A Theorem on Runge Problem for Analytic Curves (*) (**).

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Summary. – *Runge problem for analytic curves means: given a curve C of an open Ω in an analytic manifold X is it possible to find a sequence of curves $\{C_r\}$ of X which, restricted to Ω , converge to C in a suitable topology? My paper gives an affirmative answer when X is a Stein manifold of dimension three and Ω is a Runge and Stein open of X .*

Runge problem for analytic curves means: given a curve C of an open Ω in an analytic manifold X is it possible to find a sequence of curves $\{C_r\}$ of X which, restricted to Ω , converge to C in a suitable topology?

My paper gives an affirmative answer when X is a Stein manifold of dimension three and Ω is a Runge and Stein open of X .

The essential idea of the proving procedure is to try to define C as the zero set of two holomorphic functions f_1, f_2 of Ω which moreover generate in each point of C the ideal sheaf associate to C .

In this case, infact two functions f'_1, f'_2 of X «close» to f_1, f_2 define a curve C' of X «close» to C .

It's known (see [2], Corollar 2 zu Satz 9) that if C is a regular curve, C is ideally theoretically complete intersection; but there are examples of singular curves of $\mathbb{C}^3$ for which this is not true (for instance: the reducible curve composed by the three coordinates axis in $\mathbb{C}^3$).

However it's possible to prove (see Theorem 1) that under suitable hypothesis there exist two functions f_1, f_2 of Ω defining C plus in case curves of X through the singular points of C and generating the ideal sheaf of C in all its regular points.

Thus, taking two functions f'_1, f'_2 of X close to f_1, f_2 and zero on the possible more curves we can find a curve C' of X close to C in Ω .

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1. – Existence of many local birational projections.

Let fix an hermitian metric on $\mathbb{C}^n$.

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DEFINITION 1. — Let f be a holomorphic function in a neighborhood U of O in $\mathbf{C}^n$ such that $f(0) = 0$ but $f \not\equiv 0$.

A linear subspace L of $\mathbf{C}^n$ of dimension $n-1$ is said to be *regular in respect to f at O* if there exists a neighborhood $U' \subset U$ of O where f is a finite function on the fibers of the projection $\pi_L: \mathbf{C}^n \rightarrow L$ (that is $\#(\pi_L^{-1}(w) \cap f^{-1}(z) \cap U') < \infty$ for each $z \in \mathbf{C}$ and $w \in L$).

REMARK. — L is regular with respect to f in O if and only if f is regular in z_n at O for each coordinate system $(z_1, \dots, z_n)$ in $\mathbf{C}^n$ isometric with the fixed one and such that $L = \{z \in \mathbf{C}^n: z_n = 0\}$.

DEFINITION 2. — Let V be an analytic subspace of pure dimension $k < n$ of a neighborhood U of O in $\mathbf{C}^n$ containing O .

A linear subspace L of $\mathbf{C}^n$ is said to be *regular in respect of V at O* (and $\pi: \mathbf{C}^n \rightarrow L$ *regular*) if there exists a neighborhood $U' \subset U$ of O such that:

- 1) $\pi(V \cap U')$ is an analytic subspace of pure dimension k in $\pi(U')$;
- 2) $\pi^{-1}: \pi(V \cap U') \rightarrow V$ is meromorphic.

REMARK. — When the condition 1) is satisfied, the condition 2) is equivalent to claim that $\pi^*: \mathcal{M}_{\pi(V),0} \rightarrow \mathcal{M}_{V,0}$ is an isomorphism; where $\mathcal{M}_{V,0}$ is the stalk of germs of meromorphic functions of V at O (see [1]).

PROPOSITION 1. — Let V be the germ at O of an analytic subspace of pure dimension $k \leq n-1$ in $\mathbf{C}^n$.

Let r be a positive integer such that $k+1 \leq r \leq n-1$, the family of all linear subspaces of $\mathbf{C}^n$ of dimension r non regular in respect to V at O is a set of measure zero in $\text{Gr}(r, n)$.

PROOF. — Let's procede by induction from $n-1$ to $k+1$.

LEMMA. — For each function f holomorphic at O in $\mathbf{C}^n$ let f_h be the first homogeneous polynomial not identically zero in the expansion of f at O , and let $W = \{x \in \mathbf{P}^{n-1}: f_h^*(x) = 0\}$.

To each point $x \in \mathbf{P}^{n-1} - W$ corresponds (by the natural duality between $\mathbf{P}^{n-1}$ and $\text{Gr}(n-1, n)$) a linear subspace L_x of $\mathbf{C}^n$ of dimension $n-1$ regular in respect to f at O .

Thus the set of all the linear subspaces non regular in respect to f at O is a set of measure zero in $\text{Gr}(n-1, n)$.

LEMMA'S PROOF. — Since $x \in \mathbf{P}^{n-1} - W$, the normal line N to L_x is not contained in the cone $\{x \in \mathbf{C}^n: f_h(x) = 0\}$.

Let a_n be a vector of N of lenght 1 and let $a_1, \dots, a_{n-1}$ be an orthonormal base of L_x and $l: \mathbf{C}_z^n \rightarrow \mathbf{C}_z^n$ the isometry sending the base $(a_1, \dots, a_n)$ in the natural base of $\mathbf{C}^n$.

The new function $g = f \circ l$ has total order h as f and since $f_h(a_n) \neq 0$ then $g_h(0, \dots, 0, 1) \neq 0$, that is g is regular in z_n at O .

INDUCTION: $r = n - 1$. - Let $\mathfrak{F}$ be the ideal of V in $\mathcal{O}_0$ (stalk at O in $\mathcal{O}$), $\mathcal{A} = \mathcal{O}/\mathfrak{F}$ and $p: \mathcal{O} \rightarrow \mathcal{A}$ the quotient map.

Choose $f \in \mathfrak{F}$ non identically zero, the subset A of P^{n-1} of linear subspaces L' of dimension $n-1$ regular in respect to f at O has as complement W in P^{n-1} an algebraic variety.

The map $\pi': V \rightarrow L'$ is finite in a neighborhood of O (Weierstrass preparation theorem applied to f), so $V' = \pi(V)$ is an analytic subspace of dimension k in a neighborhood of O in L' (proper mapping theorem).

Let $\mathfrak{F}'$ be the ideal of V' in $\mathcal{O}'_0$, $\mathcal{A}' = \mathcal{O}'_0/\mathfrak{F}'$ and $p': \mathcal{O}'_0 \rightarrow \mathcal{A}'$.

Because the map $\pi'^*: \mathcal{A}' \rightarrow \mathcal{A}$ is injective we can consider $\mathcal{A}'$ a subspace of $\mathcal{A}$.

