Second Order Boundedness Criteria (*).

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Summary. – *See Introduction.*

1. – Introduction.

In this note we present certain conditions for boundedness of solutions and their first derivatives for a second order differential equation of the form

$$x'' + f(x, x')x' + g(x) = 0 \tag{1}$$

in which $x' = dx/dt$.

It is assumed that $f: (-\infty, \infty) \times (-\infty, \infty) \to (-\infty, \infty)$, $g: (-\infty, \infty) \to (-\infty, \infty)$, f and g are continuous, and

$$xg(x) > 0 \quad \text{and} \quad f(x, y) \geqslant 0 \quad \text{if } |x| \geqslant x_1 \tag{2}$$

for some $x_1 \geqslant 0$.

Much has been written and continues to be written about this equation. Condition (2) allows (1) to include the classical problems of relaxation oscillations of which the van der Pol and Liénard equations are examples, while selecting $x_1 = 0$ brings us to the standard oscillation problems with non-negative damping.

Excellent surveys of these problems have been written in recent years and the interested reader is referred to the books by Sansone and Conti [9] and Reissig, Sansone and Conti [8] for detailed discussions. In addition, extensive bibliographies have been offered in [3], [4], and [7]. Repeating those bibliographies here seems unwarranted.

In the literature the problem specified by (2) is generally divided into three parts:

$$xg(x) > 0 \ \text{ if } x \neq 0 \quad \text{and} \quad f(x, y) > 0\,. \tag{2)'$$

$$f(x, y) \geqslant 0 \ \text{ if } x^2 + y^2 \geqslant M > 0 \quad \text{and} \quad xg(x) > 0 \ \text{ if } |x| \geqslant M\,. \tag{2)''$$

$$f(x, y) \geqslant 0 \ \text{ if } |x| \geqslant M > 0 \quad \text{and} \quad xg(x) > 0 \ \text{ if } |x| \geqslant M\,. \tag{2)'''$$

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Problem (2)′ was surveyed by BUSHAW [5]. It governs the very concrete physical problems in which a body in equilibrium is set in motion and subjected to a restoring force which acts always to return it to the equilibrium position, together with a friction force which always acts in a manner to slow the motion. In any case, one writes (1) as the equivalent system

$$x' = y\,, \qquad y' = -f(x, y)\,y - g(x)\,. \tag{3}$$

By considering the function

$$V(x, y) = \bar{G}(x) + y^2/2 \tag{4}$$

with $\bar{G}(x) = \int_0^x g(s)\,ds$ so that along solutions of (3) the derivative of V satisfies

$$V'(x, y) = -g(x, y)\,y^2 \tag{5}$$

the author [1] pointed out that when (2)′ holds then any solution of (3) has $|y(t)|$ bounded for all future time. From this it was easy to see that all solutions of (3) under (2)′ were bounded if and only if each solution starting in Quadrants I or III subsequently crossed the x-axis [1]. A synthesis of [1] and [2] yields the following result.

THEOREM 1. – Suppose that (2)′ holds and that either

(i) $f(x, y) = h(x)r(y)$ with h and r positive functions, or

(ii) $f(x, y) = h(x)|y|^{\alpha}$ with h positive and $0 \leqslant \alpha < 1$. Then all solutions of (3) are bounded if and only if

(iii) $\int_0^{\pm\infty} [h(x) + |g(x)|]\,dx = \pm\infty$.

Problem (2)″ governs physical problems of more complexity than (2)′, yet the basic ingredients are the same. There can arise relaxation oscillations, but not of the van der Pol or Liénard type, as f must be non-negative for large $|y|$. WILLETT and WONG [10] observed that when (2)″ holds, then Equation (5) still holds and for $x^2 + y^2 \geqslant M$, $V' \leqslant 0$ so that independent of the fact that V is no longer positive definite (and hence is not a Liapunov function) it is still an immediate consequence that for any solution $(x(t), y(t))$ of (3), $|y(t)|$ is bounded. They then, of course, concluded that all solutions of (3) under (2)″ are bounded if and only if any solution of (3) entering Quadrant I or III with $|x(t)| \geqslant M$ subsequently crosses the x-axis. This work was done under the assumption that solutions of (3) are unique, but careful analysis of their work shows that assumption unnecessary. Willett and Wong also obtained a strong condition for unboundedness of solutions and an interesting result on boundedness in case $(2)''$ holds but $f(x, y) \to 0$ as $y \to 0$ in a certain monotone fashion.

