the parameters α, β, γ explicitly. The absence of x in the notation will indicate the case $x = 1$ so that

$$G_n(\zeta) = G_n(\zeta, \alpha, \beta, \gamma) = G_n(\zeta, 1; \alpha, \beta, \gamma).$$

We shall use the notation $Z_n(x; \alpha, \beta, \gamma)$ for

$$(2.10) \quad {}_2F_2 \left[\begin{matrix} -n, & \gamma + n + 1; \\ \alpha + 1, & \beta + 1; \end{matrix} x \right].$$

which is a generalization of BATEMAN'S function [13, p. 236]

$${}_2F_2 \left[\begin{matrix} -n, & 2v + 1; \\ v + \frac{1}{2}, & b + 1; \end{matrix} t \right],$$

which is itself a generalization of BATEMAN'S z -function

$$Z_n(x) = {}_2F_2 \left[\begin{matrix} -n, & n + 1; \\ 1, & 1; \end{matrix} x \right].$$

3. - Characterization of the polynomials $G_n(\zeta; \alpha, \beta, \gamma)$.

Let

$$(3.1) \quad f_n(\zeta) = \sum_{k=0}^n \frac{(-n)_k (\gamma + n + 1)_k (\zeta)_k}{k!} H(k) \quad (n = 0, 1, 2, \dots),$$

where $H(k)$ is some function of k , $H(0) = 1$, $H(1) \neq 0$. We now seek those functions $H(k)$ which will make $f_n(\zeta)$ orthogonal.

Assume that $f_n(\zeta)$ satisfies (2.9) and then equate the coefficients of $(\zeta)_k$ we get

$$(3.2) \quad \frac{(n+1)(\gamma+n+k)}{(n-k+1)(\gamma+n)} = -A_n \left\{ k + \frac{kH(k-1)}{(n-k+1)(\gamma+n+k-1)H(k)} \right\} \\ + B_n + C_n \frac{(n-k)(\gamma+n-1)}{n(\gamma+n+k-1)}.$$

Putting $k = 0, n, n+1, 1$ we get,

$$(3.3) \quad B_n + C_n = 1,$$

$$(3.4) \quad \frac{(n+1)(\gamma+2n)}{(\gamma+n)} = -A_n \left\{ n + \frac{nH(n-1)}{(\gamma+2n-1)H(n)} \right\} + B_n,$$

$$(3.5) \quad A_n = - \frac{(\gamma+2n)(\gamma+2n+1)}{\gamma+n} \frac{H(n+1)}{H(n)},$$

and

$$(3.6) \quad \frac{(n+1)(\gamma+n+1)}{n(\gamma+n)} = -A_n \left\{ 1 + \frac{1}{n(\gamma+n)} \frac{H(0)}{H(1)} \right\} + B_n \\ + \frac{(n-1)(\gamma+n+1)}{n(\gamma+n)} C_n.$$

Equations (3.3), (3.4), (3.5) and (3.6) give

$$(3.7) \quad \frac{H(n)}{nH(n+1)} - \frac{H(n-1)}{(n-1)H(n)} = 1 - \frac{H(0)}{n(n-1)H(1)}$$

Let $\mu_n = \frac{H(n)}{nH(n+1)}$; then (3.7) becomes

$$(3.8) \quad \left(\mu_n - \frac{K}{n}\right) - \left(\mu_{n-1} - \frac{K}{n-1}\right) = 1,$$

where

$$K = \frac{H(0)}{H(1)} = \frac{1}{H(1)}.$$

Solving (3.8) for μ_n we get

$$\mu_n = \frac{K}{n} + n + C,$$

where C is an arbitrary constant. We may accordingly put

$$\frac{H(n-1)}{H(n)} = \frac{1}{\lambda(n)},$$

where

$$\lambda(n) = n^2 + nC + K = (n + \alpha)(n + \beta).$$

Hence

$$H(n+1) = \frac{1}{\lambda(n)\lambda(n-1)\dots\lambda(0)}.$$

Substituting for $\lambda(n)$, $\lambda(n-1)$, ... and $\lambda(0)$ we get

$$H(n) = \frac{1}{(\alpha)_n(\beta)_n},$$

provided neither α nor β is a negative integer. Thus (3.2) can be rewritten as

$$(3.9) \quad (n+1)(\gamma+n+k) = -kA_n \left\{ (n-k+1)(\gamma+n) + \frac{(\gamma+n)(\alpha+k-1)(\beta+k-1)}{\gamma+n+k-1} \right\} \\ + (n-k+1)(\gamma+n)B_n + \frac{(n-k)(\gamma+n-1)(n-k+1)(\gamma+n)}{n(\gamma+n+k-1)}C_n.$$

This must hold for $k = 0, 1, 2, \dots, n+1$.

Solving the equation obtained from (3.9) by putting $k = 0, 1, n, n + 1$ we find that

$$(3.10) \quad A_n = - \frac{(\gamma + 2n + 1)(\gamma + 2n + 2)}{(\gamma + n + 1)(\alpha + n + 1)(\beta + n + 1)},$$

$$(3.11) \quad B_n = - \frac{A_n}{(\gamma + 2n + 2)(\gamma + 2n)} \left\{ \gamma(\alpha + n + 1)(\beta + n + 1) + \right. \\ \left. + n(\gamma + 2n + 1)(\alpha + \beta - \gamma) + n(\alpha + \beta + 2n + 2) \right\},$$

and

$$(3.12) \quad C_n = 1 - B_n.$$

It remains to verify (3.9) for $k = 2, 3, \dots, n - 1$. This can be done if we note that (3.9) is an equation of the third degree in k . Since it is satisfied by four values of k it is true for all k and in particular for $k = 0, 1, \dots, n + 1$.

We may now state the following result.

THEOREM 1. - *The set of polynomials (3.1) is orthogonal provided that*

$$H(k) = \frac{1}{(\alpha)_k(\beta)_k},$$

where α, β, γ are arbitrary constants such that α and β are not equal to 0, $-1, -2, \dots$ and $\gamma \neq -1, -2, \dots$. Thus (3.1) becomes

$$f_n(\zeta) = {}_3F_2 \left[\begin{matrix} -n, \gamma + n + 1, \zeta; \\ \alpha, \beta; \end{matrix} 1 \right].$$

We note from the proof that in case α, β or γ is a negative integer then our set of polynomials becomes a finite set of orthogonal polynomials. This is the case considered by KARLIN and MCGREGOR [11] (see also [10] and [20]).

If we put $H(k) = \frac{x^k}{(\alpha + 1)_k(\beta + 1)_k}$ we get the following.

COROLLARY. - *The polynomials (in ζ)*

$$(3.13) \quad G_n(\zeta, x) = {}_3F_2 \left[\begin{matrix} -n, \gamma + n + 1, \zeta; \\ \alpha + 1, \beta + 1; \end{matrix} x \right]$$

are orthogonal if and only if $x = 1$.

This also shows that the RICE polynomials

$$H_n(\zeta, p, v) = {}_3F_2 \left[\begin{matrix} -n, n + 1, \zeta; \\ 1, p; \end{matrix} v \right]$$

(as polynomials in ζ) are not orthogonal unless $v = 1$. In this case we get essentially the PASTERNAK polynomials (1.5).