Let write $\eta_n = p(z_n)$ we want to show that $\mathcal{A} = \mathcal{A}[\eta_n]$ and η_n is algebraic on $\mathcal{A}'$.

Since L' is regular in respect to f at O , we can find a orthonormal base $(v_1, \dots, v_{n-1}, v_n)$ of $\mathbf{C}^n$ (where $(v_1, \dots, v_{n-1})$ is an arbitrary orthonormal base of L') giving a coordinate system $(z_1, \dots, z_n)$ in $\mathbf{C}^n$ for which f is regular in z_n at O .

Then $f = hQ$, where $h \in \mathcal{O}'[z_n]$ is a Weierstrass polynomial and Q is a unity of $\mathcal{O}$, therefore $h \in \mathfrak{F}$. Say

$$h = z_n^z + \sum_{i < z} a_i z_n^i.$$

With $a_i \in \mathcal{O}'$.

Then in $\mathcal{A}$ we have $0 = \eta_n^z + \sum p(a_i) \eta_n^i$, that is η_n algebraic on $\mathcal{A}'$.

Let $a \in \mathcal{A}$, $a = p(g)$ with $g \in \mathcal{O}$, by the division algorithm $g = hH + \sum b_i z_n^i$ with $b_i \in \mathcal{O}'$; in $\mathcal{A}$ this means $a = \sum p(b_i) \eta_n^i$ that is $a \in \mathcal{O}'[\eta_n]$.

Proceeding analogously we can find $f' \in \mathfrak{F}'$, $f' \neq 0$ and L'' subspace of dimension $n-2$ of L' regular with respect to f' (almost all L'' are good).

Let write $V'' = \pi''(V')$, this is a subspace of dimension k in L'' .

Let define $\mathfrak{F}''$, $\mathcal{A}''$, p'' , π''^* , as above.

Choose $v_1, \dots, v_{n-2}$ in L'' and define η_{n-1} , we can show again $\mathcal{A}' = \mathcal{A}''[\eta_{n-1}]$ with η_{n-1} algebraic on $\mathcal{A}''$; therefore $\mathcal{A} = \mathcal{A}''[\eta_{n-1}, \eta_n]$.

Let $\mathcal{F}$ denote the quotient field of $\mathcal{A}$, $\mathcal{F}'$ the quotient field of $\mathcal{A}'$, $\mathcal{F}''$ the quotient field of $\mathcal{A}''$.

Then $\mathcal{F} = \mathcal{F}''(\eta_{n-1}, \eta_n)$.

Following the primitive element theorem (Van der Werden) is possible to check that for all the couples $(c_{n-1}, c_n) \in \mathbf{C}^2$ such that $|c_{n-1}v_{n-1} + c_nv_n| = 1$ except a finite set holds

$$\mathcal{F} = \mathcal{F}''(c_{n-1}\eta_{n-1} + c_n\eta_n).$$

Let choose $v'_{n-1} = c_{n-1}v_{n-1} + c_nv_n$ such that the linear subspace $\tilde{L}'$ generated by $v_1, \dots, v_{n-2}, v'_{n-1}$ is regular in respect to f yet.

This is possible because L'' is represented in P^{n-1} by a projective line containing the point representing L' .

Since $L' \notin W$, L'' meets W in a finite set.

Choosing $\tilde{L}'$ in L'' out of this set and out of the set of the $\tilde{L}'$ where the primitive element in the form stated above does not hold, for all these $\tilde{L}'$ we have $\mathcal{F} = \tilde{\mathcal{F}}'$.

So $\tilde{\pi}'^{-1}: \tilde{V}' \rightarrow V$ is meromorphic.

We want to show that such $\tilde{L}'$ are represented by almost all the points of $\mathbf{P}^{n-1}$.

We arrived to $\tilde{L}'$ choosing L' out of W , choosing a projective line L'' in $\mathbf{P}^{n-1}$ containing L' and regular in respect to f , almost all the L'' have this property; at last choosing $\tilde{L}'$ on L'' out of a finite set.

By the Tonelli Theorem it follows that the set C of all the points of $\mathbf{P}^{n-1}$ non regular in respect to V has measure zero, since almost all the projective lines through L' meet C in a set of measure zero.

INDUCTION. — Let's suppose we showed the theorem for each r in the interval $r_0 + 1, \dots, n - 1$ (with $r_0 \geq r + 1$) and let's show it for $r = r_0$.

Let L' be one of the spaces of dimension $r_0 + 1$ regular in respect to V and let $\pi: V \rightarrow L'$ be the natural projection.

Let's consider $V' = \pi'(V)$ in a neighborhood of O in L' .

If we apply the induction with ambient space L' instead of $\mathbf{C}^n$ and the analytic subspace V' instead of V , we can claim that almost all the linear subspaces L'' of dimension r_0 of L' are regular in respect to V' and hence to V .

Therefore almost all the points L' in $\text{Gr}(r_0 + 1, n)$ are regular in respect to V and almost all the linear subspaces of dimension 1 of $\text{Gr}(r_0 + 1, n)$ passing through L are regular in respect to V .

It is possible to find an open of $\text{Gr}(r_0 + 1, n)$ isomorphic to $\mathbf{C}^{(r_0+1)(n-r_0-1)} = \mathbf{C}^s$ such that the linear subspaces of dimension 1 in $\text{Gr}(r_0 + 1, n)$ which don't come from lines of $\mathbf{C}^s$ make a set of measure zero in $\text{Gr}(r_0, n)$.

We can also suppose that $\mathbf{C}^{s-1} = \{z \in \mathbf{C}^s: z_s = 0\}$ meets the set of all the non regular points of $\mathbf{C}^s$ in a set of measure zero.

Almost all the lines of $\mathbf{C}^s$ meet $\mathbf{C}^{s-1}$ only in one point, these lines then make a trivial bundle of base $\mathbf{C}^{s-1}$ and fiber $\mathbf{P}^{s-1}$ (in the fiber over $z_0 \in \mathbf{C}^{s-1}$ there are the lines through z_0).

Since almost all the points of almost all the fibers are regular and the bundle is almost all $\text{Gr}(r_0, n)$ we can conclude that the set of all the points of $\text{Gr}(r_0, n)$ non regular in respect to V make a set of measure zero.

2. — Existence of many distinguishing projections.

From now on A_r will indicate the r -Hausdorff measure in $\mathbf{C}^n$.

