The Classes and Characters of Certain Maximal and other subgroups of $\mathcal{O}_{2n+2}(2)$ (*).

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Summary. – *$\mathcal{O}_{2n+2}(2)$ is the group of a non-singular quadric in $PG(2n+1, 2)$. The related finite geometry is used to give a simple and systematic determination of the classes and characters of the maximal subgroup fixing a point on the quadric, of the intersection of this stabiliser with the simple subgroup of $\mathcal{O}^+_{2n+2}(2)$ of index 2, and of other subgroups. Explicit results are tabulated for groups of orders* 64, 128, 576, 960, 1152, 1920, 46080, 92160, 1290240, 2580480.

1. – Introduction.

An orthogonal group $\mathcal{O}_{2n+2}(2)$ of degree $2n+2$ over the field of two elements is the group of a non-singular quadric Ω in a $[2n+1]$, a projective space of dimension $2n+1$ over that field. For brevity we write $\mathcal{O}_{2n+2}$ for $\mathcal{O}_{2n+2}(2)$ hereafter. The maximal subgroup of the title is the stabiliser $\mathcal{M}_{2n+2}$ in $\mathcal{O}_{2n+2}$ of a point m_0 on Ω. That other large subgroup of $\mathcal{O}_{2n+2}$, the stabiliser $\mathcal{T}_{2n+2}$ of a point p_0 off Ω, is the direct product of a group of order 2 with the symplectic group $\mathrm{Sp}_{2n}(2)$, and so may be considered as well-known. Here we shall examine $\mathcal{M}_{2n+2}$, and in particular determine its classes and characters.

The tangent prime M_0 to Ω at m_0 is the join of m_0 to a $[2n-1]$ meeting Ω in a non-singular quadric Q. The group in M_0 of the cone joing m_0 to Q is a copy of $\mathcal{M}_{2n+2}$. Since the geometry of this cone may be inferred from that of Q we expect that information for $\mathcal{M}_{2n+2}$ can be obtained geometrically from properties of the group $\mathcal{O}_{2n}$ of Q. This is indeed the case; $\mathcal{M}_{2n+2}$ is the semidirect product of an elementary abelian group $\mathcal{A}_{2n}$ of order 2^{2n} with $\mathcal{O}_{2n}$. The action of the centralisers of elements of $\mathcal{O}_{2n}$ on the corresponding spaces of fixed points in the $[2n-1]$ gives the classes of $\mathcal{M}_{2n+2}$ from those of $\mathcal{O}_{2n}$. Further, similar geometric considerations allow one to write down the characters of $\mathcal{M}_{2n+2}$ from those of $\mathcal{O}_{2n}$, $\mathcal{M}_{2n}$ and $\mathcal{T}_{2n}$. The method is systematic and simple, both geometrically and arithmetically, in practice.

There are two kinds of non-singular quadric in $[2n+1]$; ruled quadrics containing $[n]$, and non-ruled quadrics containing only $[n-1]$. There correspond two kinds of $\mathcal{O}_{2n+2}$. When we wish to indicate to which type of quadric a group belongs we attach a superfix (1) or (2) according as the quadric is ruled or not.

(*) Entrata in Redazione il 7 giugno 1974.

To determine the periods and types of powers of members of $\mathcal{M}_{2n+2}$ expeditously it is helpful to have at hand similar information for $\mathcal{O}_{2n+2}$. The largest $\mathcal{O}_{2n+2}$ which has been explicitly described in the literature is $\mathcal{O}_8^{(1)}$ (**13**), and this contains $\mathcal{O}_4^{(1)}$, $\mathcal{O}_4^{(2)}$, $\mathcal{O}_6^{(1)}$, $\mathcal{O}_6^{(2)}$. We shall obtain the classes and characters of $\mathcal{M}_4^{(1)}$, $\mathcal{M}_4^{(2)}$, $\mathcal{M}_6^{(1)}$, $\mathcal{M}_6^{(2)}$, $\mathcal{M}_8^{(1)}$ of respective orders 8, 24, 1152, 1920, 2 580 480. The results for $\mathcal{M}_4^{(1)}$, the dihedral group, and $\mathcal{M}_4^{(2)}$, the symmetric group Σ_4 on 4 symbols, are familiar, but we need a brief geometric encounter with them on our way to the characters of $\mathcal{M}_6^{(1)}$ and $\mathcal{M}_6^{(2)}$. The group $\mathcal{O}_8^{(1)}$ occurs as a primitive collineation group in complex 7-space, and has been studied as such by HAMILL (**18**). All our subgroups thus occur as complex collineation groups. $\mathcal{M}_8^{(1)}$ is the group generated by projections centred on the 56 vertices of a certain complex figure β_7 (**18**, p. 69). $\mathcal{M}_6^{(1)}$ and $\mathcal{M}_6^{(2)}$ are primitive complex collineation groups which contain homologies: MITCHELL (**21**, pp. 1, 2) lists all such groups. LITTLEWOOD (**19**, p. 190; or for a book reference **20**, p. 277) obtains the characters and classes of $\mathcal{M}_6^{(1)}$ by restricting characters of $\mathcal{O}_6^{(1)}$, which is a copy of Σ_8. But as Littlewood himself says (**19**, p. 150), his procedures are tentative in nature: when applied to $\mathcal{M}_8^{(1)}$ they become involved and laborious, chiefly because the 64 classes of $\mathcal{M}_8^{(1)}$ occur in only 48 classes of $\mathcal{O}_8^{(1)}$. It was this that provoked the search for the simple and systematic method presented here.

