On the Growth of Meromorphic Solutions of Linear Differential Equations Having Arbitrary Entire Coefficients (*) .

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Summary. — *We investigate the class of linear ordinary differential equations having arbitrary entire functions as coefficients. In our main results, we obtain estimates for the growth of meromorphic solutions of such equations.*

1. — Introduction.

In this paper, we treat n -th order homogeneous linear differential equations,

$$(1) \quad L(y) = \sum_{i=0}^n f_i(z) y^{(i)} = 0 ,$$

where the coefficients $f_0, f_1, \dots, f_n$ are arbitrary entire functions.

In [11], G. VALIRON used the Valiron-Wiman theory [12; Chapter IV] to prove that when all the coefficients f_i are polynomials, then any entire solution of $L(y) = 0$ is of finite order of growth. It easily follows that when all the coefficients f_i are polynomials (and thus f_n has only finitely many zeros), then every solution of $L(y) = 0$ which is meromorphic in the plane must also be of finite order of growth.

In the case when one or more of the coefficients f_i are transcendental entire functions, the Valiron-Wiman theory can still be used to estimate the growth of an entire solution in terms of the growth of the coefficients. (In [6; pp. 202-205], M. FREI obtained this estimate in the case when f_n has no zeros. In the case when f_n has zeros, an easy modification of the proof, using a minimum modulus estimate on f_n which holds « nearly everywhere » (see § 2 below) such as in [10; p. 336] if f_n is of finite order or [4; p. 97] if f_n is of infinite order, also leads to an estimate on the growth of entire solutions in terms of the growth of the coefficients.)

In the case when one or more of the coefficients f_i are transcendental, the problem of determining the growth of meromorphic solutions in the plane of $L(y) = 0$ is more complicated than for entire solutions since the Valiron-Wiman theory is not applicable. The difference is indicated by an example constructed in [3] which shows

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that even in the first-order case, it is not possible to estimate the growth of a meromorphic solution in terms of the growth of the coefficients alone. (It was shown in [3] that given *any* increasing function $\Phi(r)$ on $(0, +\infty)$, there exists a meromorphic function $h(z)$ in the plane, which is a solution of a first-order linear differential equation $f_1 y' + f_0 y = 0$ whose coefficients f_0 and f_1 are entire functions of finite order, such that the Nevanlinna characteristic $T(r, h)$ of $h(z)$ is larger than $\Phi(r)$ at a sequence of r tending to $+\infty$. We remark that although a pole of the solution h must be a zero of the coefficient f_1 , the multiplicity of the pole of h may far exceed the multiplicity of the zero of f_1 .) It was shown in [1] that given any first-order differential equation $\Omega(z, y, y') = 0$, where Ω is a polynomial in y and y' whose coefficients are arbitrary entire functions of z , then the growth of a meromorphic solution of $\Omega = 0$ can be estimated in terms of the growth of the coefficients of Ω and the counting function for the poles of the solution. (The fact that the counting function for the poles cannot be eliminated from the estimate for the growth is indicated by the example of unrestricted growth described above.)