PROPOSITION 2. — Let D be a subset of $\mathbf{C}^n - \{0\}$ such that $A_r(D) = 0$ for $r > 2d$. Let write

$$\mathcal{L}_k = \{L \in \text{Gr}(k, n): L \cap D \neq \emptyset\}.$$

Then

$$A_r(\mathfrak{L}_k) = 0 \quad \text{for } r > 2d + 2(k-1)(n-k).$$

Therefore the Lebesgue measure in $\text{Gr}(k, n)$ $A_{2k(n-k)}(\mathfrak{L}_k) = 0$ for $d < n - k$.

PROOF. — Let write $\mathcal{A}_k = \{(x, L) \in (\mathbb{C}^n - \{0\}) \times \text{Gr}(k, n) : x \in L\}$,

$$P_1: \mathcal{A}_k \rightarrow \mathbb{C}^n - \{0\}, \quad P_2: \mathcal{A}_k \rightarrow \text{Gr}(k, n).$$

Since $\mathfrak{L}_k = P_2(P_1^{-1}(D))$ and P_2 is locally lipschitzian, it is enough to show that

$$A_r(P_1^{-1}(D)) = 0 \quad \text{for } r > 2d + 2(k-1)(n-k) \text{ in } (\mathbb{C}^n - \{0\}) \times \text{Gr}(k, n).$$

Let $x \in D$, $P_1^{-1}(x) \subset \{x\} \times \text{Gr}(k, n)$ is equivalent, by isometries in $\text{Gr}(k, n)$ to $\text{Gr}(k-1, n-1)$ and thus:

$$A_r(P_1^{-1}(x)) = 0 \quad \text{for } r > 2(k-1)(n-k).$$

Let a, b positive real numbers, $P_1^{-1}(D)$ is a measurable subset of $\mathcal{A}_k$ in respect to the measure

$$A_{2d+a} \times A_{2(k-1)(n-k)+b} = A_{2d+2(k-1)(n-k)+a+b}$$

and, using the Tonelli's Theorem, is possible to show that has measure zero.

COROLLARY 1. — Let V be an analytic subspace of an open Ω in $\mathbb{C}^n$.

Let's suppose $A_r(\bar{V}) = 0$ for each $r > 2k$, and let's fix $x_0 \in V$, almost all the affine linear subspace L of dimension $n-k-1$ through x_0 have the properties:

- 1) $\pi^{-1}(\pi(x_0)) \cap \bar{V} = \{x_0\}$;
- 2) there exists a neighborhood U of x_0 such that

$$\pi^{-1}(\pi(U)) \cap \bar{V} = U \cap V.$$

REMARK. — Let X be a submanifold of dimension n in $\mathbb{C}^N$. Almost all the linear subspaces $L \in \text{Gr}(n, N)$ have the property that there exists a neighborhood U of x_0 in X where the natural projection $\pi_L: U \rightarrow L$ gives local coordinates.

PROOF. — Let $L \in \text{Gr}(n, N)$, π_L does not give local coordinates at x_0 in X if and only if $\text{rank } J_x(\pi|_X)(x_0) < n$, that is if and only if $\dim[L \cap (T_{x_0}(X))^\perp] \geq 1$; such linear subspaces L make an analytic subspace of $\text{Gr}(n, N)$.

PROPOSITION 3. – Let Ω be an open $\mathbf{C}^n$, C a curve of Ω such that $A_r(\bar{C}) = 0$ for each $r > 2$.

For almost all the linear subspaces $L \in \text{Gr}(2, n)$ the projection $\pi: \mathbf{C}^n \rightarrow L$ has the following properties:

- 1) for each $x_i \in \text{sing}(C)$ there exists a neighborhood U_i of x_i such that $\pi(U_i \cap C)$ is a curve in $\pi(U_i)$;
- 2) $\pi^{-1}: \pi(U_i \cap C) \rightarrow C$ is meromorphic;
- 3) $\pi^{-1}(\pi(U_i)) \cap \bar{C} = U_i \cap C$.

PROOF. – It follows from Propositions 1 and 2 and the fact that $\text{sing}(C)$ is countable.

3. – Local complete intersection.

PROPOSITION 4. – Let Ω be an open of $\mathbf{C}^n$, C a curve of Ω such that $A_r(\bar{C}) = 0$ for each $r > 2$.

For each $x_i \in \text{sing}(C)$ there exists an affine linear space L_i of $\mathbf{C}^n$ such that:

- 1) $L_i \cap C = \{x_i\}$;
- 2) $L_i \cap L_j = \emptyset$ if $i \neq j$;

and for each $x_i \in \text{sing} C$ there exists an open neighborhood $A_i = U'_i \times \mathbf{C}^{n-2}$ of L_i and holomorphic functions on A_i , $h_{i1}, \dots, h_{in-1}$ such that:

- 3) $\{x \in A_i: h_{ij}(x) = 0 \text{ for each } j = 1, \dots, n-1\} = (C \cap A_i) \cup L_i$;
- 4) $h_{i1x}, \dots, h_{i,n-1,x}$ generate $\mathfrak{J}_{C,x}$ for each $x \in C \cap A_i - \{x_i\}$.

PROOF. – Let's consider $L \in \text{Gr}(2, n)$ and $\pi: \mathbf{C}^n \rightarrow L$ as in Prop. 3 and let's put on $\mathbf{C}^n$ a coordinate system such that

$$L = \{x \in \mathbf{C}^n: z_j(x) = 0 \text{ for } j = 3, \dots, n\}.$$

Let's write $U'_i = \pi(U_i)$, $C_i = U_i \cap C$, $C'_i = \pi(C_i)$.

Since $\pi^{-1}: C'_i \rightarrow C_i$ is meromorphic there exist $m_{i2}, \dots, m_{in-1}$ meromorphic on C'_i such that:

$$\pi^{-1}(x) = (z_1(x), z_2(x), m_{i2}(x), \dots, m_{in-1}(x)) \quad \text{for each } x \in C'_i.$$

Rechoosing the U'_i we can find functions $f_i, f_{i2}, \dots, f_{in-1}, g_{i2}, \dots, g_{in-1}$ holomorphic on U_i such that

$$m_{ij}(x) = f_{ij}(x)/g_{ij}(x) \quad \text{for each } x \in C'_i, x_i \in \text{sing } C, j = 2, \dots, n-1,$$

$\{x \in U'_i : f_i(x) = 0\} = C'_i$ and furthermore f_{ix} generates $\mathfrak{F}_{C'_i, x}$ for each $x \in U'_i$, and besides each $g_{ij}(x) = 0$ at most in $\pi(x_i)$ on C'_i .