It is necessary to have geometric descriptions of the classes of $\mathcal{O}_4^{(1)}$, $\mathcal{O}_4^{(2)}$, $\mathcal{O}_6^{(1)}$, $\mathcal{O}_6^{(2)}$. We deduce such descriptions from information available for $\mathcal{O}_8^{(1)}$. These descriptions have some interest in themselves. $\mathcal{O}_4^{(1)}$ of order 72 occurs as a transitive subgroup of Σ_6 in (**19**, p. 187; or **20**, p. 275), while $\mathcal{O}_4^{(2)}$ is a copy of Σ_5. CONWELL (**3**) discusses the geometry of $\mathcal{O}_6^{(1)}$ when establishing its isomorphism with Σ_8, but the classes are not mentioned. $\mathcal{O}_6^{(2)}$ is the famous cubic surface group of order 51 840, and EDGE (**12**, pp. 642, 643) has described some of its classes from our viewpoint. Frame (**14**, p. 94), HAMILL (**18**, p. 78) and EDGE (**11**, p. 146) have obtained the complete classification in various other settings. Besides the familiar characters of $\mathcal{O}_4^{(1)}$, $\mathcal{O}_4^{(2)}$ and $\mathcal{O}_6^{(1)}$ we need only those of $\mathcal{T}_4^{(1)}$, $\mathcal{T}_4^{(2)}$ and $\mathcal{T}_6^{(1)}$. Each $\mathcal{T}_4$ is the direct product of Σ_2 by Σ_3, while $\mathcal{T}_6^{(1)}$ is the direct product of Σ_2 by Σ_6, so their characters are well-known.

One advantage of our approach, not so far apparent, is that it yields results not only for $\mathcal{M}_{2n+2}$ but also for any subgroup of $\mathcal{M}_{2n+2}$ containing $\mathcal{A}_{2n}$. A Sylow 2-subgroup $\mathcal{S}_{2n+2}$ of $\mathcal{O}_{2n+2}$ is one such subgroup, and we obtain, in particular, an inductive description of its classes. We give the classes and characters of $\mathcal{S}_6^{(1)}$ and $\mathcal{S}_6^{(2)}$, each of order 2^7, and isomorphic. Another such subgroup is the stabiliser $\mathcal{C}_{2n+2}$ in $\mathcal{O}_{2n+2}$ of a tangent line to Ω through m_0. Although the classes of $\mathcal{C}_{2n+2}$ can be obtained readily from those of $\mathcal{T}_{2n}$, the determination of the characters requires a knowledge of the stabiliser of a point on a non-singular quadric in $[2n]$. Such a theory has been worked out, but its presentation must be left to another day. Although there are similarities with the results in $[2n+1]$ there are also significant differences due to the existence of a kernel for a quadric in $[2n]$ (**12**, p. 630). Since we need to give a geometric interpretation of the classes of $\mathcal{T}_6^{(1)}$ in order to find the characters of $\mathcal{M}_8^{(1)}$, we make a second use of this information and quickly determine the classes of $\mathcal{C}_8^{(1)}$ of order 92160.

$\mathcal{O}_{2n+2}$ has a subgroup $\mathcal{O}_{2n+2}^+$ of index 2 which is usually simple (**6**, p. 65). Each of $\mathcal{M}_{2n+2}$, $\mathcal{S}_{2n+2}$, $\mathcal{T}_{2n+2}$, $\mathcal{C}_{2n+2}$ has a subgroup of index 2 in $\mathcal{O}_{2n+2}^+$: we denote these by $\mathcal{M}_{2n+2}^+$, $\mathcal{S}_{2n+2}^+$, $\mathcal{T}_{2n+2}^+$, $\mathcal{C}_{2n+2}^+$ respectively. Since $\mathcal{A}_{2n}$ is in $\mathcal{O}_{2n+2}^+$ these fall within our ambit. For each of the groups mentioned above whose classes and characters are explicitly found we determine, at the same time, the classes and characters of the « half-group ». $\mathcal{M}_6^{(1)+}$ of order 576 is another primitive complex collineation group containing homologies. Miss Hamill's paper (**17**) is devoted to the classification of its operations in that representation. $\mathcal{M}_8^{(1)+}$ and $\mathcal{C}_8^{(1)+}$ have been studied in connection with triality (**7**, pp. 537, 538, 539).