In the case of equations of order higher than one, it was shown in [2] that if $\Omega(z, y, y', \dots, y^{(n)})$ is a polynomial in $y, y', \dots, y^{(n)}$ whose coefficients are arbitrary entire functions of z , then the growth of any meromorphic solution $h(z)$ of $\Omega = 0$ can be estimated in terms of the growth of the coefficients of Ω and (i) the counting functions for the zeros and poles of h if $n = 2$, or (ii) the counting functions for the zeros and poles of h and the counting functions for the zeros of certain successive logarithmic derivatives, $h_1 = h'/h, \dots, h_k = h'_{k-1}/h_{k-1}$, for some $k \leq n - 2$, if $n > 2$. In the present paper, we show that in the special case of linear differential equations, these results can be greatly improved. In § 3 below, we show that for a second-order homogeneous linear differential equation having arbitrary entire coefficients, the growth of a meromorphic solution can be estimated in terms of the growth of the coefficients and the counting function for the zeros of the solution. (Of course, the example of unrestricted growth described earlier indicates that the counting function for the zeros cannot be eliminated from the estimate.) For homogeneous linear differential equations of order higher than two, we show in § 5 that regardless of the order of the equation, the growth of a meromorphic solution $h(z)$ can be estimated in terms of the growth of the coefficients and the counting functions for the zeros of h and for the distinct zeros of h'/h . In § 7, we show that for a homogeneous linear differential equation of any order n , the growth of a meromorphic solution h can be estimated in terms of the growth of the coefficients and the counting function for the distinct zeros of $h(z) - z^m$, for some m in the set $\{0, 1, \dots, n\}$. This result is a corollary of a theorem proved in § 6, which states that the growth of a meromorphic solution of the nonhomogeneous n -th order linear differential equation $L(y) = f$ (where f is entire, $f \not\equiv 0$), can be estimated in terms of the growth of the coefficients $f, f_0, \dots, f_n$ and the counting function for the distinct zeros of the solution.

In § 8, we investigate the growth of meromorphic solutions of a special class of n -th order linear differential equations which can be reduced to second-order equations.

2. — Notation.

For a meromorphic function $g(z)$ on the plane, we will use the standard notation for the Nevanlinna functions $m(r, g)$, $N(r, g)$ and $T(r, g)$ introduced in [9; pp. 6, 12]. We will also use the notation $n(r, g)$ to denote the number of poles (counting multiplicity) of g in $|z| \leq r$, and we will denote by $\bar{n}(r, g)$ the number of distinct poles of g in $|z| \leq r$. As in [9; p. 70], we will denote by $\bar{N}(r, g)$, the function obtained by replacing $n(r, g)$ by $\bar{n}(r, g)$ in the definition of $N(r, g)$. As in [2], we will say that a certain property $P(r)$ holds «nearly everywhere in $[0, +\infty)$ » if $P(r)$ holds for all $r \geq 0$ with the possible exception of a set of finite measure.

3. — THEOREM 1. — Let $f_0(z)$, $f_1(z)$ and $f_2(z)$ be entire functions. Let $M_1(r)$ be a monotone nondecreasing function on $[0, +\infty)$ and let $A(r)$ be a monotone non-increasing function on $[0, +\infty)$ such that $M_1(r) \geq 1$ and $A(r) > 0$ for all $r \geq 0$. Let

$$(2) \quad |f_j(z)| \leq M_1(r) \quad \text{on } |z| = r \text{ for } j = 0, 1, 2 \text{ and all } r > 0.$$

In addition, let

$$(3) \quad |f_2(z)| \geq A(r) \quad \text{on } |z| = r, \text{ nearly everywhere in } [0, +\infty).$$

Let $y(z)$ be a meromorphic function on the plane which is not identically zero and which satisfies the equation,

$$(4) \quad f_2(z)y''(z) + f_1(z)y'(z) + f_0(z)y(z) = 0,$$

at each point where y is analytic. Then for any $a > 1$, there exist positive constants K , K_1 and r_0 such that for all $r > r_0$,

$$(5) \quad T(r, y) \leq K \left(rN(ar, 1/y) + r^2 \exp(K_1 G(ar) \log^+ G(ar)) \right),$$

where,

$$(6) \quad G(r) = r\bar{N}(r, 1/y) + r^2 M_1(r)/A(r).$$

Thus, the growth of the solution y can be estimated in terms of the growth of the coefficients in the equation and the counting function $N(r, 1/y)$ for the zeros of y .