Let's define the functions h_{ij} holomorphic on $A_i = \pi^{-1}(U'_i)$ in this way:

$$h_{i1}(x) = f_i(\pi(x)), \quad h_{ij}(x) = g_{ij}(\pi(x))z_{j+1}(x) - f_{ij}(\pi(x)) \quad \text{for } j = 2, \dots, n-1.$$

Therefore:

$$\{x \in A_i : h_{ij}(x) = 0 \text{ for } j = 1, \dots, n-1\} = C_i \cup L_i$$

where $L_i = \{x \in \mathbf{C}^n : z_j(x) = z_j(x_i) \text{ for each } j = 1, 2 \text{ or such that } g_{i, j-1}(\pi(x_i)) = 0\}$.

Moreover $h_{i,1x}, \dots, h_{i,n-1,x}$ generate $\mathfrak{F}_{C_x, x}$ for each $x \in A_i \cap C - \{x_i\}$ infact: let's indicate with $p_i : U'_i \rightarrow \mathbf{C}^n$ the meromorphic map

$$p_i(z_1, z_2) = \left(z_1, z_2, \frac{f_{i2}(z_1, z_2)}{g_{i2}(z_1, z_2)}, \dots, \frac{f_{i,n-1}(z_1, z_2)}{g_{i,n-1}(z_1, z_2)} \right).$$

For each $x \in C_i \cap A_i - \{x_i\}$ there exists a neighborhood U_x contained in $U_i - \{x_i\}$ such that p_i is holomorphic on $U'_x = \pi(U_x)$.

Let's write $C_x = C_i \cap U_x$, $C'_x = C'_i \cap U'_x$.

Let q holomorphic on U_x and vanishing on C_x , then $q(p(y)) = 0$ for each $y \in C'_x$ and then there exists a holomorphic function b on U'_x such that $q(p(y)) = f_i(y)b(y)$ for each $y \in U'_x$ (that is $q(p(\pi(u))) = h_{i1}(u)b(\pi(u))$ for each $u \in U_x$.)

The function $q(u) - q(p(\pi(u)))$ vanishes in $p(U'_x)$ in $U'_x \times \mathbf{C}^{n-2}$.

The rest of the proof follows from this lemma:

LEMMA. - Let U be an open of $\mathbf{C}^d \subset \mathbf{C}^n$ and $\varphi : U \rightarrow \mathbf{C}^n$ holomorphic so defined

$$\varphi(z_1, \dots, z_d) = (z_1, \dots, z_d, \varphi_{d+1}(z), \dots, \varphi_n(z)).$$

Let's call $S = \varphi(U)$ and let q be an holomorphic function on a neighborhood of S and vanishing on S .

Then for each $z \in S$ there exists a neighborhood U_z of z and functions $a_{d+1}, \dots, a_n$ holomorphic on U_z such that

$$q(y) = (z_{d+1}(y) - \varphi_{d+1}(y))a_{d+1}(y) + \dots + (z_n(y) - \varphi_n(y))a_n(y)$$

for each $y \in U_z$.

PROOF OF THE LEMMA. - The map $\psi : U \times \mathbf{C}^{n-d} \rightarrow U \times \mathbf{C}^{n-d}$ defined by

$$\psi(z_1, \dots, z_n) = (z_1, \dots, z_d, z_{d+1} - \varphi_{d+1}(z), \dots, z_n - \varphi_n(z))$$

gives locally a change of coordinates in $U \times \mathbf{C}^{n-d}$.

Moreover $\psi(S) = U \times \{0\}$ and $\psi \circ \varphi = 1_U$.

Let's write $(\psi^{-1})_z$ the inverse map of ψ from a neighborhood of $\psi(z)$ to a neighborhood of z and $q' = q \circ (\psi^{-1})_z$, q' is defined in a neighborhood of $\psi(z) \in U \times \{0\}$ and vanishing on $U \times \{0\}$, therefore there exist functions $b_{a+1}, \dots, b_n$ holomorphic in a neighborhood of $\psi(z)$ such that

$$q'(w_1, \dots, w_n) = w_{a+1} b_{a+1}(w) + \dots + w_n b_n(w)$$

therefore

$$q(z_1, \dots, z_n) = (z_{a+1} - \varphi_{a+1}(z)) b_{a+1}(\psi(z)) + \dots + (z_n - \varphi_n(z)) \cdot b_n(\psi(z))$$

in a neighborhood of z .

CONTINUATION OF THE PROPOSITION'S PROOF. – Thus by the lemma

$$q(u) - q(p(\pi(u))) = \left(z_3(u) - \frac{f_{i,3}(u)}{g_{i,3}(u)} \right) a_2(u) + \dots + \left(z_n(u) - \frac{f_{i,n-1}(u)}{g_{i,n-1}(u)} \right) a_{n-1}(u)$$

in a neighborhood of x .

Therefore:

$$q(u) = h_{i1}(u) a(u) + h_{i2}(u) \frac{1}{g_{i2}(u)} a_2(u) + \dots + h_{i,n-1}(u) \frac{1}{g_{i,n-1}(u)} a_{n-1}(u).$$

PROPOSITION 5. – Let X be a Stein manifold of dimension n , Ω an open of X and C a curve of Ω with a finite set of singularities.

For each $x_i \in \text{sing } C$ there exists an analytic subspace Y_i of X such that:

- 1) $\dim(Y_i \cap C) = 0$, $x_i \in Y_i \cap C$;
- 2) $Y_i \cap Y_j = \emptyset$ if $i \neq j$;

and an open neighborhood A_i of Y_i with $A_i \cap A_j = \emptyset$ if $i \neq j$ and holomorphic functions on A_i , $h_{i1}, \dots, h_{i,n-1}$ such that:

- 3) $\{x \in A_i : h_{ij}(x) = 0 \text{ for each } j = 1, \dots, n-1\} = (C \cap A_i) \cup Y_i$;
- 4) $h_{i1,x}, \dots, h_{i,n-1,x}$ generate $\mathfrak{J}_{C,x}$ for each $x \in C \cap A_i - \{x_i\}$.

PROOF. – Let's embed X in $\mathbb{C}^N$.

Applying the Prop. 2 and its Remark it's possible to find a n -dimensional linear subspace L of $\mathbb{C}^N$ and a neighborhood U of $\text{sing}(C)$ in X such that π_L is holomorphic and does not send a component of C to a point.

Applying the Prop. 4 to the space L , the open $\pi(U)$ and the curve $\pi(C \cap U)$ and pulling back L_i , A_i , h_{ij} to X we get the new Y_i , A_i , h_{ij} as wished.

4. – Complete intersection on compacts.

From now on we'll have a continuous needling of the results contained in [2]. The following is a brief exposition of some of them.

Let $\mathcal{F}$ be an analytic sheaf on a Stein manifold X .

Let's suppose there exist $f_1, \dots, f_r \in \Gamma(X, \mathcal{F})$ generating $\mathcal{F}_x$ for each $x \in X$.