It is convenient to recall here some notation previously used for quadrics over $GF(2)$ (**7**; **12**; **13**). As suggested by our choice of symbols above, the points of the $[2n+1]$ will be called m or p according as they are on or off Ω. Lines are of types g, c, t, s according as they meet Ω in 3, 2, 1 or no m. A t is a tangent line and a c is a chord.

2. – The groups $\mathcal{M}_{2n+2}$, $\mathcal{M}_{2n+2}^+$ $(n \geqslant 1)$.

2.1. – We may take coordinates $(x, y; z_0, z_1, \ldots, z_{2n-1}) = (x, y; \mathbf{z}')$ of the $[2n+1]$ so that Ω is given by

$$xy + Q(\mathbf{z}) = xy + z_0 z_n + z_1 z_{n+1} + \ldots + z_{n-1} z_{2n-1} + \lambda(z_{n-1}^2 + z_{2n-1}^2) = 0 , \tag{1}$$

where λ is 0 or 1 according as Ω is ruled or not (**4**, p. 197). Since the only non-zero scalar is 1 a point has a unique vector. Likewise an element of $\mathcal{O}_{2n+2}$ has a unique matrix which must fix the above form, so the group of the quadric is the orthogonal group of the quadratic form. Since the only possible eigen-value of a member $\mathbf{A}$ of $\mathcal{O}_{2n+2}$ is 1 the fixed points of $\mathbf{A}$ correspond to the fixed vectors and form a subspace. For brevity we call this the *fixed space of* $\mathbf{A}$.

$\mathcal{O}_{2n+2}$ is transitive on the m (**8**, p. 35), so we may take m_0 to have coordinates $(1, 0; \mathbf{0}')$. Then its tangent prime M_0 is given by $y = 0$, and so joins m_0 to the $[2n-1]$: $x = y = 0$. This space C_0 is the polar space with respect to Ω of the c joining m_0 to m_1 with coordinates $(0, 1; \mathbf{0}')$. Any point of C_0 has a vector of the form $(0, 0; \mathbf{z}')$: we shall call the point z and take $\mathbf{z}'$ for its coordinate vector in C_0. The section Q of Ω by C_0 is given by $Q(\mathbf{z}) = 0$, so, from (1), Q is non-singular and is ruled or non-ruled with Ω.

2.2. – Let $\mathcal{A}_{2n}$ be the subgroup of those members of $\mathcal{M}_{2n+2}$ which fix each line in M_0 through m_0. A point of M_0 is either fixed by an element $\mathbf{A}$ of $\mathcal{A}_{2n}$ or is taken to the third point on its join to m_0. Hence $\boldsymbol{A}$ must have the form

$$\boldsymbol{A} = \begin{pmatrix} 1 & \nu & \boldsymbol{\beta}' \\ 0 & 1 & \boldsymbol{0}' \\ \boldsymbol{0} & \boldsymbol{\alpha} & \boldsymbol{I}_{2n} \end{pmatrix}.$$

We readily see from (1) that $\boldsymbol{A}$ fixes Ω if and only if

$$\nu = Q(\boldsymbol{\alpha}) \quad \text{and} \quad \boldsymbol{\beta}' = (\alpha_n, \alpha_{n+1}, \ldots, \alpha_{2n-1}, \alpha_0, \alpha_1, \ldots, \alpha_{n-1}). \tag{2}$$

If $\boldsymbol{\alpha} = \boldsymbol{0}$ then $\boldsymbol{\beta} = \boldsymbol{0}$. Otherwise $\boldsymbol{\beta}$ is the prime coordinate vector of the polar $[2n-2]$ in C_0 of α with respect to Q. We write this $\boldsymbol{A}$ as $(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})$. Matrix multiplication gives

$$(\boldsymbol{\alpha}_1, \boldsymbol{I}_{2n})(\boldsymbol{\alpha}_2, \boldsymbol{I}_{2n}) = (\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2, \boldsymbol{I}_{2n}), \tag{3}$$

so, there being 2^{2n} choices for $\boldsymbol{\alpha}$, we have

LEMMA 1. – *$\mathcal{A}_{2n}$ is an elementary abelian group of order 2^{2n}.*

Any point not in $\boldsymbol{M}_0$ has for its vector $(\mu, 1; \boldsymbol{\alpha}')$ for some μ and $\boldsymbol{\alpha}$. The point is on Ω if and only if $\mu = Q(\boldsymbol{\alpha})$, in which case $(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})$ takes it to m_1. Every line through m_0 but not in $\boldsymbol{M}_0$ is a c containing, besides m_0, just 1 m and 1 p. Since each point of C_0 is polar to m_0 the lines through m_0 in $\boldsymbol{M}_0$ are g or t according as they meet C_0 in m or p. We deduce