PROOF. — Given $a > 1$, let $\sigma > 1$ be such that $\sigma^2 = a$. Let $w = y'/y$. Then w satisfies the first-order equation,

$$(7) \quad f_2 w^2 + f_2 w' + f_1 w + f_0 = 0.$$

In view of (2) and (3), it follows from [1; § 3] or [2; § 3] that there exist positive constants K_2 and $r_1 > 1$ such that for all $r > r_1$,

$$(8) \quad T(r, w) \leq K_2(J(\sigma r)),$$

where,

$$(9) \quad J(r) = \log^+ M_1(r) + r^2 M_1(r)/A(r) + rN(r, w).$$

Clearly $n(t, w) = \bar{n}(t, y) + \bar{n}(t, 1/y)$ for all $t \geq 0$, so it easily follows that,

$$(10) \quad N(r, w) = \bar{N}(r, y) + \bar{N}(r, 1/y) \quad \text{for all } r > 0.$$

Since the coefficients f_0, f_1 and f_2 are entire, it follows easily from (4) that if y has a pole at a point z_0 , then f_2 must have a zero at z_0 . Of course, the multiplicity of the pole of y may far exceed the multiplicity of the zero of f_2 , but in any case we have $\bar{n}(t, y) \leq \bar{n}(t, 1/f_2)$ for all $t \geq 0$. It easily follows that there is a constant $K_3 \geq 0$ such that

$$(11) \quad \bar{N}(r, y) \leq \bar{N}(r, 1/f_2) + K_3 \quad \text{for all } r > 0.$$

Now by Jensen's formula, it follows [9; p. 11] that $T(r, 1/f_2) = T(r, f_2) + O(1)$ as $r \rightarrow \infty$, and since f_2 is entire, it thus follows from (2), [9; p. 24] and (11), that there are positive constants $r_2 > r_1$ and K_4 such that

$$(12) \quad \bar{N}(r, y) \leq \log^+ M_1(r) + K_4 \quad \text{for all } r > r_2.$$

Let $r_3 > 1$ be a value of r at which (2) and (3) are valid. Then clearly $M_1(r_3)/A(r_3) \geq 1$, and since M_1/A is nondecreasing we have $M_1(r)/A(r) \geq 1$ for all $r \geq r_3$. Thus there exists $r_4 > r_2$ such that,

$$(13) \quad rK_4 \leq r^2 M_1(r)/A(r) \quad \text{for } r > r_4.$$

Now $\log^+ M_1(r) \leq M_1(r)$ for all $r \geq 0$, and since $A(r)$ is nonincreasing, we thus have,

$$(14) \quad \log^+ M_1(r) \leq A(r_4) M_1(r)/A(r) \quad \text{for } r \geq r_4.$$

Letting $r_5 > r_4$ be a value of r for which $(r+1)A(r_4) + 2r^2 \leq 3r^2$ if $r > r_5$, it now follows from (9), (10), (12), (13) and (14) that for all $r > r_5$,

$$(15) \quad J(r) \leq 3G(r),$$

where $G(r)$ is as in (6).

Now let $u = 1/y$ and let $v = -w$. Then $v = u'/u$, so by [2; § 7], there exist positive constants $r_6 > r_5$, C and C_1 such that for all $r > r_6$,

$$(16) \quad T(r, u) \leq C(rN(\sigma r, u)) + r^2 \exp(C_1 \Psi(\sigma r)),$$

where,

$$(17) \quad \Psi(r) = T(r, v) + N(r, v)(\log r + \log^+ N(r, v)).$$

Since $v = -w$, clearly there exists $r_7 > r_6$ such that,

$$(18) \quad \Psi(r) \leq T(r, w)(2 \log r + \log^+ T(r, w)) \quad \text{for } r > r_7.$$

In view of (8) and (15) and the fact that $G(r) \geq r^2$ for $r > r_3$ (since $M_1(r)/A(r) \geq 1$ for $r > r_3$), it follows from (18), that there exists $r_8 > r_7$ such that,

$$(19) \quad \Psi(r) \leq 12K_2 G(\sigma r) \log^+ G(\sigma r) \quad \text{for all } r > r_8.$$

Since $T(r, y) = T(r, u) + O(1)$ by Jensen's formula and since $\sigma^2 = a$ and $N(\sigma r, u) \leq N(ar, u)$, it easily follows from (16) and (19) that there exists $r_0 > r_8$ such that (5) holds for all $r > r_0$, where $K = 2C$ and $K_1 = 12K_2 C_1$. This completes the proof of Theorem 1.