Problem $(2)'''$ has been considered to be far more difficult than the others and the results are considerably less sweeping. The main exception to this is the work by GRAEF [7] when $f(x, y) = h(x)y$ and a forcing function is included. He allows $M > 0$ and shows that Condition (iii) of Theorem 1 is very nearly necessary and sufficient for boundedness even with a periodic forcing function. However, when f does not have this simple form, then the usual conditions used for boundedness take on a harsh form of point by point inequalities, rather than «averaging» conditions of the type of (iii). Furthermore, a theorem stating that $(2)'''$ implies $|y(t)|$ bounded is missing from the literature and such a theorem is, in fact, false. Thus, investigators have been unable to reduce boundedness to showing that solutions in Quadrants I and III cross the x-axis. It is generally required that the investigator construct intricate Jordan curves bounding solutions.

REMARK 0. – The present investigation began under the assumption that $(2)''$ holds and our intent was to improve the results of WILLETT and WONG [10] showing boundedness in the difficult cases resulting from $f(x, y)$ becoming wery small along lines given by $y =$ constant. Under $(2)''$ boundedness is trivial if $\int_0^{\pm\infty} g(s)\,ds = +\infty$ and so we assumed that either $\bar{G}(\infty) < \infty$ or $\bar{G}(-\infty) < \infty$. (In fact, Willett and Wong had assumed both of these integrals to be finite.) Under that assumption, we discovered that it was then possible to make a mild continuation hypothesis allowing us to treat the more general case (2). This is a surprise in that under $(2)''$, $\bar{G}(\pm\infty) = +\infty$ implies boundedness, but under (2) the condition of $\bar{G}(\pm\infty) = +\infty$ can actually *cause* unboundedness. The following example and Lemma 2 prove this statement.

EXAMPLE 1. – Let $g(x) = x$ and define $S = \{(x, y): y > 0 \text{ and } |x| \leqslant 2\}$, $U = \{(x, y): y > 1 \text{ and } |x| \leqslant 1$, and let H be the complement of S. Suppose that $f(x, y)$ is a continuous function satisfying $f(x, y) = 0$ on H, $f(x, y) \leqslant 0$ on S, and $f(x, y) \leqslant -1$ on U. Under these conditions, if $(x(t), y(t))$ is the solution of (3) starting at $x(0) = 2$ and $y(0) = 2$, then $x^2(t) + y^2(t) = 8$ as the solution traverses the circle in the clockwise direction until $x = -2$ and $y = 2$. At that time, $x^2(t) + y^2(t)$ increases until $x(t) = 2$ and $y(t) > 2$, say at $t = t_1 > 0$. Then $x^2(t) + y^2(t) = 4 + y^2(t_1)$ until $x(t) = -2$ and $y(t) > 0$. The pattern is repeated infinitely often. In fact, the function $V(x, y) = (x^2 + y^2)/2$ satisfies $V' \geqslant 0$ along solutions of (3) with V' positive on S. It is easily seen that the solution spirals off to infinity.

We next note that under (2), solutions of (3) can have finite escape time. This would happen, for example, if $f(x, y) = (x^2 - 1)y^3$ as elementary investigation will show. The solution starting at $x_0 = 0$ and $y_0 > 0$ but large will have the property that $y(t) \to \infty$ before $x(t)$ reaches $\frac{1}{2}$, independent of $g(x)$.

2. – Continuation and boundedness.

Our first result embodies a continuation hypothesis which is generally not directly verifiable from the f and g. As the proof is long and cumbersome, we omit it and offer instead a complete proof of a weaker result which is readily verifiable and which may indicate to the interested reader just how to construct a proof of the first result.