At a point $x \in X$ the least number of generators of $\mathcal{F}_x$ is given by the dimension on $\mathbb{C}$ of the vector space $L_x(\mathcal{F}) = \mathcal{F}_x / \mathcal{M}_x \cdot \mathcal{F}_x$.

Globally the existence, for a fixed integer s , of s global sections of $\mathcal{F}$: $g_1, \dots, g_s$ generating $\mathcal{F}$ in each point is equivalent to the existence of a global holomorphic section for a kind of bundle $E(\mathcal{F}, f, s) \rightarrow X$ called Endromisbundel.

The Endromisbundel is an open of $X \times \mathbb{C}_r$ obtained subtracting analytic spaces defined by $f_1, \dots, f_r$ and by the integers $\{\dim L_x(\mathcal{F})\}_{x \in X}$.

Let's define for each positive integer k :

$$Y_k = \{x \in X : \dim L_x(\mathcal{F}) \geq k\}$$

the Y_k make a decreasing sequence of analytic subspaces of X , empty from $r+1$; on $X_x = Y_k - Y_{k+1}$ the Endromisbundel is an holomorphic fiber bundle with fiber the topological space of matrices $F_{r,s,k}$ homotopic to the space W_{sk} of the unitary hortonormal k -frames of $\mathbb{C}^s$.

An important result of FORSTER and RAMSPOTT (Satz 5 and 6 of [2]) claims that if X is Stein the existence of holomorphic sections for the Endromisbundel is equivalent to the existence of continuous sections.

The problem of the least number of generators for the sheaf in each point of X , is then reduced to a purely topological problem with main ingredients the spaces X_k and the fibers W_{sk} .

THEOREM 1. – Let X be a Stein manifold of dimension $n \geq 3$, Ω a Stein open of X and C a curve of Ω .

Let Ω' be a Stein open relatively compact in Ω and let's write $C' = C \cap \Omega'$.

For each $x_i \in \text{sing}(C') = \{x_1, \dots, x_s\}$ there exists an analytic subspace Y_i of X such that:

- 1) $\dim(Y_i \cap C) = 0$, $x_i \in Y_i \cap C$;
- 2) $Y_i \cap Y_j = \emptyset$ if $i \neq j$;

and there exists functions $h_1, \dots, h_{n-1}$ holomorphic on Ω' such that:

- 3) $\{x \in \Omega' : h_j(x) = 0 \text{ for } j = 1, \dots, n-1\} = C' \cup \bigcup_{i=1}^s (Y_i \cap \Omega')$;
- 4) $h_{1,x}, \dots, h_{n-1,x}$ generate $\mathcal{F}_{C',x}$ for each $x \in \text{reg}(C')$.

PROOF. – Let's consider Y_i , A_i , h_{ij} as in Prop. 5.

Let's define:

$$\Phi_i: \mathcal{O}_{|A_i}^{n-1} \rightarrow \mathcal{O}_{|A_i} \quad \text{by} \quad \Phi_i(f_1, \dots, f_{n-1}) = \sum_{j=1}^{n-1} h_{ij} f_j$$

the image

$$\mathcal{F}_i = \Phi_i(\mathcal{O}_{|A_i}^{n-1})$$

is coherent subsheaf of $\mathcal{O}_{|A_i}$ and

$$\mathfrak{F}_{i,x} = \begin{cases} \mathcal{O}_x & \text{if } x \in A_i - (C \cup Y_i), \\ \mathfrak{F}_{C,x} & \text{if } x \in C \cap A_i - \{x_i\}. \end{cases}$$

Let's define a sheaf $\mathfrak{F}$ in $\mathcal{O}_\Omega$ in this way: $\mathfrak{F}_x = \mathfrak{F}_{i,x}$ if $x \in A_i$ and $\mathfrak{F}_x = \mathfrak{F}_{C,x}$ if $x \in \Omega - \bigcup_{i=1}^s Y_i$.

The sheaf $\mathfrak{F}$ is coherent and since $\bar{\Omega}'$ is compact in Ω Stein, there exists a chain of syzygies of $\mathfrak{F}$ in Ω' :

$$\dots \rightarrow \mathcal{O}_{|\Omega}^r \rightarrow \mathcal{O}_{|\Omega'}^r \xrightarrow{f} \mathfrak{F}_{|\Omega'} \rightarrow 0$$

The components of $f = (f_1, \dots, f_r)$ therefore generate $\mathfrak{F}_x$ in each point of Ω' .

Since we are interested in finding $h_1, \dots, h_{n-1} \in \Gamma(\Omega', \mathcal{O})$ with the same property we have to show that there exists a continuous section of the Endromisbündel $E(\Omega', \mathfrak{F}, f, n-1)$ on Ω' .

The functions $h_{i1}, \dots, h_{in-1}$ generate $\mathfrak{F}_x$ for each $x \in A$, so on $A'_i = A_i \cap \Omega'$ (Stein) $h_{i1}|_{A'_i}, \dots, h_{in-1}|_{A'_i}$ define ([2], Hilfssatz 3) a continuous section of

$$E(A'_i, \mathfrak{F}|_{A'_i}, f|_{A'_i}, n-1) = E(\Omega', \mathfrak{F}, f, n-1)|_{A'_i}.$$

Let $A' = \bigcup_{i=1}^s A'_i$.

Choose a triangulation on Ω' which induces a triangulation on C' and on $Y' = \bigcup_{i=1}^s (Y_i \cap \Omega')$.

Refining the triangulation we can find a neighborhood V of Y' which is a polyedron contained in A' and such that $V \cap C'$ is contractible on $\{x_1, \dots, x_s\}$.

To extend our continuous section from $V \cap C' - Y'$ to $C' - Y'$ it is enough (see [7], page 174):

$$H^{q+1}(C' - Y', (V \cap C') - Y'; \pi_q(W_{n-1, n-1})) = 0$$

for each $q \geq 1$, because $E(\mathfrak{F}, f, n-1)|_{C' - Y'}$ is a fiber bundle with fiber $F_{r, n-1, n-1}$ homotopic to $W_{n-1, n-1}$ (see [2], Satz 1, page 148).

Now

$$H^{q+1}(C' - Y', V \cap C' - Y') \simeq H^{q+1}(C', V \cap C') \simeq H^{q+1}(C') \quad \text{for } q \geq 0$$

by the excision first and by the contractibility of $V \cap C'$ on $\{x_1, \dots, x_s\}$ after; at last since C' is a curve $H^{q+1}(C') = 0$ for each $q \geq 1$.

Let W be a subpolyedron of Ω' which is a neighborhood of $C' \cup Y'$ contractible on $C' \cup Y'$, if W is small enough the section of $E(\mathfrak{F}, f, n-1)|_{C' \cup Y'}$ can be extended to W (see [2], Hilfssatz 2, page 147 and proof of Satz 9, page 161).