We also remark that the polynomials (3.13) (in which ζ is considered as a parameter and x is the variable) are orthogonal if and only if $\zeta = \alpha + 1$. In this case we get essentially the JACOBI polynomials (2.3). For proof see [1] where it was proved that the only orthogonal polynomials of the form

$${}_pF_q \left[\begin{matrix} -n, \gamma + n + 1, \alpha_1, \dots, \alpha_p; \\ \beta_1, \beta_2, \dots, \beta_q; \end{matrix} x \right]$$

are given by

$${}_2F_1 \left[\begin{matrix} -n, \gamma + n + 1; \\ \alpha + 1; \end{matrix} x \right].$$

We note that $G_n(\zeta)$ satisfies the three term recurrence relation

$$G_{n+1}(\zeta) = (A_n\zeta + B_n)G_n(\zeta) + C_nG_{n-1}(\zeta),$$

$A_nC_n \neq 0$ with A_n, B_n, C_n given by (3.10), (3.11) and (3.12).

In case $\gamma = \alpha = \beta = 0, x = 1$ we know that

$$G_n(\zeta, 1; 0, 0, 0) = F_n(\zeta)$$

and the orthogonality relation is given by (1.3). On the other hand if $\gamma = \alpha = 0, x = 0$ we have

$$G_n(\zeta + 1, 1; 0, \beta, 0) = F_n^\beta(1 - \beta + 2\zeta)$$

and the orthogonality relation is given by CARLITZ (see Formula (1.7)).

In the more general case WEBER and ERDÉLYI [20] and ERDÉLYI et al [8] gave explicit orthogonality relations.

It might be of some interest to consider the related polynomials;

$$g_n(\zeta) = G_n(\zeta, x; \alpha, \beta, \gamma - n)$$

in analogy with the pseudo-LAGUERRE polynomials

$$(3.16) \quad L_n^{(\alpha-n)}(x) = \frac{(\alpha - n + 1)}{n!} {}_1F_1 \left[\begin{matrix} -n; \\ \alpha - n + 1; \end{matrix} x \right].$$

SHIVELY [16] proved that the polynomials (3.16) are not orthogonal by proving they cannot satisfy a recurrence relation of the form (2.10). A similar negative result holds for the $g_n(\zeta)$. We state the following theorem

THEOREM 3. - *The set of polynomials $G_n(\zeta, x; \alpha, \beta, \gamma - n)$ are not orthogonal for any values of α, β, γ and x independent of n .*

4. - Contiguous relations.

In this section we shall obtain a number of contiguous relations which are satisfied by $G_n(\zeta, x; \alpha, \beta, \gamma)$ and by $G_n(\zeta; \alpha, \beta, \gamma)$. We begin by stating the following three formulas;

$$(4.1) \quad (\zeta - \alpha)G_n(\zeta, x; \alpha, \beta, \gamma) = \zeta G_n(\zeta + 1, x; \alpha, \beta, \gamma) - \alpha G_n(\zeta, x; \alpha - 1, \beta, \gamma),$$

$$(4.2) \quad (\alpha + 1) \{ 2\zeta - \beta - 1 - (\alpha - \gamma + \zeta)x \} G_n(\zeta, x; \alpha, \beta, \gamma)$$

$$= (\alpha + 1)(1 - x)\zeta G_n(\zeta + 1, x; \alpha, \beta, \gamma)$$

$$+ (\alpha + 1)(\zeta - \beta - 1)G_n(\zeta - 1, x; \alpha, \beta, \gamma)$$

$$- x(\gamma - \alpha + n)(\alpha + n + 1)G_n(\zeta, x; \alpha + 1, \beta, \gamma)$$

and

$$(4.3) \quad (\gamma + n + 1) \{ G_n(\zeta - 1, x; \alpha, \beta, \gamma + 1) - G_n(\zeta - 1, x; \alpha, \beta, \gamma) \}$$

$$= (\zeta - 1) \{ G_n(\zeta - 1, x; \alpha, \beta, \gamma) - G_n(\zeta, x; \alpha, \beta, \gamma) \}.$$

The proof of (4.1) and (4.2) follow immediately from equations (15) and (30) given in RAINVILLE [14], while (4.3) is a consequence of the following relations:

$$(4.4) \quad \frac{n(\zeta - 1)}{(\alpha + 1)(\beta + 1)} {}_3F_2 \left| \begin{matrix} -n + 1, \gamma + n + 2, \zeta \\ \alpha + 2, \beta + 2 \end{matrix}; x \right|$$

$$= G_n(\zeta - 1, x; \alpha, \beta, \gamma) - G_n(\zeta - 1, x; \alpha, \beta, \gamma + 1),$$

and

$$(4.5) \quad \frac{n(\gamma + n + 1)}{(\alpha + 1)(\beta + 1)} {}_3F_2 \left| \begin{matrix} -n + 1, \gamma + n + 2, \zeta \\ \alpha + 2, \beta + 2 \end{matrix}; x \right|$$

$$= G_n(\zeta - 1, x; \alpha, \beta, \gamma) - G_n(\zeta, x; \alpha, \beta, \gamma).$$

Both formulas (4.4) and (4.5) can be easily verified by equating coefficients of powers of x .

The contiguous relations which are satisfied by $G_n(\zeta)$ are

$$(4.6) \quad n(\gamma + n + 1)G_n(\zeta) = (\zeta - \alpha - 1)(\zeta - \beta - 1)[G_n(\zeta - 1) - G_n(\zeta)]$$

$$+ \zeta(\zeta + \gamma - \alpha - \beta)[G_n(\zeta + 1) - G_n(\zeta)],$$

and

$$\begin{aligned}
 (4.7) \quad & \zeta(\zeta + \gamma - \alpha - \beta) G_n(\zeta + 1) \\
 &= \left[\zeta^2 + \zeta(n + \gamma - \alpha - \beta) + \frac{n}{\gamma + 2n} \left\{ \alpha\beta + (\gamma + n)(\gamma - \alpha - \beta + n) \right\} G_n(\zeta) \right. \\
 &\quad \left. - \frac{n}{\gamma + 2n} [\alpha\beta + (\gamma + n)(\gamma - \alpha - \beta + n)] G_{n-1}(\zeta) \right].
 \end{aligned}$$