LEMMA 1. – Let (2) hold with $\bar{G}(\infty)<\infty$ or $\bar{G}(-\infty)<\infty$. Suppose that for each $y_0>0$ the maximal solution $y(x, x_0, y_0)$ of

$$y(dy/dx) = -f(x, y)y - g(x) \tag{3$'$}$$

with $y(x_0, x_0, y_0)=y_0$ and $-x_1 \leqslant x_0 < x_1$ can be continued as a solution for $x \geqslant x_0$ until $y(x, x_0, y_0)=0$ or until $x=x_1$; then there exists $K(x_0, y_0)>0$ such that each solution $(x(t), y(t))$ of (3) defined at $t=t_0$ with $x(t_0)=x_0$ and $y(t_0)=y_0$ satisfies $y(t) \leqslant K(x_0, y_0)$ on its maximal right interval of existence past t_0. Suppose that for each $y_0<0$ the minimal solution $y(x, x_0, y_0)$ of (3)$'$ with $y(x_0, x_0, y_0)=y_0$ and $-x_1 < x_0 \leqslant x_1$ can be continued as a solution for $x \leqslant x_0$ until $y(x, x_0, y_0)=0$ or until $x=-x_1$; then there exists $K(x_0, y_0)>0$ such that each solution $(x(t), y(t))$ of (3) defined at $t=t_0$ satisfies $y(t) \geqslant -K(x_0, y_0)$ on its maximal right-interval of definition.

LEMMA 2. – Let (2) hold and suppose there is a continuous function $q\colon (-\infty, \infty) \to (0, \infty)$ such that for $-x_1 \leqslant x \leqslant x_1$ we have

$$\text{(Q)}\qquad \begin{cases} q(y) \geqslant -f(x, y)y - g(x) & \text{for } y>0\,, \\ q(y) \geqslant f(x, y)y + g(x) & \text{for } y<0\,, \\ \displaystyle\int_0^{\pm\infty} [y/q(y)]\,dy = +\infty\,, \quad \text{and} \quad \bar{G}(\infty)<\infty & \text{or } \bar{G}(-\infty)<\infty\,. \end{cases}$$

Then for each (x_0, y_0) there exists $K(x_0, y_0)$ such that any solution $(x(t), y(t))$ defined at any t_0 with $x(t_0)=x_0$ and $y(t_0)=y_0$ can be continued as a solution of (3) for all $t \geqslant t_0$ and for all such t we have $|y(t)| \leqslant K(x_0, y_0)$.

PROOF. – Notice first that if we can show that $|y(t)|$ is bounded so long as $(x(t), y(t))$ is defined, then we can conclude that all solutions can be continued for all future time. To see this, note that $|y(t)| \leqslant K$ implies from (3) that $|x'(t)| \leqslant K$ and so $|x(t)| \leqslant |x(t_0)| + K(t-t_0)$. It is known [6; p. 61] that a solution $(x(t), y(t))$ on $[t_0, T)$ can fail to be defined past T only if $x^2(t)+y^2(t) \to \infty$ as $t \to T^-$. Our inequalities prohibit this behavior.

Our proof consists of finding a Liapunov function with several curves of discontinuities and matching together the « level » curves to form either one or two curves bounding $y(t)$.

Let (x_0, y_0) be given. We will find $K(x_0, y_0)$.

Case I. Suppose $\bar{G}(\infty) < \infty$.

We define

$$G(x) = \begin{cases} \int_{x_1}^{x} g(s)\,ds & \text{if } x \geqslant x_1 \\ \int_{-x_1}^{x} g(s)\,ds & \text{if } x \leqslant x_1 \end{cases}$$

and define

$$R(x, y) = G(x) + y^2/2 \ .$$

Case I(a). Suppose $\bar{G}(-\infty) < \infty$.