LEMMA 2. – *$\mathcal{A}_{2n}$ acts transitively on the c through m_0, the m off $\boldsymbol{M}_0$, and the p off $\boldsymbol{M}_0$.*

2.3. – We consider the stabliser in $\mathcal{M}_{2n+2}$ of m_1, or, equivalently, of the chord $m_0 m_1$. Using the fact that each member of this stabiliser fixes C_0, we quickly see from (1) that it consists of all $\begin{pmatrix} 1 & 0 & \boldsymbol{0}' \\ 0 & 1 & \boldsymbol{0}' \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{a} \end{pmatrix}$ with $\boldsymbol{a}$ in the group $\mathcal{O}_{2n}$ of Q. We denote $\begin{pmatrix} 1 & 0 & \boldsymbol{0}' \\ 0 & 1 & \boldsymbol{0}' \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{a} \end{pmatrix}$ by $(\boldsymbol{0}, \boldsymbol{a})$: this is consistent with our earlier notation if $\boldsymbol{a} = \boldsymbol{I}_{2n}$. The stabiliser, the set of all these $(\boldsymbol{0}, \boldsymbol{a})$, may, without confusion, also be called $\mathcal{O}_{2n}$. We may prove

LEMMA 3. – *$\mathcal{M}_{2n+2}$ is the semidirect product $\mathcal{A}_{2n}\mathcal{O}_{2n}$.*

PROOF. – If $\boldsymbol{A}$ is in $\mathcal{M}_{2n+2}$ then there is, by Lemma 2, an $(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})$ such that $(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})\boldsymbol{A}$ fixes m_1 and so is in $\mathcal{O}_{2n}$. Hence $\mathcal{M}_{2n+2}$ is $\mathcal{A}_{2n}\mathcal{O}_{2n}$. Moreover, $\mathcal{A}_{2n}$ is normal in $\mathcal{M}_{2n+2}$ and, as a glance at the matrices $(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})$ and $(\boldsymbol{0}, \boldsymbol{a})$ shows, intersects $\mathcal{O}_{2n}$ trivially. Hence the product is semidirect, and the Lemma is proven.

We shall write $(\boldsymbol{\alpha}, \boldsymbol{a})$ for $(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})(\boldsymbol{0}, \boldsymbol{a})$: this conforms with the previous notation.

2.4. – Let L be the fixed space in C_0 of $\boldsymbol{a}$ in $\mathcal{O}_{2n}$. We denote the polar space of L with respect to Q by L'. If L is an $[r]$ then L' is a $[2n-r-2]$. In § 3 it will be necessary to distinguish the $(\boldsymbol{\alpha}, \boldsymbol{a})$ with α not in L' from the other $(\boldsymbol{\alpha}, \boldsymbol{a})$. We now give some preparatory lemmas for the former set. Corresponding results for the latter set will arise as corollaries of the discussion of conjugacy in § 3. First we need

LEMMA 4. – *The non-zero vectors of* $\operatorname{Im}(\boldsymbol{a} + \boldsymbol{I}_{2n})$ *are the coordinates of the points of* L'.

PROOF. – If L is an $[r]$ then $\boldsymbol{a} + \boldsymbol{I}_{2n}$ has rank $2n-r-1$, so the points of C_0 with coordinate vectors in $\operatorname{Im}(\boldsymbol{a} + \boldsymbol{I}_{2n})$ form a $[2n-r-2]$, say N. Since L' is a $[2n-r-2]$ we need only show that N is contained in L', which is the intersection of all primes of C_0 that are fixed by $\boldsymbol{a}$. If $\boldsymbol{l}$ is the prime coordinate of a fixed prime then $\boldsymbol{l}'\boldsymbol{a} = \boldsymbol{l}'$. Hence, for any $\boldsymbol{z}$,

$$\boldsymbol{l}'(\boldsymbol{a} + \boldsymbol{I}_{2n})\boldsymbol{z} = \boldsymbol{l}'\boldsymbol{a}\boldsymbol{z} + \boldsymbol{l}'\boldsymbol{z} = \boldsymbol{l}'\boldsymbol{z} + \boldsymbol{l}'\boldsymbol{z} = 0 .$$

So each point of N lies in every fixed prime and thus in L', and the result is proved.