Formulas (4.6) and (4.7) generalize similar relations given by PASTERNAK [12, (5.2) and (5.5)]. Instead of proving the above two formulas we shall prove the more general relations satisfied by

$${}_3F_2 \left[\begin{matrix} a, & b, & c; \\ d, & e; \end{matrix} 1 \right],$$

which are believed to be new. These are

$$\begin{aligned}
 (4.8) \quad & c(c + a + b + 1 - d - e) {}_3F_2 \left[\begin{matrix} a + 1, & b + 1, & c + 1; \\ d + 1, & e + 1; \end{matrix} 1 \right] \\
 & - (c - d)(c - e) {}_3F_2 \left[\begin{matrix} a + 1, & b + 1, & c; \\ d + 1, & e + 1; \end{matrix} 1 \right] \\
 & + de {}_3F_2 \left[\begin{matrix} a, & b, & c; \\ d, & e; \end{matrix} 1 \right] = 0
 \end{aligned}$$

and

$$\begin{aligned}
 (4.9) \quad & c(c + 1 + b + a - d - e) {}_3F_2 \left[\begin{matrix} a, & b, & c + 1; \\ d, & e; \end{matrix} 1 \right] \\
 & - (c^2 + (b - d - e + 1)c + \frac{a}{a - b + 1} [de + b(b - d - e)]) {}_3F_2 \left[\begin{matrix} a, & b, & c; \\ d, & e; \end{matrix} 1 \right] \\
 & + \frac{a}{a - b + 1} [de + b^2 - b(d + e)] {}_3F_2 \left[\begin{matrix} a + 1, & b - 1, & c; \\ d, & e; \end{matrix} 1 \right] = 0.
 \end{aligned}$$

To prove (4.8) we first rewrite it in the following way:

$$\begin{aligned}
 (4.10) \quad & c(c + a + b + 1 - d - e) \left\{ {}_3F_2 \left[\begin{matrix} a, & b, & c + 1; \\ d, & e; \end{matrix} 1 \right] - {}_3F_2 \left[\begin{matrix} a, & b, & c; \\ d, & e; \end{matrix} 1 \right] \right\} \\
 & + (c - d)(c - e) \left\{ {}_3F_2 \left[\begin{matrix} a, & b, & c - 1; \\ d, & e; \end{matrix} 1 \right] - {}_3F_2 \left[\begin{matrix} a, & b, & c; \\ d, & e; \end{matrix} 1 \right] \right\} \\
 & + ab {}_3F_2 \left[\begin{matrix} a, & b, & c; \\ d, & e; \end{matrix} 1 \right] = 0.
 \end{aligned}$$

We note that the sum of the first two terms on the left hand side of (4.10) is

$$(4.11) \quad (c + a + b + 1 - d - e) \sum_{k=0}^{\infty} \frac{k(a)_k(b)_k(c)_k}{k! (d)_k(e)_k} \\ - (c^2 - (d + e)c + de) \sum_{k=0}^{\infty} \frac{k(a)_k(b)_k(c)_{k-1}}{k! (d)_k(e)_k}.$$

But

$$(4.12) \quad c(c)_k = (c)_{k+1} - k(c)_k,$$

$$(4.13) \quad c^2(c)_k = (c)_{k+2} - (2k + 1)(c)_{k+1} + k^2(c)_k.$$

Hence (4.11) becomes

$$(c + a + b + 1 - d - e) \sum_{k=0}^{\infty} \frac{k(a)_k(b)_k(c)_k}{k! (d)_k(e)_k} \\ - \sum_{k=0}^{\infty} \frac{k(a)_k(b)_k}{k! (d)_k(e)_k} [(c)_{k+1} - (2k - 1)(c)_k + (k - 1)^2(c)_{k-1}] \\ + (d + e) \sum_{k=0}^{\infty} \frac{k(a)_k(b)_k}{k! (d)_k(e)_k} [(c)_k - (k - 1)(c)_{k-1}] \\ - de \sum_{k=0}^{\infty} \frac{k(a)_k(b)_k(c)_{k-1}}{k! (d)_k(e)_k}.$$

Simplifying we get

$$- ab {}_3F_2 \left[\begin{matrix} a, & b, & c; \\ & d, & e; \end{matrix} 1 \right]$$

which completes the proof of (4.8).

In a similar way we can prove (4.9), that is by using (4.12), (4.13) and

$$(4.14) \quad c^2(c + 1)_k = (c)_{k+2} - (k + 1)(c)_{k+1}$$

and equating coefficients of $(c)_k$.

5. - Operational formulas.

Let D denote the differential operator $d/d\zeta$. By induction on k we can show that

$$(5.1) \quad \left(\frac{1 + \alpha + D}{2} \right)_k \operatorname{sech}^{\alpha+1} \zeta = (1 + \alpha)_k \operatorname{sech}^{\alpha+1} \zeta \left(\frac{1 - \tanh \zeta}{2} \right)^k.$$

This implies

$$\begin{aligned} & {}_{p+1}F_{q+1} \left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_p, \frac{1+\alpha+D}{2} \\ \beta_1, \beta_2, \dots, \beta_q, 1+\alpha \end{matrix} ; x \right] \operatorname{sech}^{\alpha+1} \zeta \\ &= \operatorname{sech}^{\alpha+1} {}_pF_q \left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_p, \frac{1}{2} x(1 - \tanh \zeta) \\ \beta_1, \beta_2, \dots, \beta_q \end{matrix} \right]. \end{aligned}$$

In particular we obtain

$$G_n \left(\frac{1+\alpha+D}{2}, x \right) \operatorname{sech}^{\alpha+1} = \operatorname{sech}^{\alpha+1} \zeta {}_2F_1 \left[\begin{matrix} -n, \gamma+n+1 \\ 1+\beta \end{matrix} ; \frac{1}{2} x(1 - \tanh \zeta) \right].$$

Then, making use of (2.2) we get

$$(5.2) \quad G_n \left(\frac{1+\alpha+D}{2}, x \right) \operatorname{sech}^{\alpha+1} \zeta = \frac{n!}{1+\beta)_n} \operatorname{sech}^{\alpha+1} \zeta P_n^{(\beta, \gamma-\beta)}(1-x+x \tanh \zeta).$$

Another operational formula can be obtained in terms of the operator [2]

$$\theta = x + x^2 d/dx.$$

We have from [2, (1.2)]

$$\theta^n(x^{\alpha+k}) = (\alpha+k+1)_n x^{\alpha+k+n},$$

so that

$$(5.3) \quad {}_2F_2 \left[\begin{matrix} -n, \zeta; \theta \\ \alpha+1, \beta+1 \end{matrix} ; \right] x^{\gamma+n-1} = x^{\gamma+n+1} G_n(\zeta, x),$$

and

$$(5.4) \quad {}_2F_2 \left[\begin{matrix} -n, \gamma+n+1; t\theta \\ \alpha+1, \beta+1 \end{matrix} ; \right] x^{\zeta-1} = x^{\zeta-1} G_n(\zeta, tx).$$

Both (5.3) and (5.4) can be verified easily. For example to verify (5.3) we see that the left hand side is

$$\begin{aligned} & \sum_{k=0}^n \frac{(-n)_k (\zeta)_k}{k! (\alpha+1)_k (\beta+1)_k} \theta^k \{ x^{\gamma+n+1} \} \\ &= \sum_{k=0}^n \frac{(-n)_k (\zeta)_k (\gamma+n+1)_k}{k! (\alpha+1)_k (\beta+1)_k} x^{\gamma+n+k+1} = x^{\gamma+n+1} G_n(\zeta, x). \end{aligned}$$

In a similar manner we can show that

$$(5.5) \quad {}_0F_1 \left[\begin{matrix} -n, \zeta \\ \alpha + 1 \end{matrix}; x \right] = x^{\gamma+n} (\gamma + 1)_n G_n(\zeta, x; \alpha, \gamma, \gamma).$$