Then consider the set of points S_1 in Quadrant IV satisfying $R(x, y) = d_1$ where $d_1 = G(\infty) + G(-\infty) + y_0^2/2$ and $x_1 \leqslant x < \infty$. We have $R'(x, y) \leqslant 0$ for $x \geqslant x_1$ and so no solution of (3) crosses S_1 from above. Now S_1 intersects the line $x = x_1$ at a point (x_1, y_1). Clearly, $y_1 < y_0$. Define $W(x, y) = x + \int_0^y [s/q(s)]\,ds$ for $y < 0$ and $-x_1 \leqslant x \leqslant x_1$. We have $W' = y - [f(x, y)y + g(x)]y/q(y) \leqslant 0$ along solutions of (3) on its domain of definition by (Q). Let S_2 denote the set of points with $y < 0$ satisfying $W(x, y) = W(x_1, y_1)$ for $-x_1 \leqslant x \leqslant x_1$. No solution of (3) crosses S_2 from above. Also, S_2 intersects the line $x = -x_1$ at a point $(-x_1, y_2)$ with $y_2 < y_1 < -|y_0|$. Now define a set S_3 in Quadrant III for $-\infty < x \leqslant -x_1$ by $R(x, y) = R(-x_1, y_2)$. We have $R'(x, y) \leqslant 0$ for $x \leqslant -x_1$. By choice of d_1, (x_0, y_0) lies above $S_1 \cup S_2 \cup S_3$ and $(x(t), y(t))$ cannot cross that set for increasing t.

In a similar manner, let S_4 be the set in Quadrant II for $-\infty < x \leqslant -x_1$ with $R(x, y) = d_1$. Define $y_3 > 0$ by $R(-x_1, y_3) = d_1$. Define $Z(x, y) = -x + \int_0^y [s/q(s)]\,ds$ for $y > 0$ and $-x_1 \leqslant x \leqslant x_1$. Let S_5 be the set of points with $y > 0$, $-x_1 \leqslant x \leqslant x_1$, and $Z(x, y) = Z(-x_1, y_3)$. Then on its domain of definition we have $Z' = -y - [f(x, y)y + g(x)]y/q(y) \leqslant 0$ by (Q). Determine $y_4 > 0$ by $Z(x_1, y_4) = Z(-x_1, y_3)$. Finally, let S_6 be the set of points with $y > 0$, $x \geqslant x_1$, and $R(x, y) = R(x_1, y_4)$. We have $R' \leqslant 0$ for $x \geqslant x_1$. Thus, the set $S_4 \cup S_5 \cup S_6$ is an upper bound for $(x(t), y(t))$. We pick $K(x_0, y_0) = \max[-y_2, y_4]$. This completes the proof of Case I(a).

Case I(b). Let $G(-\infty) = \infty$.

Then let $d_2 = G(\infty) + G(-x_1 - |x_0|) + y_0^2/2$ and consider the set M_1 for $x \geqslant x_1$ and $y < 0$ defined by $R(x, y) = d_2$. No solution of (3) crosses M_1 from above. Determine $y_1 < 0$ by $R(x_1, y_1) = d_2$. Continue from (x_1, y_1) with a set M_2 defined by $W(x, y) = W(x_1, y_1)$ to a point $(-x_1, y_2)$. Continue from $(-x_1, y_2)$ to $(-x_1, -y_2)$ with a set M_3 given by $R(x, y) = R(-x_1, y_2)$. Now determine the set M_4 by $Z(x, y) = Z(-x_1, -y_2)$ with $y > 0$ for $-x_1 \leqslant x \leqslant x_1$. Finish the curve with a set M_5

defined by $R(x, y) = R(x_1, y_4)$ for $y_4 > 0$ and y_4 the solution of $Z(x_1, y_4) = Z(-x_1, -y_2)$. The set M_5 is defined for $x_1 \leqslant x < \infty$. Now $M_1 \cup \ldots \cup M_5$ is a horseshoe shaped set, open to the right, bounding $(x(t), y(t))$ from above, from below, and on the left. Pick $K(x_0, y_0) = \max[-y_2, y_4]$. This completes the proof of Case I.