LEMMA 5. – *If* α *is a point of* C_0 *off* L' *then the space of fixed points of* $(\boldsymbol{\alpha}, \boldsymbol{a})$ *is the join of* m_0 *to the intersection of* L *with the polar prime of* α.

PROOF. – The matrix of $(\boldsymbol{\alpha}, \boldsymbol{a})$ is, from §§ 2.2, 2.3, $\begin{pmatrix} 1 & \nu & \boldsymbol{\beta}'\boldsymbol{a} \\ 0 & 1 & \boldsymbol{0}' \\ \boldsymbol{0} & \boldsymbol{\alpha} & \boldsymbol{a} \end{pmatrix}$ where $\boldsymbol{\beta}$ and ν are as in (2). Hence the point $(x, y; \boldsymbol{z})$ is fixed if and only if

$$\nu y + \boldsymbol{\beta}'\boldsymbol{a}\boldsymbol{z} = 0 ; \qquad \boldsymbol{\alpha} y + \boldsymbol{a}\boldsymbol{z} = \boldsymbol{z} .$$

By Lemma 4 and the second equation $y = 0$, so the fixed points are in M_0. Then the equations become

$$\boldsymbol{\beta}'\boldsymbol{a}\boldsymbol{z} = 0 ; \qquad \boldsymbol{a}\boldsymbol{z} = \boldsymbol{z} ;$$

which are equivalent to

$$\boldsymbol{\beta}'\boldsymbol{z} = 0 ; \qquad \boldsymbol{a}\boldsymbol{z} = \boldsymbol{z} .$$

Since $\boldsymbol{\beta}$ is the coordinate vector of the polar prime of α we have the result.

We may notice that if $\boldsymbol{\alpha} = \boldsymbol{0}$ or α is in L' then the same argument shows that the fixed points of $(\boldsymbol{\alpha}, \boldsymbol{a})$ in M_0 are those in the join of m_0 to L.

2.5. – One immediate consequence of Lemma 5 is that *if* $\alpha \neq \mathbf{0}$ *then the fixed space of* $(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})$ *is the polar* $[2n-1]$ *of the line* $m_0\alpha$ *with respect to* Ω. In the terminology of (**10**) this line is the *axis* of the involution $(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})$. Further, by (**10**, pp. 62, 65) we see that each $(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})$, and hence $\mathcal{A}_{2n}$, is contained in $\mathcal{O}^+_{2n+2}$.

The Dickson invariant of $(\mathbf{0}, \boldsymbol{a})$ in $\mathcal{O}_{2n+2}$ is the same as that of $\boldsymbol{a}$ in $\mathcal{O}_{2n}$. This follows immediately from equation (21) of (**6**, p. 65); see also (**4**, p. 206). Thus (**6**, p. 65) $\mathcal{O}^+_{2n+2}$ intersects $\mathcal{O}_{2n}$ in $\mathcal{O}^+_{2n}$. Hence $\mathcal{M}^+_{2n+2}$ intersects $\mathcal{O}_{2n}$ in $\mathcal{O}^+_{2n}$ and so, by Dedekind's rule, we have

LEMMA 6. – *$\mathcal{M}^+_{2n+2}$ is the semidirect product $\mathcal{A}_{2n}\mathcal{O}^+_{2n}$.*

We may now justify the title by proving

THEOREM 1. – *$\mathcal{M}_{2n+2}$ is maximal in $\mathcal{O}_{2n+2}$, and $\mathcal{M}^+_{2n+2}$ is maximal in $\mathcal{O}^+_{2n+2}$.*

PROOF. – A point m on Q is moved to the third point of the line mm_0 by those $(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})$ with α not conjugate to m. Thus, since $\mathcal{O}_{2n}$ is transitive on the points of Q, $\mathcal{M}_{2n+2}$ is transitive on the m of M_0 other than m_0. This set can, by § 2.2, only be empty if Q has no points. From (1) this is so only if $n=1$ and Q is non-ruled. Hence, by Lemma 2, $\mathcal{M}^{(2)}_4$ has two orbits on Ω and all other $\mathcal{M}_{2n+2}$ have three orbits: m_0; the other m in M_0; the m off M_0.