4. - LEMMA. - Let $\Omega(z, y, y', \dots, y^{(n)})$ be a polynomial in $y, y', \dots, y^{(n)}$, whose coefficients are meromorphic functions of z in the plane. For each $r > 0$, let $\Delta(r)$ be the maximum of the Nevanlinna characteristics of the coefficients of Ω . For each nonnegative integer q , let Ω_q denote the homogeneous part of Ω of total degree q in the indeterminates $y, y', \dots, y^{(n)}$. Let $y_0(z)$ be a meromorphic function in the plane which is not identically zero and which satisfies the equation $\Omega = 0$, but which for some nonnegative integer q does not satisfy the equation $\Omega_q = 0$. Then for any $a > 1$, there exist positive constants K and r_0 such that for all $r > r_0$, we have

$$(20) \quad T(r, y_0) \leq K(F(ar)),$$

where,

$$(21) \quad F(r) = \bar{N}(r, 1/y_0) + \bar{N}(r, y_0) + \Delta(r) + \log r.$$

PROOF. - Let $w_0 = y'_0/y_0$. Then it is easy to see by induction (see [7; p. 73]), that for each $j \geq 1$, $y_0^{(j)}/y_0$ can be written as a polynomial in $w_0, w'_0, \dots, w_0^{(j-1)}$ with nonnegative integer coefficients. It follows that for each nonnegative integer q , $\Omega_q(z, y_0(z), \dots, y_0^{(n)}(z))/(y_0(z))^q$ can be written $R_q(z, w_0(z), \dots, w_0^{(n-1)}(z))$, where $R_q(z, w_1, \dots, w_n)$ is a polynomial in $w_1, \dots, w_n$, whose coefficients are linear combina-

tions (with constant coefficients) of the coefficients of Ω_a . If m is the maximum nonnegative integer with the property that y_0 is not a solution of the equation $\Omega_m = 0$, then since y_0 satisfies $\Omega = 0$, we have

$$(22) \quad \sum_{a=0}^m R_a(z, w_0(z), \dots, w_0^{(n-1)}(z))(y_0(z))^a \equiv 0,$$

at each point where y_0 is not zero or infinity. Using the elementary rules for calculating with the Nevanlinna characteristic [9; p. 14], it follows [8; p. 108] that,

$$(23) \quad T(r, y_0) \leq \sum_{a=0}^m T(r, R_a(z, w_0, \dots, w_0^{(n-1)})) + O(1),$$

as $r \rightarrow +\infty$. Now by [7; p. 55], for each $j \geq 1$,

$$(24) \quad T(r, w_0^{(j)}) = O(T(r, w_0)),$$

nearly everywhere in $[0, +\infty)$ as $r \rightarrow +\infty$. In addition, by [9; p. 63] and (10), we have,

$$(25) \quad T(r, w_0) = O(\bar{N}(r, y_0) + \bar{N}(r, 1/y_0) + \log(rT(r, y_0))),$$

nearly everywhere in $[0, +\infty)$ as $r \rightarrow +\infty$. Using the elementary rules for calculating with the Nevanlinna characteristic [9; p. 14], and recalling the definitions of R_a and $\Delta(r)$, it follows easily from (23), using (24) and (25), that there is a constant K such that (20) holds with $a = 1$, nearly everywhere in $[0, +\infty)$. Since $T(r, y_0)$ and $F(r)$ are monotone nondecreasing functions, the conclusion of the lemma follows immediately from either [1; Lemma C, p. 68] or [2; § 2].