To extend the section from $W - (C' \cup Y')$ to $\Omega' - (C' \cup Y')$ is enough:

$$H^{q+1}(\Omega' - (C' \cup Y'), W - (C' \cup Y'), \pi_q(W_{n-1,1})) = 0 \quad \text{for each } q \geq 1,$$

because $E(\mathfrak{F}, f, n-1)|_{\Omega' - (C' \cup Y')}$ is a fiber bundle with fiber $F_{r,n-1,1}$ homotopic to $W_{n-1,1}$.

Since $\pi_q(W_{n-1,1}) = 0$ for $q \leq 2n-4$ is sufficient to prove that

$$H^{q+1}(\Omega' - (C' \cup Y'), W - (C' \cup Y')) \simeq H^{q+1}(\Omega', W) \simeq H^{q+1}(\Omega', C' \cup Y')$$

vanishes for $q \geq 2n-3$.

From the exact sequence

$$\dots \rightarrow H^q(C' \cup Y') \rightarrow H^{q+1}(\Omega', C' \cup Y') \rightarrow H^{q+1}(\Omega') \rightarrow \dots$$

since $H^q(C' \cup Y') \simeq H^{q+1}(\Omega') = 0$ for $q \geq n$ (see [6]), we can conclude

$$H^{q+1}(\Omega', C' \cup Y') = 0 \quad \text{for } q \geq n.$$

Therefore if $n \geq 3$ $H^{q+1}(\Omega', C' \cup Y', \pi_q(W_{n-1,1})) = 0$ for each $q \geq 1$, thus there exists a holomorphic section of $E(\mathfrak{F}, f, n-1)$ on Ω' .

5. - Convergence of analytic subsets.

PROPOSITION 6. - Let Ω be an n -dimensional manifold.

Let $f, f_\nu: \Omega \rightarrow \mathbb{C}^{n-d}$ holomorphic maps ($\nu \in N$) with

$$f = \lim_{\nu \rightarrow \infty} f_\nu.$$

Let's write $X = \{x \in \Omega: f(x) = 0\}$, $X_\nu = \{x \in \Omega: f_\nu(x) = 0\}$.

If X has pure dimension d , then for each compact K in Ω there exists an integer ν_K such that for $\nu \geq \nu_K$: $X_\nu \cap \overset{\circ}{K}$ has dimension d .

LEMMA. - For each $x_0 \in X$, there exists a neighborhood U_{x_0} and an integer ν_{x_0} such that for $\nu \geq \nu_{x_0}$: $X_\nu \cap U_{x_0}$ has pure dimension d .

LEMMA'S PROOF. - We can change coordinates at x_0 such that $x_0 = 0$ and there exists a polycylinder $\Delta^{(n)}(0, r)$ where the projection $\pi: \Delta^{(n)}(0, r) \cap X \rightarrow \Delta^{(d)}(0, r)$ is a finite map.

We can also suppose that on $D = \{z \in \mathbb{C}^n: (z_1, \dots, z_d) \in \Delta^{(d)}(0, r) \text{ and } |z_i| = r_i \text{ for } d+1 \leq i \leq n\}$ the lower bound of $|f|$ is a positive number δ .

For $\nu \geq \nu_{x_0}$ big enough we have $\|f - f_\nu\| < \delta$ so that $\inf_D |f_\nu| > 0$.

Then called $\Delta_\nu = (\{y\} \times \mathbb{C}^{n-d}) \cap \Delta^{(n)}$ for each $y \in \Delta_y$, $X_\nu \cap \bar{\Delta}_y$ is all inside Δ_y .

and therefore is compact and finite, that is

$$\dim(X_v \cap \Delta_y) = 0.$$

Then $\dim(X_v \cap \Delta^{(n)}(0, r)) \geq d$ (see [3], Cor. 8, page 113).

But $\dim(X \cap_v \Delta^{(n)}(0, r)) \leq d$, so $\dim(X_v \cap \Delta^{(n)}(0, r)) = d$ for $v \geq v_{x_0}$.

From now on we will make a continuous referement to two papers ([4] and [5]) of KING.

Most of the notations used here are taken from these papers.

As in KING [4] if f is a holomorphic map from a manifold Ω to $\mathbf{C}^r$ we denote:

$$\begin{aligned}\eta(f) &= \frac{1}{2\pi i} \partial \log |f|^2, \\ \omega(f) &= d\eta(f) = \bar{\partial}\eta(f), \\ R(f) &= b[\eta(f) \wedge (\omega(f)^{n-k-1})] - [(\omega(f))^{n-k}] = [X],\end{aligned}$$

if the zero set X of f is a pure k -dimensional subset of Ω .

PROPOSITION 7. – Let Ω be a n -dimensional manifold

$$f_v: \Omega \rightarrow \mathbf{C}^r (v \in \tilde{N} = N \cup \{\infty\})$$

holomorphic maps such that $f_\infty = \lim_{v \rightarrow \infty} f_v$.

Let's suppose the $X_v = \{x \in \Omega: f_v(x) = 0\}$ are pure $(n-r)$ dimensional subsets of Ω .

Then

$$R(f_\infty) = \lim_{v \rightarrow \infty} R(f_v) \quad \text{in } \mathcal{D}_{2(n-r)}(\Omega).$$

PROOF. – First of all we need the following:

LEMMA 1. – For each $x_0 \in X_\infty$ there exists a neighborhood P_{x_0} and a costant C_{x_0} such that:

$$C_{x_0} \geq M(R(f_v)_{P_{x_0}}) \quad \text{for each } v \in N.$$

where $M(T)$ is the mass of T (see [5], page 192 and 201).

PROOF. OF THE LEMMA. – Let's choose at x_0 a clear coordinate system for X_∞ (see [5], page 201-202).

For each choose $I = (i_1, \dots, i_d)$ with $d = n-r$ and $1 \leq i_1 \leq \dots \leq i_d \leq n$ we can find a polycylinder P_I such that the map $\pi_I: P_I \cap X_\infty \rightarrow \mathbf{C}^d$ is finite and $X \cap \partial_I P_I = \emptyset$ where $\partial_I P_I = \partial P_I - \{z \in \mathbf{C}^n: |z_{i_\alpha}| = \delta_{i_\alpha} \text{ for } \alpha = 1, \dots, d\}$

Then $\inf_{\partial_I P_I} |f_\infty| = \sigma_I > 0$.