Case II. Suppose $G(\infty) = \infty$.

Then it must be (by (Q)) that $G(-\infty) < \infty$. One repeats the proof of Case I(b) starting in the left half-plane and constructs a horseshoe shaped curve opening to the left which bounds $(x(t), y(t))$. The reader should experience no difficulty in filling in the details. A sketch is advisable.

LEMMA 3. – Let (2) and (Q) hold. Then every solution of (3) is bounded if and only if each solution of (3) entering Quadrants I or III with $|x(t_0)| \geqslant x_1$ subsequently crosses the x-axis.

PROOF. – Given the result of Lemma 2, this is essentially contained in both [1] and [10]. If $(x(t), y(t))$ is any solution of (3), then there is a constant K with $|y(t)| \leqslant K$ for all future time. We first suppose that any solution crosses the x-axis as required. Now the maximal solution of $y(dy/dx) = -f(x, y)y - g(x)$ through $(\max[|x(t_0)|, x_1], K)$ for x increasing crosses the x-axis at some $(x_2, 0)$, forming a curve which, together with the line from $(x_2, 0)$ to $(x_2, -K)$ bounds $x(t)$ on the right, as $x' = y < 0$ in Quadrant IV. A similar construction in the left half-plane is accomplished by taking a solution of the same equation through $(-\max[|x(t_0)|, x_1], -K)$ intersecting the x-axis at some $(-x_3, 0)$, and continuing with a line from $(-x_3, 0)$ to $(-x_3, K)$, which bounds the solution on the left. Thus, the x-axis intersection requirement implies boundedness.

Suppose there is a solution $(x(t), y(t))$ entering Quadrant *I* with $x(t_0) > x_1$ which does not cross the x-axis. As $x' = y > 0$ and $y' \leqslant 0$, if $x(t)$ is not unbounded, then $x(t) \to X$ and $y(t) \to L$ as $t \to \infty$. Clearly, $L = 0$ as $x' = y$ and we say $x(t) \to X < \infty$. Thus, $y' \to -g(X)$ and so for large t we have $y' \leqslant -g(X)/2$. An integration of this last inequality implies that $y(t)$ becomes negative, a contradiction. This completes the proof.

We next offer a sufficient condition for unboundedness of solutions of (3) which is, in fact, a fairly trivial generalization of Theorem 3.4 of [10]. It is therefore presented without proof. The only real change being that we have replaced zero with a positive number c.

LEMMA 4. – Let (2) hold. If there exist numbers $c \geqslant 0$ and $\varepsilon > 0$ such that

$$(a) \quad \int_{x_1}^{\infty} \Big(\max_{c \leqslant y \leqslant c+\varepsilon} f(x, y) \Big) dx < \infty, \text{ or}$$

$$(b) \quad \int_{-x_1}^{-\infty} \Big(\min_{-c-\varepsilon \leqslant y \leqslant -c} f(x, y) \Big) dx > -\infty,$$

then (3) has unbounded solutions.

In the following results on boundedness, we make use of the fact that if there are solutions in the Quadrant I (or III) with $|x(t)| > x_1$ which do not cross the x-axis, then $y(t)$ converges to a constant. Thus, conditions for boundedness need only be given on arbitrarily narrow strips, say on $c \leqslant y \leqslant c + \varepsilon$ and $x_1 \leqslant x < \infty$.

For the next theorem we need the following conditions, arrived at in part by reversing the hypotheses of Lemma 4. For convenience in notation, we now define

$$F(\infty, x, c) = [2(G(\infty) - G(x) + c^2/2)]^{\frac{1}{2}} \quad \text{for } x \geqslant x_1,$$

and

$$F(-\infty, x, c) = [2(G(-\infty) - G(x) + c^2/2)]^{\frac{1}{2}} \quad \text{for } x \leqslant -x_1 .$$

Condition (C): $G(\infty) < \infty$ and for each $c \geqslant 0$ there exists $\varepsilon > 0$ such that either

(i) $\displaystyle\int_{x_1}^{\infty}\Big(\min_{c \leqslant y \leqslant c+\varepsilon} f(x, y)\Big)\, dx = +\infty$, or

(ii) for each fixed $x \geqslant x_1$, $f(x, y)$ is nondecreasing in y for $c \leqslant y \leqslant c + \varepsilon$ and $\displaystyle\int_{x_1}^{\infty} f(x, F(\infty, x, c))\, dx = +\infty$.