5. — THEOREM 2. — Let $f_0(z), f_1(z), \dots, f_n(z)$ be entire functions and let f_n be not identically zero. For each $r > 0$, let $\Phi(r)$ denote the maximum of $\{T(r, f_j) : j = 0, \dots, n\}$. Let $y_0(z)$ be a nonconstant meromorphic function on the plane which satisfies the linear differential equation,

$$(26) \quad \sum_{j=0}^n f_j y^{(j)} = 0,$$

at each point where y_0 is analytic. Let $w_0 = y'_0/y_0$. Then for any $a > 1$, there exist positive constants K, K_1 and r_0 such that for all $r > r_0$,

$$(27) \quad T(r, y_0) \leq K \left(rN(ar, 1/y_0) + r^2 \exp [K_1 H(ar) \log(rH(ar))] \right),$$

where,

$$(28) \quad H(r) = \bar{N}(r, 1/y_0) + \bar{N}(r, 1/w_0) + \Phi(r) + \log r.$$

Thus the growth of the solution y_0 can be estimated in terms of the growth of the coefficients and the counting functions for the zeros of y_0 and the distinct zeros of y'_0/y_0 .

PROOF. — Since y_0 satisfies equation (26), it is clear that w_0 satisfies an equation,

$$(29) \quad A(z, w, w', \dots, w^{(n-1)}) = 0,$$

where A is a polynomial in $w, w', \dots, w^{(n-1)}$, whose homogeneous part of maximum total degree in $w, w', \dots, w^{(n-1)}$ is $f_n w^n$, and where the coefficients of A are linear combinations (with integer coefficients) of $f_0, \dots, f_n$. Thus, if $a > 1$, and we let $\sigma > 1$ be such that $\sigma^2 = a$, then by § 4, there exist positive constants K_2 and r_1 , such that for all $r > r_1$, we have

$$(30) \quad T(r, w_0) \leq K_2 F(\sigma r),$$

where

$$(31) \quad F(r) = \bar{N}(r, 1/w_0) + \bar{N}(r, w_0) + \Phi(r) + \log r.$$

Since the coefficients of (26) are entire, it is easy to see that if y_0 has a pole at a point z_0 , then f_n must have a zero at z_0 . Thus $\bar{n}(t, y_0) \leq \bar{n}(t, 1/f_n)$ for all $t \geq 0$. From this and Jensen's formula (i.e. $T(r, 1/f_n) = T(r, f_n) + O(1)$), it follows using the definition of $\Phi(r)$, that there are positive constants $r_2 > r_1$ and K_3 such that for $r > r_2$,

$$(32) \quad \bar{N}(\sigma r, y_0) \leq \Phi(\sigma r) + K_3.$$

Since $\bar{N}(r, w_0) = \bar{N}(r, y_0) + \bar{N}(r, 1/y_0)$ (see (10)), it easily follows from (30), (31) and (32), that there exists $r_3 > r_2$ such that for all $r > r_3$,

$$(33) \quad T(r, w_0) \leq 2K_2 H(\sigma r),$$

where H is given by (28).

Now let $u = 1/y_0$ and $v = -w_0$. Then $v = u'/u$, so by [2; § 7], there exist positive constants $r_6 > r_3$, C and C_1 such that (16) holds for $r > r_6$, where $\Psi(r)$ is given by (17) and thus obeys (18) (where w is the function w_0) for all r greater than some number $r_7 > r_6$. Since $\sigma^2 = a$, it easily follows from (18) and (33) that there exists $r_8 > r_7$ such that for all $r > r_8$, we have,

$$(34) \quad \Psi(\sigma r) \leq 6K_2 H(ar) \log(rH(ar)).$$

Since $T(r, y_0) = T(r, u) + O(1)$, it easily follows from (16) and (34) that (27) holds for all r greater than some number r_0 , where $K = 2C$ and $K_1 = 6K_2 C_1$. This concludes the proof of Theorem 2.