Suppose that the subgroup $\mathcal{H}$ of $\mathcal{O}_{2n+2}$ strictly contains $\mathcal{M}_{2n+2}$. There must be elements of $\mathcal{H}$ moving m_0. Hence for the case of $\mathcal{M}^{(2)}_4$ the group $\mathcal{H}$ is transitive on the m of Ω. For the other $\mathcal{M}_{2n+2}$ there are at most two orbits under $\mathcal{H}$. Suppose there are two. Then either the m in M_0 other than m_0 or all the m of M_0 form an orbit under $\mathcal{H}$. But the m in M_0 other than m_0 span M_0: to see this observe from (1) that if Q is ruled then the vertices of the simplex of reference in C_0 lie on Q; while if Q is non-ruled and $n>1$ all but two of these vertices are on Q, and the third points of the join of the other two to the point z with $z_0=z_n=1$ and $z_i=0$ otherwise are m. Hence, in either case, every element of $\mathcal{H}$ fixes M_0 and thus m_0. This is impossible. Hence $\mathcal{H}$ has one orbit of m. The indices of $\mathcal{M}_{2n+2}$ in $\mathcal{H}$ and $\mathcal{O}_{2n+2}$ are thus equal, each being the number of m on Ω. Hence $\mathcal{H}$ is $\mathcal{O}_{2n+2}$ and $\mathcal{M}_{2n+2}$ is maximal in $\mathcal{O}_{2n+2}$.

We may repeat the proof for $\mathcal{M}^+_{2n+2}$ using the transitivity of $\mathcal{O}^+_{2n}$ on the m of C_0; see (**9**, p. 419).

2.6. – Each subgroup of $\mathcal{M}_{2n+2}$ containing $\mathcal{A}_{2n}$ is the semidirect product of $\mathcal{A}_{2n}$ and the intersection of the subgroup with $\mathcal{O}_{2n}$.

The $\mathcal{T}_{2n+2}$ fixing a t through m_0 contains $\mathcal{A}_{2n}$ and meets $\mathcal{O}_{2n}$ in the stabiliser $\mathcal{F}_{2n}$ of the point p_0 of intersection of the t with C_0. Associated with p_0 is a unique transvection $\{p_0\}$ whose fixed space is the polar $[2n-2]$ of p_0 with respect to Q. $\mathcal{F}_{2n}$ is (**9**, p. 421) the direct product $\langle\{p_0\}\rangle \times \mathcal{F}^+_{2n}$, where $\mathcal{F}^+_{2n}$ is a copy of $\mathrm{Sp}_{2n-2}(2)$. Thus *$\mathcal{T}_{2n+2}$ is the semidirect product $\mathcal{A}_{2n}\mathcal{F}_{2n}$* and *$\mathcal{T}^+_{2n+2}$ is $\mathcal{A}_{2n}\mathcal{F}^+_{2n}$*.

A Sylow 2-subgroup of $\mathcal{M}_{2n+2}$ must contain the normal $\mathcal{A}_{2n}$, and so is the product of $\mathcal{A}_{2n}$ with an $\mathcal{S}_{2n}$. But $\mathcal{M}_{2n+2}$ has index $(2^{n+1}\mp 1)(2^n \pm 1)$ in $\mathcal{O}_{2n+2}$, the upper sign being taken if and only if Ω is ruled, since this is the number of m (**22**, p. 302). Hence $\mathcal{A}_{2n}\mathcal{S}_{2n}$ *is an* $\mathcal{S}_{2n+2}$. Likewise $\mathcal{A}_{2n}\mathcal{S}^+_{2n}$ *is an* $\mathcal{S}^+_{2n+2}$. Although we shall not use the information it is worth remarking that the usual description of Borel subgroups as stabilisers of flags follows immediately by induction.

3. – The conjugacy classes.

3.1. – We obtain the classes of the group $\mathcal{A}_{2n}\mathcal{G}$ where $\mathcal{G}$ is a subgroup of $\mathcal{O}_{2n}$. Any subgroup $\mathcal{G}$ of $\mathcal{O}_{2n}$ gives a semidirect product $\mathcal{A}_{2n}\mathcal{G}$ containing $\mathcal{A}_{2n}$. We prove

THEOREM 2. Let $\mathcal{C}$ be a conjugacy class of $\mathcal{G}$. Suppose that the space of fixed points in C_0 of an element of $\mathcal{C}$ is an $[r]$ and that the centraliser in $\mathcal{G}$ of that element has orbits of sizes $\sigma_1, \sigma_2, \dots, \sigma_l$ on the $[r-1]$ of the fixed space. Then $l+1$ classes $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_l$ of $\mathcal{A}_{2n}\mathcal{G}$ arise from $\mathcal{C}$. $\mathcal{C}_0$ containing $\mathcal{C}$ has size $2^{2n-r-1}|\mathcal{C}|$, and for $i\geqslant 1$ $\mathcal{C}_i$ has size $2^{2n-r-1}\sigma_i|\mathcal{C}|$.

PROOF. – The members of $\mathcal{A}_{2n}\mathcal{G}$ are those $(\boldsymbol{\alpha}, \boldsymbol{a})$ with $(\boldsymbol{0}, \boldsymbol{a})$ in $\mathcal{G}$. All $(\boldsymbol{0}, \boldsymbol{a})$ mentioned in this proof are in $\mathcal{G}$, and all geometry is in C_0 with respect to Q.