Condition (D): $G(-\infty) < \infty$ and for each $c \geqslant 0$ there exists $\varepsilon > 0$ such that either

(i) $\displaystyle\int_{-x_1}^{-\infty}\Big(\max_{-c \geqslant y \geqslant -c-\varepsilon} f(x, y)\Big)\, dx = -\infty$, or

(ii) for each fixed $x \leqslant -x_1$, $f(x, y)$ is nonincreasing in y for $-c \geqslant y \geqslant -c-\varepsilon$ and $\displaystyle\int_{-x_1}^{-\infty} f(x, -F(-\infty, x, c))\, dx = -\infty$.

THEOREM 2. – Let (2) and (Q) hold. Suppose that either $G(\infty) = \infty$ or (C) holds and suppose that either $G(-\infty) = \infty$ or (D) holds. Then all solutions of (3) are bounded.

PROOF. – We show that any solution entering Quadrant I with $x(t_1) > x_1$ crosses the x-axis.

The derivative of the function $R(x, y) = G(x) + y^2/2$ along solutions of (3) yields $R' \leqslant 0$ for $x(t) \geqslant x_1$. If $G(\infty) = \infty$, then clearly $x(t)$ is bounded so that the argument used in the proof of Lemma 3 brings the solution to the x-axis. We then suppose that $G(\infty) < \infty$ so that (C) holds and $R(x(t), y(t)) \to d$ as $t \to \infty$. In fact, $d \geqslant G(\infty)$ otherwise we would again have $x(t)$ bounded. Thus, as $y' \leqslant 0$, $y(t) \to c \geqslant 0$ and so by (C) there exists $\varepsilon > 0$ such that either (i) or (ii) holds. Also, there exists $t_2 \geqslant t_1$

with $c \leqslant y(t) \leqslant c+\varepsilon$ for $t \geqslant t_2$. If (i) holds, then $y' \leqslant -f(x,y)x'$ and so

$$y(t) \leqslant y(t_2) - \int_{t_2}^{t} f(x(s), y(s))x'(s)\,ds \leqslant$$
$$\leqslant y(t_2) - \int_{t_2}^{t}\Big(\min_{c\leqslant y\leqslant c+\varepsilon} f(x(s), y)\Big)x'(s)\,ds = y(t_2) - \int_{x(t_2)}^{x(t)}\Big(\min_{c\leqslant y\leqslant c+\varepsilon} f(s, y)\Big)\,ds .$$

As $x(t)$ is not bounded, the last integral and (C) (i) shows that $y(t)$ becomes negative.

We now suppose that (C) (i) fails and (C) (ii) holds. Again, we have $c \leqslant y(t) \leqslant \leqslant c+\varepsilon$ for $t \geqslant t_2$. As $R' \leqslant 0$, $y(t) \to c$, and $R(x(t), y(t)) \to G(\infty) + c^2/2$ for $x(t)$ unbounded, we have $y(t) > F(\infty, x, c)$ for $t \geqslant t_2$. As f is monotone in y for fixed x and for $t \geqslant t_2$, we obtain

$$y' \leqslant -f(x, F(\infty, x, c))\,x' .$$

An integration from t_2 to t yields $y(t) \leqslant y(t_2) - \int_{x(t_2)}^{x(t)} f(s, F(\infty, s, c))ds$ which shows that $y(t)$ becomes negative if $x(t) \to \infty$ in view of (C)(ii). Thus, in any case the solution crosses the x-axis. A similar argument in Quadrant III using $G(-\infty) = \infty$ or (D) completes the proof.