Set $P'_I = \pi_I P_I$ in $\mathbf{C}^d$, for each $x \in P'_I$ let write $L_x = \{z \in \mathbf{C}^n : z_{i_\alpha} = z_{i_\alpha}(x) \text{ for } \alpha = 1, \dots, d\}$, then $\#(L_x \cap X_\infty \cap P_I) = s_I(x, f_\infty)$ is finite and

$$s_I(x, f_\infty) = \int_{L_x \cap \partial_I P_I} \frac{\lambda_I(f_\infty)}{|f_\infty|^{4(n-r)+2}}$$

for the Bochner-Martinelli formula, where $\lambda_I(f_\infty)$ is a form which coefficients are polynomials in f_∞ and its derivatives.

Thus $s_I(x, f_\infty)$ is an holomorphic function with positive integral values, then a constant $s_I(f_\infty)$, for P_I is connected.

For ν big enough $\inf_{\partial_I P_I} |f_\nu - f_\infty| < \sigma_\nu$, so $s_I(f_\nu)$ is defined and for ν bigger than a suitable ν_I and we have $s_I(f_\nu) = s_I(f_\infty)$.

Let put $P_{x_0} = \bigcap_I P_I$; if w is the usual Kähler form associated to an hermitian metric on Ω :

$$\begin{aligned} M(R(f_\nu)|_{P_{x_0}}) &= \int_{P_{x_0}} R(f_\nu)(w^d) = \int_{P_{x_0}} [- (\omega(f_\nu)^{n-d})] \wedge w^d \leq \\ &\leq \sum_I \int_{P_I} [- (\omega(f_\nu)^{n-d}) \wedge (dz_I \wedge d\bar{z}_I)] = \sum_I \int_{P'_I} \left(\int_{L_x \cap P_I} [- (\omega(f_\nu)^{n-d})] dz_I \wedge d\bar{z}_I \right) = \\ &= \sum_I \int_{P'_I} \left(\int_{L_x \cap \partial_I P_I} - \eta(f_\nu) \wedge \omega(f_\nu)^{n-d-1} \right) dz_I \wedge d\bar{z}_I = \\ &= \sum_I s_I(f_\nu) \text{vol}(P'_I) = \sum_I s_I(f_\infty) \text{vol}(P'_I) = C'_{x_0}. \end{aligned}$$

For $C_{x_0} = \max \{C'_{x_0}, M(R(f_1)|_{P_{x_0}}), \dots, M(R(f_{\nu-1})|_{P_{x_0}})\}$ we have $M(R(f_\nu)|_{P_{x_0}}) \leq C_{x_0}$ for each $\nu \in N$.

Now we can easily prove the:

LEMMA 2. - For each compact K of Ω there exists a constant C_K such that

$$M(R(f_\nu)|_K) \leq C_K \quad \text{for each } \nu \in N.$$

PROOF OF THE LEMMA 2. - The opens P_x where $x \in X_\infty \cap K$ cover $X_\infty \cap K$, then there exists $x_1, \dots, x_m \in X_\infty \cap K$ such that

$$A = \bigcup_{\alpha=1}^m P_{x_\alpha} \supset X_\infty \cap K.$$

Then $\inf_{\bar{K} \setminus A} |f_\infty| = \tau > 0$ and for ν bigger than a suitable $\tilde{\nu}$, from $\sup |f_\nu - f_\infty| < \tau$, it follows $X_\nu \cap K \subset A$.

Thus

$$M(R(f)|_K) \leq \sum_{\alpha=1}^m M(R(f_\nu)|_{P_{x_\alpha}}) \leq \sum_{\alpha} C_{x_\alpha} = C'_K.$$

For $C_K = \max \{C'_K, M(R(f_1)|_{\bar{K}}), \dots, M(R(f_{r-1})|_{\bar{K}})\}$ we have $M(R(f_v)|_{\bar{K}}) < C_K$ for each $v \in N$.

CONTINUATION OF THE PROOF OF PROPOSITION 7. – Since the sequence $\{R(f_v)\}$ is bounded in mass on each compact and $b\{R(f_v)\} = \{0\}$, the sequence is bounded on $I_{2k}^{\text{loc}}(\Omega)$ (see [5], page 196) and relatively compact.

Therefore there exists a subsequence $\{R(f_{v_k})\}$ converging to $T \in I_{2k}^{\text{loc}}(\Omega)$.

This current has support in X_∞ .

In fact let $x \notin X_\infty$ and U_x a compact neighborhood of x disjoint from X_∞ .

For k bigger than a suitable k' we have $\sup_{U_x} |f_{v_k} - f_\infty| < \inf_{U_x} |f_\infty|$ and for each form $\varphi \in A_{2k}^0(\bar{U}_x)$ $T(\varphi) = \lim_{k \rightarrow \infty} R(f_{v_k})(\varphi) = \lim_{k \rightarrow \infty} 0 = 0$.

Therefore for [5], Proposition 3.1.3, page 200 $T = \sum \gamma_i [X_{\infty,i}]$ where the γ_i are positive integers and the $X_{\infty,i}$ are the irreducible components of X_∞ .

From [4], page 47 $R(f_\infty) = \sum c_i [X_{\infty,i}]$.

Let's consider $x_i \in \text{reg}(X_{\infty,i})$ and choose in a neighborhood U of x_i a coordinate system such that

$$X_{\infty,i} \cap U = \mathbf{C}^d \cap U \text{ and } z_j(x_i) = 0 \quad \text{for } j = 1, \dots, n.$$

Let's choose at 0 a polycylinder P which meets X_∞ only in $\mathbf{C}^d$.

Let's call L the orthogonal space to $\mathbf{C}^d$ in 0

$$\begin{aligned} \gamma_i = M(T|_{L \cap P}) &= T|_{L \cap P}(1) = \lim_{k \rightarrow \infty} R(f_{v_k})(1) = \\ &= \lim_{k \rightarrow \infty} \int_{L \cap P} [-\omega(f_{v_k})^{n-d}] = \lim_{k \rightarrow \infty} \int_{L \cap \partial P} [-\eta(f_{v_k}) \wedge \omega(f_{v_k})^{n-d-1}] = \\ &= \int_{L \cap \partial P} [-\eta(f_\infty) \wedge \omega(f_\infty)^{n-d-1}] = c_i. \end{aligned}$$

The last equality comes from [4], Lemma 3, page 51-54.

Therefore $T - R(f_\infty) = 0$ in $\Omega - \text{sing}(X)$, that is $\text{supp}(T - R(f_\infty)) \subset \text{sing}(X)$ and for the support measure theorem (pag. 199) of [5]: $T - R(f_\infty) = 0$.

This shows also that every convergent subsequence of $\{R(f_v)\}$ converges to $R(f_\infty)$ and therefore all the sequence converges to $R(f_\infty)$.