Now let $\delta = y \, d/dy$. It is well known that

$$(\delta)_n y^\alpha = (\alpha)_n y^\alpha.$$

Hence we get

$$(5.6) \quad G_n(\delta, x) y^{\alpha+1} = \frac{n!}{(\beta + 1)_n} y^{\alpha+1} P_n^{(\beta, \gamma-\beta)}(1-2x).$$

An interesting application of (5.2) when $x = 1$ is to obtain the three-term recurrence relation satisfied by $G_n(\zeta)$ from the corresponding three-term recurrence relation for the JACOBI polynomial [19, p. 71]:

$$(5.7) \quad P_{n+1}^{(\beta, \gamma-\beta)}(x) = (a_n x + b_n) P_n^{(\beta, \gamma-\beta)}(x) - c_n P_{n-1}^{(\beta, \gamma-\beta)}(x)$$

where

$$\begin{aligned} a_n &= \frac{(\gamma + 2n + 1)(\gamma + 2n + 2)}{2(n + 1)(\gamma + n + 1)}, \\ b_n &= \frac{\gamma(2\beta - \gamma)(\gamma + 2n + 1)}{2(n + 1)(\gamma + n + 1)(\gamma + 2n)}, \\ c_n &= \frac{(\beta + n)(\gamma - \beta + n)(\gamma + 2n + 2)}{(n + 1)(\gamma + n + 1)(\gamma + 2n)}. \end{aligned}$$

To see how this can be done we first get from (5.2) for $x = 1$

$$\begin{aligned} (5.8) \quad & \frac{(\beta + 1)_n}{n!} D G_n \left(\frac{1 + \alpha + D}{2} \right) \operatorname{sech}^{x+1} \zeta \\ &= -(\alpha + 1) \operatorname{sech}^{x+1} \zeta \tanh \zeta P_n^{(\beta, \gamma-\beta)}(\tanh \zeta) \\ &+ \operatorname{sech}^{x+1} \zeta (1 - \tanh^2 \zeta) \frac{d}{d(\tanh \zeta)} P_n^{(\beta, \gamma-\beta)}(\tanh \zeta), \end{aligned}$$

where $D = d/d\zeta$.

But we have [19, p. 72]

$$\begin{aligned} & (1 - u^2) \frac{d}{du} P_n^{(\beta, \gamma-\beta)}(u) \\ &= -n \left\{ u + \frac{\gamma - 2\beta}{\gamma + 2n} \right\} P_n^{(\beta, \gamma-\beta)}(u) + \frac{2(\beta + n)(\gamma - \beta + n)}{\gamma + 2n} P_n^{(\beta, \gamma-\beta)}(u). \end{aligned}$$

Hence (5.8) becomes

$$\begin{aligned} & \operatorname{sech}^{-x-1} \zeta \frac{(1+\beta)_n}{n!} D G_n \left(\frac{1+\alpha+D}{2} \right) \operatorname{sech}^{x+1} \zeta \\ &= -(\alpha+n+1) \tanh \zeta P_n^{(\beta, \gamma-\beta)}(\tanh \zeta) + n \frac{2\beta-\gamma}{\gamma+2n} P_n^{(\beta, \gamma-\beta)}(\tanh \zeta) \\ & \quad + \frac{2(\beta+n)(\gamma-\beta+n)}{\gamma+2n} P_{n-1}^{(\beta, \gamma-\beta)}(\tanh \zeta). \end{aligned}$$

Substituting for $\tanh \zeta P_n^{(\beta, \gamma-\beta)}(\tanh \zeta)$ from (5.7) we get after some simplification

$$\begin{aligned} & a_n \frac{n+1}{(1+\beta+n)} D G_n \left(\frac{1+\alpha+D}{2} \right) \operatorname{sech}^{x+1} \zeta \\ &= -(\alpha+n+1) \frac{(n+1)!}{(\beta+1)_{n+1}} \operatorname{sech}^{x+1} \zeta P_n^{(\beta, \gamma-\beta)}(\tanh \zeta) \\ & \quad + \frac{(n+1)!}{(\beta+1)_{n+1}} \left[b_n(\alpha+n+1) + a_n \frac{n(2\beta-\gamma)}{2n+\gamma} \right] \operatorname{sech}^{x+1} \zeta P_n^{(\beta, \gamma-\beta)}(\tanh \zeta) \\ & \quad + \frac{(n+1)!}{(\beta+1)_{n+1}} \left[\frac{2(n+\beta)(n+\gamma-\beta)}{2n+\gamma} a_n - (\alpha+n+1)c_n \right] \operatorname{sech}^{x+1} \zeta P_{n-1}^{(\beta, \gamma-\beta)}(\tanh \zeta) \end{aligned}$$

By using (5.2) for $x=1$, we get from this

$$\begin{aligned} (5.9) \quad G_{n+1} \left(\frac{\zeta+\alpha+1}{2} \right) &= \frac{-(n+1)}{(\alpha+n+1)(\beta+n+1)} a_n \zeta G_n \left(\frac{\zeta+\alpha+1}{2} \right) \\ & \quad + \frac{n+1}{(\alpha+n+1)(\beta+n+1)} \left[(\alpha+n+1)b_n + \frac{n(2\beta-\gamma)}{2n+\gamma} a_n \right] G_n \left(\frac{\zeta+\alpha+1}{2} \right) \\ & \quad + \frac{n(n+1)}{(\alpha+n+1)(\beta+n+1)(\beta+n)} \left[\frac{2(n+\beta)(n+\gamma+\beta)}{2n+\gamma} a_n - (\alpha+n+1)c_n \right] G_{n-1} \left(\frac{\zeta+\alpha+1}{2} \right). \end{aligned}$$

Substituting for a_n , b_n and c_n from (5.7) in (5.9) and simplifying we get the required three term relation.

6. - Generating relation.

It is possible to obtain various generating relations involving $G_n(\zeta, x)$ or $G_n(\zeta)$. To begin with we state the formula

$$\begin{aligned} (6.1) \quad & \sum_{n=0}^{\infty} \frac{(\gamma+1)_n}{n!} G_n(\zeta, x) t^n \\ &= (1-t)^{-\gamma-1} {}_3F_2 \left[\begin{matrix} \frac{1}{2}(\gamma+1), \frac{1}{2}(\gamma+2), \zeta; \\ \alpha+1, \beta+1; \end{matrix} \frac{-4xt}{(1-t)^2} \right]. \end{aligned}$$

This can be proved in the following manner. We have

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \frac{(\gamma+1)_n}{n!} G_n(\zeta, x) t^n \\
 &= \sum_{n=0}^{\infty} \frac{(\gamma+1)_n}{n!} t^n \sum_{k=0}^A \frac{(-n)_k (\gamma+n+1)_k (\zeta)_k}{k! (\alpha+1)_k (\beta+1)_k} x^k \\
 &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(-1)^k (\gamma+1)_{n+k} (\zeta)_k x^k t^n}{(n-k)! k! (\alpha+1)_k (\beta+1)_k} \\
 &= \sum_{k=0}^{\infty} \frac{(\zeta)_k (-x)^k t^k}{k! (\alpha+1)_k (\beta+1)_k} \sum_{n=0}^{\infty} \frac{(\gamma+1)_{n+2k} t^n}{n!} \\
 &= (1-t)^{-\gamma-1} {}_3F_2 \left[\begin{matrix} \frac{1}{2}(\gamma+1), \frac{1}{2}(\gamma+2), \zeta; \\ \alpha+1, \beta+1; \end{matrix} \frac{-4xt}{(1-t)^2} \right].
 \end{aligned}$$