REMARK 1. – Theoerm 2 is tailored for the case in which f becomes small along lines $y = c$. If $f \to 0$ as $y \to 0$ of order greater than y, then (ii) can be improved at $c = 0$.

CONDITION (E). – Let (C) hold for all $c > 0$ and suppose that for each fixed $x \geqslant x_1$, $f(x, y)/y$ is nondecreasing for $0 < y \leqslant \varepsilon$ for some $\varepsilon > 0$ and

$$\int_{x_1}^{\infty} g(s) \exp \int_{x_1}^{s} \big[f(u, F(\infty, u, 0))/F(\infty, u, 0)\big]\,du\,ds = +\infty .$$

Condition (F): Let (D) hold for each $c > 0$ and suppose that for each $x \leqslant -x_1$, $f(x, y)/y$ is a nonincreasing function of y for $-\varepsilon \leqslant y < 0$ for some $\varepsilon > 0$ and

$$\int_{-x_1}^{-\infty} g(s) \exp \int_{-x_1}^{s} \big[f(u, F(-\infty, u, 0))/F(-\infty, u, 0)\big]\,du\,ds = -\infty .$$

THEOREM 3. – Let (2) and (Q) hold. Suppose that $G(\infty) = \infty$ or (E) holds and suppose $G(-\infty) = \infty$ or (F) holds. Then all solutions of (3) are bounded.

PROOF. – Let $(x(t), y(t))$ be a solution of (3) with $x(t_1) \geqslant x_1$ and $y(t) > 0$. We show that the solution crosses the x-axis. If $G(\infty) = \infty$, then $x(t)$ is bounded on the right and one easily argues once more that the solution crosses the x-axis. Hence, we suppose (E) holds and $x(t)\to\infty$. By the proof of Theorem 2, if $y(t)>0$ for all $t\geqslant t_1$, then there exists t_2 with $y(t)\leqslant\varepsilon$ for all $t\geqslant t_2$. From (3) we obtain

$$2y\,dy/dx=-2[f(x,y)/y]y^2-2g(x)$$

which is linear in y^2 and so

$$y^2(x)\exp\int_{x(t_2)}^{x(t)}\big[2f(s,y(s))/y(s)\big]\,ds=y^2(x(t_2))-\int_{x(t_2)}^{x(t)}2g(s)\exp\int_{x(t_2)}^{s}\big[2f(u,y(u))/y(u)\big]\,du\,ds\leqslant$$

$$\leqslant y^2(x(t_2))-\int_{x(t_2)}^{x(t)}2g(s)\exp\int_{x(t_2)}^{s}\big[2f(u,F(\infty,u,0))/F(\infty,u,0)\big]\,du\,ds$$

by the assumed monotonicity. As $x(t)\to\infty$, a contradiction is obtained from the assumed divergence of the integral in (E). A similar argument in Quadrant III completes the proof.

REMARK 2. – Theorem 3 improves results by WILLETT and WONG [10; Ths. 3.2 and 3.3] in three ways. First, (Q) reduces their (2)″ considerably. Second, they require that (C) (i) and (D) (i) hold on arbitrarily wide strips instead of our arbitrarily narrow strips. Third, our integral conditions in (E) and (F) are averaging type conditions which replace their pointwise requirement:

(E)′ There exists $\varepsilon>0$ such that for each $c\geqslant 1$, there exists x_c such that

$$f\Big(x,\big[2c(\bar G(\infty)-\bar G(x))\big]^{\frac12}\Big)\big[2c(\bar G(\infty)-\bar G(x))\big]^{\frac12}\geqslant[(1+\varepsilon)c-1]g(x)\qquad\text{for } x\geqslant x_c\,.$$

We next offer a result which interpolates in a certain fashion between the conditions of Theorems 2 and 3. Subsequently, we give example showing that all of these theorems are independent.