6. – Approximation theorems.

THEOREM 2. – Let X be a three-dimensional Stein manifold and Ω a Runge and Stein open of X , C a curve in Ω .

There exists a sequence $\{T_v\} \subset Z_1^+(X)$ of holomorphic positive 1-cycles such that $\lim_{v \rightarrow \infty} (T_v|_\Omega) = [C]$ in $\mathcal{D}_2(\Omega)$.

PROOF. – Let $\{\Omega_n\}$ be a sequence of relatively compact Stein and Runge in Ω (and then in X) opens which invades Ω .

It is enough to show that for each n there exists $\{T_{r_n}\} \subset Z_1^+(X)$ such that $T_{r_n}|_{\Omega_n} \rightarrow [C \cap \Omega_n]$.

Let's write $\Omega_n = \Omega'$, as in Theorem 1 there exist two functions h_1, h_2 holomorphic on Ω' such that:

$$\{x \in \Omega' : h(x) = 0\} = (C \cap \Omega') \cup \bigcup_{i=1}^r (Y_i \cap \Omega').$$

Therefore $R(h) = c[C \cap \Omega'] + c_1[Y_1 \cap \Omega'] + \dots + c_r[Y_r \cap \Omega']$.

But $c = 1$; infact let's choose a regular point x_0 in C , because h_1 and h_2 generate $\mathfrak{J}_C$ in a neighborhood of x_0 , we can choose a coordinate system at x_0 such that $h_1 = z_1, h_2 = z_2$ and thus $c = 1$ follows from the classical Bochner-Martinelli formula.

Let's put $\mathfrak{J} = (\mathfrak{J}_{x_1})^{c_1} \cap \dots \cap (\mathfrak{J}_{x_r})^{c_r}$, since $h \in \Gamma(\Omega', \mathfrak{J})$, for the theorem 11 at p. 241 of [3] there exists a sequence $\{h_\nu\} \subset (\Gamma(X, \mathfrak{J}))^2$ converging to h in Ω' .

Thus $R(h) = \varprojlim_{\nu \rightarrow \infty} R(h_\nu)|_{\Omega'}$.

Let's denote by $T_\nu \in Z_1^+(X)$ the part of $R(h_\nu)$ whose support is not in $Y_1 \cup \dots \cup Y_r$ except for a zero-dimensional set; we have:

$$R(h_\nu) = T_\nu + c_{1\nu}[Y_1] + \dots + c_{r\nu}[Y_r]$$

where $c_{i\nu} \geq c_i$ for each $\nu \geq 0$ and $1 \leq i \leq r$.

Let's choose a point $x_i \in (\text{reg } Y_i) \cap \Omega'$, let $d_i = \dim_{x_i}(Y_i)$; let's take at x_i a coordinate set (z_1, z_2, z_3) and a polycylinder P with center x_i such that $Y_i \cap P = \{x \in P : z_j(x) = 0 \text{ for each } j \geq d_i + 1\}$ and let's denote by L the horthogonal linear space to Y_i in x_i ; for the Bochner-Martinelli formula if ν is big enough we have:

$$c_i = \int_{L \cap \partial P} \frac{\lambda(h)}{|h|^{14}} \quad \text{and} \quad c_{i\nu} \leq \int_{L \cap \partial P} \frac{\lambda(h_\nu)}{|h_\nu|^{14}}$$

where $\lambda(h)$ is a form whose coefficients are polynomials in h and its derivatives.

For the continuity of the integration we have $c_i \geq c_{i\nu}$ and then $c_i = c_{i\nu}$.

Therefore:

$$R(h_\nu) = T_\nu + c_1[Y_1] + \dots + c_r[Y_r]$$

and

$$[C \cap \Omega'] = \varprojlim_{\nu \rightarrow \infty} (T_\nu|_{\Omega'}).$$

THEOREM 3. — Let X be a three-dimensional Stein manifold, Ω a Runge-Stein open of X and C an irriducible curve of Ω .

There exists a sequence $\{C_\nu\}$ of irriducible curves of X such that

$$\varprojlim_{\nu \rightarrow \infty} [C_\nu \cap \Omega] = [C].$$

PROOF. — It's enough to prove the theorem for each open Ω' relatively compact in Ω .

Let's fix a relatively compact open B in Ω' containing the singular points of $C \cap \Omega'$.

Let now x be a regular point of C .

Changing coordinates we can make C a line.

Let fix a polycylinder U_x at x , if $\{T_\nu\}$ is a sequence of 1-cycles converging to C as in Theorem 2, for ν big enough we have $\text{supp } T_\nu \cap U_x$ regular and each plane perpendicular to C meets (in U_x) $\text{supp } T$ in a simple point by the Bochner-Martini formula.

So $\text{supp } T_\nu \cap U_x$ is connected with multiplicity one, there exists an irreducible curve $C_{\nu x}$ of X such that $T|_{U_x} = [C_{\nu x} \cap U_x]$ for each ν bigger than a suitable ν_x .

It's possible (by instance, triangularizing $\text{reg } (C)$ connected) to find a sequence of connected compact invading $\text{reg } (C)$.

Then, for $(\Omega' - B) \cap \text{reg } C$ is relatively compact in $\text{reg } C$, is contained in a connected compact K of $\text{reg } (C)$.

We can find $x_1, \dots, x_m \in K$ such that $U = \bigcup_{i=1}^m U_{x_i} \supset K$.

Therefore $C \cap \Omega' \subset U \cup B$.

And every time $U_i \cap U_j \neq \emptyset$ we can find $x_{ij} \in U_i \cap U_j$, a neighborhood $V_{ij} \subset U_i \cap U_j$ of x_{ij} and an integer ν_{ij} such that for $\nu \geq \nu_{ij}$ $(\text{supp } T_\nu) \cap V_{ij}$ is not empty and irreducible.

Since U is connected (K is connected) for $\nu \geq \max\{\nu_i, \nu_{ij}\}$ the irreducible curve representing T_ν in each U_{x_i} must be the same, then

$$T_\nu|_U = [C_\nu \cap U].$$

For ν big enough, we have $(\text{supp } T_\nu) \cap \Omega' \subset (U \cup B) \cap \Omega'$.

To prove it, a way is to remember (Theorem 2) that the $\text{supp } T_\nu$ are zero-sets of holomorphic maps going to f with zero-set C .

Therefore $(T_\nu - [C_\nu])|_U = 0$, then $\text{supp } (T_\nu - [C_\nu]) \subset B$, that is

$$\dim (\text{supp } [(T_\nu - [C_\nu])|_{\Omega'}]) = 0$$

and the measure support theorem (page 199 of [5]) shows $T_\nu = [C_\nu]$ on Ω' .

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