Another way of proving (6.1) which might be of some interest is to start by proving in the same way we proved (6.1) the generating relation

$$\begin{aligned}
 (6.2) \quad & \sum_{n=0}^{\infty} \frac{(\gamma+1)_n}{n!} Z_n(x; \alpha, \beta, \gamma) t^n \\
 &= (1-t)^{-\gamma-1} {}_2F_2 \left[\begin{matrix} \frac{1}{2}(\gamma+1), \frac{1}{2}(\gamma+2); \\ \alpha+1, \beta+1; \end{matrix} \frac{-4xt}{(1-t)^2} \right],
 \end{aligned}$$

where $Z_n(x; \alpha, \beta, \gamma)$ is defined by (2.10).

If we now replace x by $x\theta$ in (6.2), where $\theta = y(y+D)$, $D = d/dy$ and then operate on $y^{\gamma-1}$ we get (6.1).

Another generating function of slightly different nature is given by

$$(6.3) \quad \sum_{r=0}^{\infty} G_n(-r, x) t^r = \frac{1}{1-t} {}_3F_2 \left[\begin{matrix} -n, \gamma+n+1, 1; \\ \alpha+1, \beta+1; \end{matrix} \frac{-xt}{1-t} \right].$$

The proof is as follows. We have

$$\begin{aligned}
 \sum_{r=0}^{\infty} G_n(-r, x) t^r &= \sum_{r=0}^{\infty} t^r \sum_{k=0}^n \frac{(-n)_k (\gamma+n+1)_k (-r)_k}{k! (\alpha+1)_k (\beta+1)_k} x^k \\
 &= \sum_{k=0}^n \frac{(-n)_k (\gamma+n+1)_k (-1)^k x^k t^k}{k! (\alpha+1)_k (\beta+1)_k} \sum_{r \geq k} \frac{r!}{(r-k)!} t^{r-k} \\
 &= \sum_{k=0}^n \frac{(-n)_k (\gamma+n+1)_k (-xt)^k k!}{k! (\alpha+1)_k (\beta+1)_k} \sum_{r=0}^{\infty} \frac{(1+k)}{r!} t^r \\
 &= (1-t)^{-1} {}_3F_2 \left[\begin{matrix} -n, \gamma+n+1, 1; \\ \alpha+1, \beta+1; \end{matrix} \frac{-xt}{1-t} \right].
 \end{aligned}$$

We can prove in a similar way the following generating relations:

$$(6.4) \quad \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} G_n(-r, x) = e^{-t} Z(xt; \alpha, \beta, \gamma),$$

and

$$(6.5) \quad \sum_{r=0}^{\infty} \frac{(\beta+1)_r}{r!} G_n(-r, x) t^r = (1-t)^{-\beta-1} {}_2F_1 \left[\begin{matrix} -n, \gamma+n+1; \\ \alpha+1; \end{matrix} \middle| \frac{-xt}{1-t} \right].$$

We should like to point out that RICE [15] gave the general case for (6.4) and (6.5).

In addition to the above generating relations we have also the following relation satisfied by $G_n(\zeta)$

$$(6.6) \quad \sum_{n=0}^{\infty} \frac{(\beta+1)_n}{n! (\gamma-\alpha+1)_n} G_n(\zeta) t^n \\ = {}_1F_1 \left[\begin{matrix} \zeta + \gamma - \alpha - \beta; \\ \gamma - \alpha + 1; \end{matrix} \middle| -t \right] {}_1F_1 \left[\begin{matrix} \alpha - \zeta + 1; \\ \alpha + 1; \end{matrix} \middle| t \right].$$

To prove (6.6) we use the BARNES transformation [3]

$${}_3F_2 \left[\begin{matrix} a_1, a_2, a_3; \\ b_1, b_2; \end{matrix} \middle| 1 \right] = \frac{\Gamma(b_2)\Gamma(b_1+b_2-a_1-a_2-a_3)}{\Gamma(b_2-a_1)\Gamma(b_1+b_2-a_2-a_3)} \\ \cdot {}_3F_2 \left[\begin{matrix} a_1, b_1-a_2, b_1-a_3; \\ b_1, b_1+b_2-a_2-a_3; \end{matrix} \middle| 1 \right].$$

We get, after some simplification,

$$(\beta+1)_n G_n(\zeta) = (-1)^n (\zeta + \gamma - \alpha - \beta)_n {}_3F_2 \left[\begin{matrix} -n, \alpha - \gamma - n, \alpha - \zeta + 1; \\ \alpha + 1, \beta + \alpha - \gamma - \zeta - n + 1; \end{matrix} \middle| 1 \right] \\ = \sum_{k=0}^{\infty} \frac{(-1)^{n+k} (\gamma - \alpha + 1)_n (\zeta - \alpha - \beta + \gamma)_{n-k} (\alpha - \zeta + 1)_k n!}{(\alpha + 1)_k k! (n-k)! (\gamma - \alpha + 1)_{n-k}}.$$

Hence

$$\sum_{n=0}^{\infty} \frac{(\beta+1)_n}{n! (\gamma-\alpha+1)_n} G_n(\zeta) t^n \\ = \sum_{n=0}^{\infty} \frac{(\zeta + \gamma - \alpha - \beta)_n (-t)^n}{n! (\gamma - \alpha + 1)_n} \cdot \sum_{k=0}^{\infty} \frac{(\alpha - \zeta + 1)_k}{k! (\alpha + 1)_k} t^k \\ = {}_1F_1 \left[\begin{matrix} \zeta + \gamma - \alpha - \beta; \\ \gamma - \alpha + 1; \end{matrix} \middle| -t \right] {}_1F_1 \left[\begin{matrix} \alpha - \zeta + 1; \\ \alpha + 1; \end{matrix} \middle| t \right].$$

Now let us recall the generating relation for the JACOBI polynomials [15, p. 270]

$$(6.7) \quad \sum_{n=0}^{\infty} P_n^{(\alpha, \beta)}(x) t^n = F_4(\beta + 1, \alpha + 1; \alpha + 1, \beta + 1; \frac{1}{2} t(x-1), \frac{1}{2} t(x+1)),$$

where $F_4(a, b; c, c'; x, y)$ is defined by (2.7).