REMARK. — With the notation as in Theorem 2, it is clear that $\bar{n}(t, 1/w_0) \leq \bar{n}(t, 1/y'_0)$ for all $t \geq 0$. It follows that the estimate (27) is valid for all sufficiently large r if $H(r)$ is replaced by,

$$(35) \quad H_1(r) = \bar{N}(r, 1/y_0) + \bar{N}(r, 1/y'_0) + \Phi(r) + \log r.$$

6. — THEOREM 3. — Let $f, f_0, \dots, f_n$ be entire functions and let f and f_n be not identically zero. For each $r > 0$, let $\Delta(r)$ be the maximum of the numbers $T(r, f)$, $T(r, f_0), \dots, T(r, f_n)$. Let $y_0(z)$ be a meromorphic function on the plane which is not identically zero and which satisfies the nonhomogeneous linear differential equation,

$$(36) \quad \sum_{i=0}^n f_i y^{(i)} = f,$$

at each point where y_0 is analytic. Then for any $a > 1$, there exist positive constants r_0 and K_0 such that for all $r > r_0$, we have

$$(37) \quad T(r, y_0) \leq K_0(\bar{N}(ar, 1/y_0) + \Delta(ar) + \log r).$$

Thus the growth of the solution y_0 of the nonhomogeneous equation can be estimated in terms of the growth of the coefficients and the counting function for the distinct zeros of y_0 .

PROOF. — By § 4, if $a > 1$, there are constants K and r_1 such that for all $r > r_1$, inequality (20) holds where $F(r)$ is given by (21). Since the coefficients of (36) are entire functions, clearly if y_0 has a pole at a point z_0 , then f_n has a zero at z_0 . Thus, $\bar{n}(t, y_0) \leq \bar{n}(t, 1/f_n)$ for all $t \geq 0$.

From this and Jensen's formula, it follows (as in (32)) using the definition of $\Delta(r)$ that for all r greater than some number r_2 , we have

$$(38) \quad \bar{N}(ar, y_0) \leq \Delta(ar) + K_1,$$

where K_1 is a constant. In view of (20), (21) and (38), it is clear that (37) holds for all r greater than some number r_0 , if we take $K_0 = 2K$. This concludes the proof of Theorem 3.

7. — COROLLARY. — Let $f_0, f_1, \dots, f_n$ be entire functions and let f_n be not identically zero. For each $r > 0$, let $\Phi(r)$ denote the maximum of the numbers $T(r, f_0), \dots,$

$T(r, f_n)$. Let $y_0(z)$ be a meromorphic function on the plane which is not identically zero and which satisfies the linear differential equation,

$$(39) \quad L(y) = \sum_{j=0}^n f_j y^{(j)} = 0,$$

at each point where y_0 is analytic. Then,

(A) If g is any entire function which is not a solution of equation (39), then for any $a > 1$, there exist positive constants K and r_0 , such that for all $r > r_0$, we have

$$(40) \quad T(r, y_0) \leq KW(ar),$$

where

$$(41) \quad W(r) = \bar{N}(r, 1/(y_0 - g)) + \Phi(r) + T(r, g) + \log r.$$

(B) There exists a nonnegative integer $m \leq n$ with the property that for any $a > 1$, there exist positive constants K and r_0 such that for all $r > r_0$ we have,

$$(42) \quad T(r, y_0) \leq K(\bar{N}(ar, 1/(y_0 - z^m)) + \Phi(ar) + \log r).$$

PROOF. — *Part (A).* Let g be an entire function such that $L(g) \neq 0$. Let $u_0 = y_0 - g$. Then u_0 solves the nonhomogeneous equation,

$$(43) \quad L(u) = -L(g).$$

In view of the definition of $\Phi(r)$ and the fact that for $j \geq 1$, $T(r, g^{(j)}) = O(T(r, g))$ nearly everywhere in $[0, +\infty)$ as $r \rightarrow \infty$ by [7; p. 55], it easily follows that there is a constant $K_1 > 1$ such that nearly everywhere in $[0, +\infty)$,