Condition (G): Let $G(\infty)<\infty$ and suppose that for each $c\geqslant 0$ there exists $\varepsilon>0$ such that either

(i) $\displaystyle\int_{x_1}^{\infty}\Big(\min_{c\leqslant y\leqslant c+\varepsilon}f(x,y)\Big)dx=\infty$, or

(ii) there is a function $r\colon(c,c+\varepsilon]\to(0,\infty)$ such that $r(y)$ is continuous and positive if $y>c$; $\displaystyle\int_{y}^{c+\varepsilon}[ds/r(s)]\overset{\text{def}}{=}B(c+\varepsilon)-B(y)$ exists for $c\leqslant y\leqslant c+\varepsilon$ with $B(c)$ finite; for fixed $x>x_1$ then $f(x,y)/r(y)$ is nondecreasing for $c<y\leqslant c+\varepsilon$; and

$$\int_{x_1}^{\infty}\big[f(x,F(\infty,x,c))/r(F(\infty,x,c))\big]\,dx=+\infty\,.$$

Condition (H): Let $G(-\infty)<\infty$ and suppose that for each $c\geqslant 0$ there exists $\varepsilon>0$ such that either

(i) $\int_{-x_1}^{\infty}\Big(\max_{-c\geqslant y\geqslant -c-\varepsilon} f(x,y)\Big)\,ds=-\infty$, or

(ii) there is a function $r\colon[-c-\varepsilon, c)\to(0,\infty)$ such that $r(y)$ is continuous and positive if $y<-c$; $\int_{-c-\varepsilon}^{y}[ds/r(s)]\overset{\text{def}}{=}B(y)-B(-c-\varepsilon)$ exists for $-c\geqslant y\geqslant -c-\varepsilon$ with $B(-c)$ finite; for fixed $x<x_1$ then $f(x,y)/r(y)$ is nonincreasing in y; and $\int_{-x_1}^{-\infty}\Big[f(x, F(-\infty,x,c))/r(-F(-\infty,x,c))\Big]\,dx=-\infty$.

Theorem 4. – Let (2) and (Q) hold. Suppose $G(\infty)=\infty$ or (G) holds and suppose $G(-\infty)=\infty$ or (H) holds. Then all solutions of (3) are bounded.

Proof. – We show, once more, that a solution $(x(t),y(t))$ of (3) with $y(t_1)>0$ and $x(t_1)>x_1$ subsequently crosses the x-axis. This result is clear if $G(\infty)=\infty$ and so we assume that (G) holds and that $x(t)\to\infty$. Thus, we assume that the solution does not cross the x-axis so that $R(x(t),y(t))\to d$ and $y(t)\to c\geqslant 0$. If (G) (i) holds for this c, then a contradiction is obtained as in the previous proof. Thus, we assume that (G) (ii) holds and we take t_2 so large that $c+\varepsilon\geqslant y(t)>c$ for $t\geqslant t_2$. From (3) we have

$$y'/r(y)=-[f(x,y)/r(y)]\,y-g(x)/r(y)$$

so along the solution we obtain

$$y'/r(y)\leqslant-\Big[f(x,F(\infty,x,c))/r(F(\infty,x,c))\Big]\,x'\,.$$

An integration will yield a contradiction. A similar argument in Quadrant III will complete the proof.

Examples. – Let $g(x)=1/x^2$ for $x\geqslant 1$ so that $G(\infty)-G(x)=1/x$ for $x\geqslant 1$. Let $f(x,y)=\Big[1/(1+|x|)\Big]h(y)$ for $h(y)\geqslant 0$.

(*a*) If $h(y)=1+[\sin^2(1/y)]y^2$, then (C) (i) holds, but (E) and (G) (ii) fail.

(*b*) If $h(y)=\sqrt{|y|}$, then (C) fails, (E) fails, and (G) holds with $r(y)=\sqrt{|y|}$.

(*c*) If $h(y)=|y|$, then (C) fails, (E) holds, and (G) fails.

Remark 3. – If solutions are unique, then our boundedness conditions translate into existence of periodic solutions. If (2)$'$ holds, then the boundedness results become results on global asymptotic stability.

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