It is easy to see that

$$(6.8) \quad \int_0^1 t^{\zeta-1} (1-t)^{\beta-\zeta} F_4(a, \beta+1; c, c', -t, 1-t) dt \\ = B(\zeta, B-\zeta+1) F_2(a; \zeta, \beta-\zeta+1; c, c'; -t, t).$$

Now in (6.7) replace α by β , β by $\gamma - \beta$, x by $(1-2x)$ and then multiply both sides by $x^{\zeta-1}(1-x)^{\beta-\zeta}$ and integrate from 0 to 1. We get

$$(6.9) \quad \sum_{n=0}^{\infty} \int_0^1 x^{\zeta-1} (1-x)^{\beta-\zeta} P_n^{(\beta, \gamma-\beta)}(1-2x) t^n dx \\ = \int_0^1 x^{\zeta-1} (1-x)^{\beta-\zeta} F_4(\gamma - \beta + 1, \beta + 1; \beta + 1, \gamma - \beta + 1; -xt, t(t-x)) dx.$$

If we use (1.13) on the left hand side and (6.8) on the right hand side of (6.9) we get the following generating relation

$$(6.10) \quad \sum_{n=0}^{\infty} \frac{(\beta+1)_n}{n!} G_n(\zeta; \beta, \beta, \gamma) t^n \\ = F_2(\gamma - \beta + 1; \zeta, \beta - \zeta + 1; \beta + 1, \gamma - \beta + 1; -t, t).$$

This may be compared with (6.1).

The following are generating relations for the polynomials $G_n(\zeta, x; \alpha, \beta, \gamma - n)$:

$$(6.11) \quad \sum_{n=0}^{\infty} G_n(\zeta, x; \alpha, \beta, \gamma - n) \frac{v^n}{n!} = e^{v^2} {}_2F_2 \left[\begin{matrix} \gamma + 1, \zeta \\ \alpha + 1, \beta + 1 \end{matrix}; -xv \right]$$

and

$$(6.12) \quad \sum_{n=0}^{\infty} G_n(\zeta, x; \alpha, \beta, \gamma - n) \frac{(\lambda)_n}{n!} v^n = (1-\lambda)^{-n} {}_3F_2 \left[\begin{matrix} \lambda, \gamma + 1, \zeta \\ \alpha + 1, \beta + 1 \end{matrix}; -\frac{xv}{1-v} \right].$$

The proof of these formulas is similar to the proof of (6.3).

7. - Integral representations.

It is well known that for a non-negative integer r and for $\operatorname{Re}(\zeta) > 0$, $\operatorname{Re}(\alpha + 1) > 0$,

$$(7.1) \quad \frac{(\zeta)_r}{(\alpha + 1)_r} = \frac{\Gamma(\alpha + 1)}{\Gamma(\zeta)\Gamma(\alpha - \zeta + 1)} \int_0^1 t^{\zeta+r-1}(1-t)^{\alpha-\zeta} dt.$$

Hence using (7.1) and the definition of the JACOBI polynomial (2.2), we get

$$(7.2) \quad B(\zeta, \alpha - \zeta + 1)G_n(\zeta, x) = \frac{n!}{(\beta + 1)_n} \int_0^1 t^{\zeta-1}(1-t)^{\alpha-\zeta} P_n^{(\beta, \gamma-\beta)}(1-2xt) dt.$$

Putting $u = t/(1-t)$ in (7.2) we get

$$(7.3) \quad \begin{aligned} & B(\zeta, \alpha - \zeta + 1)G_n(\zeta, x) \\ &= \frac{n!}{(\beta + 1)_n} \int_0^\infty u^{\zeta-1}(1+u)^{-\alpha-1} P_n^{(\beta, \gamma-\beta)}\left(1 - \frac{2xu}{1+u}\right) du. \end{aligned}$$

Since the integral in (7.3) is absolutely convergent for $\operatorname{Re}(\zeta) > 0$, $\operatorname{Re}(\alpha) > -1$ we can use MELLIN'S inversion formula [18, p. 41] to get

$$(7.4) \quad \begin{aligned} & \frac{(\beta + 1)_n}{n!} (1+u)^{-\alpha-1} P_n^{(\beta, \gamma-\beta)}\left(1 - \frac{2xu}{1+u}\right) \\ &= \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} B(\zeta, \alpha - \zeta + 1)G_n(\zeta, x)u^{-\zeta} d\zeta, \end{aligned}$$

where $\operatorname{Re}(\alpha + 1) > \sigma > 0$. This equation reduces, when $\gamma = \beta = 0$ to formula (1.7) given by RICE [15].

The analogues of (7.2) and (7.3) are also given by RICE [15].

If ζ and $\alpha + 1$ are interchanged in (7.1), the resulting integral may be used to prove

$$(7.5) \quad \begin{aligned} & \frac{n!}{(\beta + 1)_n} P_n^{(\beta, \gamma-\beta)}(1-2x) \\ &= \frac{1}{B(\alpha + 1, \zeta - \alpha - 1)} \int_0^1 t^\alpha(1-t)^{\zeta-\alpha-2} G_n(\zeta, xt) dt. \end{aligned}$$

Another expression for $\frac{(\rho)_r}{(\delta)_r}$, $R_e(\rho) > 0$, $R_e(\delta) > 0$, can be obtained by using the formula given in [20, p. 263]:

$$\frac{\Gamma(p+q-1)}{\Gamma(p)\Gamma(q)} = \frac{2^{p+q+1}}{\pi} \int_0^{\frac{\pi}{2}} \cos^{p+q+2\theta} \cos(p-q)\theta d\theta$$

for all p and q such that $p+q > 1$. We find that

$$(7.6) \quad \frac{(\rho)_r}{(\delta)_r} = \frac{2\Gamma(\delta)\Gamma(\rho-\delta+1)}{\pi\Gamma(\rho)} \cdot \int_0^{\frac{\pi}{2}} (2\cos \theta)^{\rho+r-1} \cos(r+2\delta-\rho-1)\theta d\theta,$$

where $\rho > 0$, $\delta > 0$.

For real ζ , α formula (7.6) can be used to prove

$$(7.7) \quad G_n(\zeta, \alpha) = \frac{2\Gamma(\alpha+1)\Gamma(\zeta-\alpha)n!}{\pi\Gamma(\zeta)(\beta+1)_n} \cdot \operatorname{Re} \int_0^{\frac{\pi}{2}} (1 + e^{-2i\theta})^{\zeta-1} e^{2ix\theta} P_n^{(\beta, \gamma-\beta)}(1-2x-2xe^{2i\theta}) d\theta$$

$$= \frac{2\Gamma(\alpha+1)\Gamma(\zeta-\alpha)n!}{\pi\Gamma(\zeta)(\beta+1)_n} \int_0^{2\pi} (1 - e^{-i\Phi})^{\zeta-1} P_n^{(\beta, \gamma-\beta)}(1-2x-2xe^{2i\Phi}) d\Phi.$$

If we let $\alpha=1$, $\gamma=\alpha=0$ and $\alpha=m$ in (7.7) we get the formula (7.8) given by PASTERNAK [12].