Matrix multiplication gives

$$(\boldsymbol{\alpha}_1, \boldsymbol{a}_1)(\boldsymbol{\alpha}_2, \boldsymbol{a}_2) = (\boldsymbol{\alpha}_1 + \boldsymbol{a}_1\boldsymbol{\alpha}_2, \boldsymbol{a}_1\boldsymbol{a}_2)\,, \tag{4}$$

so the inverse of $(\boldsymbol{\alpha}, \boldsymbol{a})$ is $(\boldsymbol{a}^{-1}\boldsymbol{\alpha}, \boldsymbol{a}^{-1})$. Consequently

$$(\boldsymbol{\alpha}, \boldsymbol{a})^{-1}(\boldsymbol{\alpha}_1, \boldsymbol{a}_1)(\boldsymbol{\alpha}, \boldsymbol{a}) = \big(\boldsymbol{a}^{-1}(\boldsymbol{\alpha} + \boldsymbol{\alpha}_1 + \boldsymbol{a}_1\boldsymbol{\alpha}), \boldsymbol{a}^{-1}\boldsymbol{a}_1\boldsymbol{a}\big)\,. \tag{5}$$

Let $(\boldsymbol{0}, \boldsymbol{a}_1)$ be in $\mathcal{C}$ and have conjugates $(\boldsymbol{0}, \boldsymbol{a}_j)$ in $\mathcal{G}$ for $1 \leqslant j \leqslant |\mathcal{C}|$. Take $(\boldsymbol{0}, \boldsymbol{b}_j)$ in $\mathcal{G}$ so that $\boldsymbol{b}_j^{-1}\boldsymbol{a}_1\boldsymbol{b}_j = \boldsymbol{a}_j$. The $(\boldsymbol{0}, \boldsymbol{b}_j)$ form a set of coset representatives of the centralizer $\mathcal{K}$ of $(\boldsymbol{0}, \boldsymbol{a}_1)$ in $\mathcal{G}$. Now by (5)

$$(\boldsymbol{0}, \boldsymbol{b}_j)^{-1}(\boldsymbol{b}_j\boldsymbol{\alpha}, \boldsymbol{a}_1)(\boldsymbol{0}, \boldsymbol{b}_j) = (\boldsymbol{\alpha}, \boldsymbol{b}_j^{-1}\boldsymbol{a}_1\boldsymbol{b}_j) = (\boldsymbol{\alpha}, \boldsymbol{a}_j)\,.$$

Hence, by enumeration, each $(\boldsymbol{\alpha}, \boldsymbol{a}_1)$ has $|\mathcal{C}|$ conjugates by the $(\boldsymbol{0}, \boldsymbol{b}_j)$ and these 2^{2n} sets contain each $(\hat{\boldsymbol{\alpha}}, \boldsymbol{a}_j)$ just once. These are all the elements of $\mathcal{A}_{2n}\mathcal{G}$ arising from $\mathcal{C}$.

Two of these sets are conjugate in $\mathcal{A}_{2n}\mathcal{G}$ if and only if the corresponding $(\boldsymbol{\alpha}, \boldsymbol{a}_1)$ are. If $(\boldsymbol{\alpha}_1, \boldsymbol{a}_1)$ is conjugate to q of its fellows then $(\boldsymbol{\alpha}_1, \boldsymbol{a}_1)$ has $q|\mathcal{C}|$ conjugates in $\mathcal{A}_{2n}\mathcal{G}$. If the conjugate of $(\boldsymbol{\alpha}_1, \boldsymbol{a}_1)$ by $(\boldsymbol{\alpha}, \boldsymbol{a})$ is $(\boldsymbol{\alpha}_2, \boldsymbol{a}_1)$ then from (5) the element $(\boldsymbol{0}, \boldsymbol{a})$ is in $\mathcal{K}$. From now on $(\boldsymbol{0}, \boldsymbol{a})$ will be an element of this centraliser. Since $(\boldsymbol{\alpha}, \boldsymbol{a}) = (\boldsymbol{\alpha}, \boldsymbol{I}_{2n})(\boldsymbol{0}, \boldsymbol{a})$ we may consider the conjugates of $(\boldsymbol{\alpha}_1, \boldsymbol{a}_1)$ by the $(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})$ and then the conjugates of these by $\mathcal{K}$. We write L for the fixed space of $(\boldsymbol{0}, \boldsymbol{a}_1)$ in C_0.