$$(44) \quad T(r, -L(g)) \leq K_1(\Phi(r) + T(r, g) + 1).$$

Let $a > 1$, and let $\sigma > 1$ be such that $\sigma^2 = a$. Since both sides of (44) are monotone nondecreasing on $(0, +\infty)$, it follows from [1; p. 68], that there exists $r_1 > 0$ such that for all $r > r_1$, we have

$$(45) \quad T(r, -L(g)) \leq K_1(\Phi(\sigma r) + T(\sigma r, g) + 1).$$

Hence if we denote by $\Delta(r)$ the maximum of the characteristics of coefficients of the equation (43), then since $K_1 > 1$, it follows from (45) and the definition of $\Phi(r)$ that,

$$(46) \quad \Delta(r) \leq K_1(\Phi(\sigma r) + T(\sigma r, g) + 1) \quad \text{for } r > r_1.$$

But by Theorem 3 (§ 6), there exist positive constants $r_2 > r_1$ and K_2 such that for $r > r_2$,

$$(47) \quad T(r, u_0) \leq K_2(\bar{N}(\sigma r, 1/u_0) + A(\sigma r) + \log r).$$

Since $T(r, y_0) \leq T(r, u_0) + T(r, g) + O(1)$, the conclusion of Part (A) now follows immediately from (47) and (46).

Part (B). It is impossible that all of the $n+1$ functions $1, z, z^2, \dots, z^n$ be solutions of equation (39), since the n -th order equation (39) cannot have more than n linearly independent solutions in a neighborhood of a point where f_n does not vanish. Thus (40) holds with $g = z^m$ for some m in $\{0, 1, \dots, n\}$. Since $T(r, z^m) = m(\log r)$ for $r > 1$, Part (B) clearly follows from (40).

8. - THEOREM 4. - Let f_0, f_1 and f_2 be entire functions. Let $M_1(r)$ and $A(r)$ be as in the hypothesis of Theorem 1 (§ 3) satisfying (2) and (3). Let $y_0(z)$ be a transcendental meromorphic function on the plane which satisfies the equation,

$$(48) \quad f_2 y_0^{(n)} + f_1 y_0^{(n-1)} + f_0 y_0^{(n-2)} = 0,$$

at each point where y_0 is analytic. Then for any $a > 1$, there exist positive constants C, C_1 and r_0 such that for all $r > r_0$, we have

$$(49) \quad T(r, y_0) \leq C(rN(ar, 1/y_0^{(n-2)}) + r^2 \exp(C_1 J(ar) \log^+ J(ar))),$$

where,

$$(50) \quad J(r) = r\bar{N}(r, 1/y_0^{(n-2)}) + r^2 M_1(r)/A(r).$$

Thus for special equations of the form (48), the growth of a meromorphic solution y_0 can be estimated in terms of the growth of the coefficients and the counting function for the zeros of $y_0^{(n-2)}$.

PROOF. - Given $a > 1$, let $\sigma > 1$ be such that $\sigma^{n-1} = a$. Let $u_0 = y_0^{(n-2)}$. Then u_0 satisfies equation (4), so there are constants K, K_1 and r_1 such that (5) and (6) hold for $r > r_1$ with u_0 replacing y and σ replacing a . By [5; p. 172], there are constants K_2 and $r_0 > r_1$ such that for $r > r_0$, we have

$$(51) \quad T(r, y_0^{(j)}) \leq K_2 T(\sigma r, y_0^{(j+1)}) \text{ for } j = 0, 1, \dots, n-3.$$

Thus $T(r, y_0) \leq K_2^{n-2} T(\sigma^{n-2} r, u_0)$ for $r > r_0$, which together with (5) and (6) (for u_0 in place of y and σ in place of a) yields the desired estimate (49) for $r > r_0$, where C and C_1 are appropriate constants.

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