From (7.1) we get

$$\frac{k!}{(\alpha+1)_k} = \alpha \int_0^1 t^k (1-t)^{\alpha-1} dt.$$

Hence

$$(7.8) \quad G_n(\zeta, \alpha; \beta, 0) = \alpha \int_0^1 (1-t)^{\alpha-1} H_n(\zeta, \beta+1, tx) dt.$$

In formula (7.2) take $x=1$, $\zeta = \frac{1}{2}(\alpha + z + 1)$ and $1 - 2t = u$, we get

$$(7.9) \quad B\left(\frac{1}{2}\alpha + \frac{1}{2}z + \frac{1}{2}, \frac{1}{2}\alpha - \frac{1}{2}z + \frac{1}{2}\right) G_n\left(\frac{z + \alpha + 1}{2}\right) \\ = \frac{n! 2^{-\alpha}}{(\beta + 1)_n} \int_{-1}^{+1} \left(\frac{1-u}{1+u}\right)^{\frac{1}{2}z} (1-u^2)^{\frac{1}{2}\alpha - \frac{1}{2}} P_n^{(\beta, \gamma-\beta)}(u) du.$$

Inverting this integral by using FOURIER inversion formula [18, p. 17], we get

$$(7.11) \quad \operatorname{sech}^{x+1} x P_n^{(\beta, \gamma-\beta)}(\tanh x) \\ = \frac{2^{\alpha(\beta+1)_k}}{2\pi n!} \int_{-\infty}^{+\infty} e^{ixz} B\left(\frac{1}{2}\alpha + \frac{1}{2}iz + \frac{1}{2}, \frac{1}{2}\alpha - \frac{1}{2}iz + \frac{1}{2}\right) G_n\left(\frac{\alpha + iz + 1}{2}\right) dz.$$

Formulas (7.8), (7.9) and (7.10) reduce to those of BATEMAN [4] in case $\gamma = \alpha - \beta = 0$, and to those of PASTERNAK [12] in case $\gamma = \beta = 0$ and $\alpha = n$.

8. - Expansion formulas.

Let the BESSEL function $J_\mu(z)$ be defined by means of

$$J_\mu(z) = \frac{\left(\frac{1}{2}z\right)}{\Gamma(\mu+1)} {}_0F_1\left[\begin{matrix} - \\ \mu+1 \end{matrix}; -\frac{z^2}{4}\right].$$

It follows [6, p. 130] that

$$(8.1) \quad \left(\frac{z}{2}\right)^{\mu-v} k^v J_v(kz) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu+n)(\mu+2n)}{n! \Gamma(v+1)} {}_2F_1\left[\begin{matrix} -n, \mu+n \\ v+1 \end{matrix}; k^2\right] J_{\mu+2n}(z)$$

where k is arbitrary, $\mu, v, \mu - v \neq -1, -2, -3, \dots$.

Put in (8.1) $k^2 = tx$, $v = \alpha$, multiply both sides by $t^{\alpha-1}(1-t)^{\beta-\alpha}$ and then integrate on both sides from 0 to 1 get

$$(8.2) \quad \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2}z\right)^{\mu+2r}}{r! \Gamma(\alpha+r+1)} \int_0^1 t^{\alpha+r-1} (1-t)^{\beta-\alpha} dt \\ = \sum_{n=0}^{\infty} \frac{\Gamma(\mu+n)(\mu+2n)}{n! \Gamma(\alpha+1)} J_{\mu+2n}(x) \int_0^1 t^{\alpha-1} (1-t)^{\beta-\alpha} \\ \cdot {}_2F_1\left[\begin{matrix} -n, \mu+n \\ \alpha+1 \end{matrix}; tx \right] dt.$$

We note that integrating (8.1) term by term is permissible because both sides are absolutely convergent for all x and z .

Now the integral on the left hand side of (8.2) has the value

$$\left(\frac{1}{2}z\right)^{\mu+1} {}_1F_2\left[\begin{matrix} \zeta; \\ \alpha+1, \beta+1; \end{matrix} -\frac{z^2x}{4}\right]$$

while the integral on the right hand side is evaluated by means of (7.2). We thus get

$$(8.3) \quad \left(\frac{1}{2}z\right)^{\mu+1} {}_1F_2\left[\begin{matrix} \zeta; \\ \alpha+1, \beta+1; \end{matrix} -\frac{z^2x}{4}\right] \\ = \sum_{n=0}^{\infty} \frac{\Gamma(\mu+n+1)(\mu+2n+1)}{n!} G_n(\zeta, x; \alpha, \beta, \mu) J_{\mu+2n+1}(2x).$$

If we put $\alpha=0$, $\mu=1=p$, formula (8.3) becomes

$$(8.4) \quad \left(\frac{1}{2}z\right) {}_1F_2\left[\begin{matrix} \zeta; \\ 1, p; \end{matrix} -\frac{xz^2}{4}\right] \\ = \sum_{n=0}^{\infty} (2n+1) H_n(\zeta; p, x) J_{2n+1}(x).$$

This formula was not given by RICE [15]. If we specialize further we can get similar new relations involving the BATEMAN and PASTERNAK polynomials. We may also compare (8.4) with the following formula which is due to RICE [15]:

$$(8.5) \quad e^{iz} {}_1F_1\left[\begin{matrix} \zeta; \\ p; \end{matrix} -2izx\right] = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} i^n (2n+1) H_n(\zeta, \rho, x) J_{n+\frac{1}{2}}(z).$$

However we can generalize (8.5) to get

$$(8.6) \quad e^{iz} {}_2F_2\left[\begin{matrix} 1, \zeta; \\ \alpha+1, \beta+1; \end{matrix} -2izx\right] \\ = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} i^n (2n+1) G_n(\zeta, x; \alpha, \beta, 0) J_{n+\frac{1}{2}}(z).$$

This can be proved by replacing x by tx in (8.5) and using the operational formula (7.2).

Another generalization of (8.5) can be obtained in the following way.

Consider the formula [8, p. 213]

$$(8.7) \quad e^{ixz} = (2iz)^{-\frac{1}{2}\gamma-1} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n+1)}{\Gamma(\gamma+2n+1)} M_{k,m}(2iz) P_n^{(\beta, \gamma-\beta)}(x)$$

where $k = \frac{1}{2}\gamma - \beta$, $m = n + \frac{1}{2}(\gamma - 1)$ and [21, p. 337]

$$M_{k, m}(z) = z^{m+\frac{1}{2}} e^{-\frac{1}{2}z} {}_1F_1 \left[\begin{matrix} m - k + \frac{1}{2} \\ 2m + 1 \end{matrix} ; z \right].$$

Now replace x by $1 - 2xt$ and use (7.2) we get

$$\begin{aligned} (8.8) \quad e^{iz} {}_1F_1 \left[\begin{matrix} \zeta \\ \alpha + 1 \end{matrix} ; -2zx \right] \\ = (2iz)^{-\frac{1}{2}\gamma-1} \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + n + 1)(\beta + 1)_n}{n! \Gamma(\gamma + 2n + 1)} M_{k, \frac{1}{2}\gamma + n + \frac{1}{2}}(2iz) G_n(\zeta, x). \end{aligned}$$

Putting $\gamma = \beta = 0$ so that $k = 0$ in (8.8) we get (8.5).

Equating coefficients of $\left(\frac{Z}{2}\right)^n$ in (8.3) we get

$$(8.9) \quad x^n(\zeta)_n = (\alpha + 1)_n(\beta + 1)_n \sum_{k=0}^n \frac{(-n)_k(\gamma + 2k + 1)(\gamma + 1)_k}{k! (\gamma + 1)_{n+k+1}} G_k(\zeta, x).$$

Another proof of this formula can be obtained by operating with formula (7.2) on the formula [13, p. 262]

$$(1 - x)^n = 2^n(\alpha + 1)_n \sum_{k=0}^{\infty} \frac{(-n)_k(\alpha + \beta + 2k + 1)(\alpha + \beta + 1)_k}{(\alpha + 1)_k(\alpha + \beta + 1)_{n+k+1}} P_k^{(\alpha, \beta)}(x)$$

after letting $\alpha \rightarrow \beta$, $\theta \rightarrow \beta - \gamma$, $x \rightarrow 1 - 2xt$.

Formula (8.9) suggests some further possibilities. In the first place we have the multiplication formula

$$\begin{aligned} (8.10) \quad G_m(\zeta, xy; \alpha, \beta, \gamma) &= \sum_{s=0}^m \binom{m}{s} \frac{(\gamma + m + 1)_s}{(\gamma + s + 1)_s} G_s(\zeta, x) \\ &\quad \cdot y^s \sum_r (-1)^r \binom{m-s}{r} \frac{(\gamma + m + s + 1)_r}{(\gamma + 2s + 2)_r} y^r, \end{aligned}$$

and more generally

$$\begin{aligned} (8.11) \quad G_m(\zeta, xy; \alpha, \beta, \gamma') &= \sum_{s=0}^m \binom{m}{s} \frac{(\gamma' + m + 1)_s}{(\gamma + 2s + 1)_s} G_s(\zeta, x) \\ &\quad \cdot y^s \sum_r (-1)^r \binom{m-s}{r} \frac{(\gamma' + m + s + 1)_r}{(\gamma + 2s + 2)_s} y^r, \end{aligned}$$

To obtain (8.10) first multiply (8.9) by

$$\frac{(-m)_n(\gamma + m + 1)_n}{(\alpha + 1)_n(\beta + 1)_n} y_n$$

and sum over n .

When $|w| > |x\zeta|$, it can be shown that

$$(8.12) \quad \frac{1}{w - x\zeta} = \sum_{n=0}^{\infty} \frac{(\zeta)_n x^n}{(w)_{n,x}},$$

where

$$(w)_{n,x} = w(w+x)(w+2x) \dots (w+nx).$$

Substitute for $x^n(\zeta)_n$ from (8.9) and simplify, we get

$$(8.13) \quad \frac{1}{w - x\zeta} = \sum_{k=0}^{\infty} \frac{(-1)^k(\gamma + 2k + 1)(\gamma + 1)_k}{k!} G_k(\zeta, x) \\ \cdot \sum_{n=k}^{\infty} \frac{n! (\alpha + 1)_n(\gamma + 1)_n}{(n-k)! (\gamma + 1)_{n+k+1} (w)_{n,k}}.$$

If we put $x=1$ we get BATEMAN'S formula [4]. Also from (8.12) for $x=1$ we get

$$(8.14) \quad \frac{1}{w - \zeta} = \sum_{n=0}^{\infty} \frac{(\zeta)_n}{(w)_{n+1}} \quad (|w| > |\zeta|).$$

Now let $x=1$ in (8.9) and substitute in (8.4). We get

$$(8.15) \quad \frac{1}{w - \zeta} = \sum_{k=0}^{\infty} \frac{(-1)^k(\gamma + 2k + 1)(\alpha + 1)_k(\beta + 1)_k(\gamma + 1)_k}{(w)_{k+1}(\gamma + 1)_{2k+1}} G_k(\zeta, x) \\ \cdot x^{-k} {}_3F_2 \left[\begin{matrix} k+1, \alpha+k+1, \beta+k+1; \\ w+k+1, \gamma+2k+1; \end{matrix} \frac{1}{x} \right].$$

The function

$${}_3F_2 \left[\begin{matrix} k+1, \alpha+k+1, \beta+k+1; \\ w+k+1, \gamma+2k+2; \end{matrix} \frac{1}{x} \right]$$

which occurs in the right member of (8.15) has an interesting connection with $Q_n^{(\alpha, \beta)}(x)$. This we shall give in § 9.

9. - Miscellaneous results.

We note that our polynomials $G_n(\zeta, x)$ belong to the class of polynomials with a generating function of the form

$$(9.1) \quad (1-t)^{-c} \Psi\left(\frac{-4xt}{(1-t)^2}\right) = \sum_{n=0}^{\infty} f_n(x) t_n$$

where

$$\Psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n, \quad \gamma_0 \neq 0.$$

This remark is suggested by (6.1). Direct application of Theorem 48 in [13, p. 137] leads to the following relations.

$$(9.2) \quad (\gamma + n) \left[x \frac{\partial}{\partial x} G_n(\zeta, x) - n G_n(\zeta, x) \right] \\ = -n(\gamma + n) G_{n-1}(\zeta, x) - xn G_{n-1}(\zeta, x),$$

$$(9.3) \quad \frac{(\gamma + 1)_n}{n!} x \frac{\partial}{\partial x} G_n(\zeta, x) - \frac{(\gamma + 1)_n}{(n+1)!} G_n(\zeta, x) \\ = -(\gamma + 1) \sum_{k=0}^{n-1} \frac{(\gamma + 1)_k}{k!} G_k(\zeta, x) \\ - 2x \sum_{k=0}^{n-1} \frac{(\gamma + 1)_k}{k!} \frac{\partial}{\partial x} G_k(\zeta, x)$$

and

$$(9.4) \quad \frac{(\gamma + 1)_n}{n!} x \frac{\partial x}{\partial} G_n(\zeta, x) - \frac{(\gamma + 1)_n}{n!} G_n(\zeta, x) \\ = \sum_{k=0}^{n-1} (-1)^{n+k} (\gamma + 2k + 1) G_k(\zeta, x).$$

If we replace x in (2.5) by $1 - 2xt^{-1}$, α , by β and β by $\gamma - \beta$ we get

$$Q_n^{(\beta, \gamma-\beta)}\left(1 - \frac{2x}{t}\right) = \frac{\Gamma(\beta + n + 1)\Gamma(\gamma - \beta + n + 1)}{2\Gamma(\gamma + 2n + 2)} \\ \cdot (-x)^{-\beta-n-1} t^{\gamma+n+1} (t-x)^{\beta-1} {}_2F_1\left[\begin{matrix} n+1, \beta+n+1 \\ \gamma+2n+2 \end{matrix}; \frac{t}{x}\right].$$

From this and (8.1) we obtain

$$\begin{aligned}
 (9.5) \quad & {}_3F_2 \left[\begin{matrix} n+1, \beta+n+1, \alpha+n+1; \\ \gamma+2n+2, w+n+1 \end{matrix} ; \frac{1}{x} \right] \\
 &= \frac{2(-x)^{\beta+n-1}}{B(\beta+n+1, \gamma-\beta+n+1) B(\alpha+n+1, w-\alpha)} \\
 &\quad \cdot \int_0^1 t^{\alpha-\gamma-1} (t-x)^{\gamma-\beta} (1-t)^{w-\alpha-1} \cdot Q_n^{(\beta, \gamma-\beta)} \left(1 - \frac{2x}{t} \right) dt.
 \end{aligned}$$

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