The conjugate of $(\boldsymbol{0}, \boldsymbol{a}_1)$ by $(\boldsymbol{\alpha}, \boldsymbol{a})$ is, by (5), $\big(\boldsymbol{a}^{-1}(\boldsymbol{\alpha} + \boldsymbol{a}_1\boldsymbol{\alpha}), \boldsymbol{a}_1\big)$ or $\big((\boldsymbol{a}_1 + \boldsymbol{I}_{2n})\boldsymbol{a}^{-1}\boldsymbol{\alpha}, \boldsymbol{a}_1\big)$. Hence, by Lemma 4, the other $(\boldsymbol{\alpha}, \boldsymbol{a}_1)$ conjugate to $(\boldsymbol{0}, \boldsymbol{a}_1)$ are those with α in L'.

We obtain a class $\mathcal{C}_0$ of $\mathcal{A}_{2n}\mathcal{G}$ containing $2^{2n-r-1}|\mathcal{C}|$ members, since the $[2n-r-2]$ L' has $2^{2n-r-1}-1$ points. $\mathcal{C}_0$ contains $(\mathbf{0}, \boldsymbol{a}_1)$ and thus $\mathcal{C}$.

Suppose, now, that α_1 in C_0 is off L'. From (5) the conjugate of $(\boldsymbol{\alpha}_1, \boldsymbol{a}_1)$ by $(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})$ is $(\boldsymbol{\alpha}_1 + (\boldsymbol{a}_1 + \boldsymbol{I}_{2n})\boldsymbol{\alpha}, \boldsymbol{a}_1)$. The 2^{2n-r-1} vectors $\boldsymbol{\alpha}_1 + (\boldsymbol{a}_1 + \boldsymbol{I}_{2n})\boldsymbol{\alpha}$ are the coordinates of those points of the join of α_1 to L' which are not in L'. The polar $[2n-r-2]$ of all these points meet L in the same $[r-1]$. These points are those in the polar space of this $[r-1]$ but not in L'. Thus the $2^{2n}-2^{2n-r-1}$ such $(\boldsymbol{\alpha}_1, \boldsymbol{a}_1)$ with α_1 off L' fall into $2^{r+1}-1$ sets by conjugation by $\mathcal{A}_{2n}$.

Since the conjugate of $(\boldsymbol{\alpha}_1, \boldsymbol{a}_1)$ by $(\mathbf{0}, \boldsymbol{a})$ is $(\boldsymbol{a}^{-1}\boldsymbol{\alpha}_1, \boldsymbol{a}_1)$ the combination of these sets to give full classes of $\mathcal{A}_{2n}\mathcal{G}$ is determined by the action of $\mathcal{K}$. Two sets combine if and only if there is an element of $\mathcal{K}$ taking one of the corresponding $[r-1]$ of L to the other. The proof of the Theorem is thus complete.

Since each class of $\mathcal{A}_{2n}\mathcal{G}$ arises from one of $\mathcal{G}$ this Theorem 2 gives all the classes of $\mathcal{A}_{2n}\mathcal{G}$.

3.2. – From the detail of the above proof and Lemma 5 we have

COROLLARY 1. – *a) The fixed space of* $(\boldsymbol{\alpha}_1, \boldsymbol{a}_1)$ *in* $\mathcal{C}_0$ *is the join of* L *to a polar* c *through* m_0.

b) The fixed space of $(\boldsymbol{\alpha}_1, \boldsymbol{a}_1)$ *in* $\mathcal{C}_i$ *with* $i \geqslant 1$ *is the join of* m_0 *to the corresponding* $[r-1]$ *of the orbit in* L.

3.3. – With $\mathcal{G} = \mathcal{S}_{2n}$ Theorem 2 gives an *inductive determination of the classes of* $\mathcal{S}_{2n+2}$ *in terms of those of* $\mathcal{S}_{2n}$ (§ 2.6). The $\mathcal{S}_2$ each have order 2 so the induction starts. Similarly, since the $\mathcal{S}_2^+$ are trivial, Theorem 2 gives the classes of $\mathcal{S}_{2n+2}^+$ inductively. This is, however, more a theoretical than a practical result: associated with a flag there is a large number of orbits of primes in C_0, and the discussion becomes intricate.

3.4. – Theorem 2 gives, by Lemmas 3, 6, the *classes of* $\mathcal{M}_{2n+2}$ *and* $\mathcal{M}_{2n+2}^+$ *from those of* $\mathcal{O}_{2n}$ *and* $\mathcal{O}_{2n}^+$ respectively. As we shall see the result is of practical use for these groups; for many centralisers the orbits are those $[r-1]$ of L which have the same kind of section with Q.

3.5. – We may obtain some information about powers and periods of members of $\mathcal{M}_{2n+2}$. Suppose that $(\mathbf{0}, \boldsymbol{a})$ has fixed space L in C_0. If α is in L' then $(\boldsymbol{\alpha}, \boldsymbol{a})$ is conjugate to $(\mathbf{0}, \boldsymbol{a})$ and so has the same period. On the other hand we have

LEMMA 7. – If α is not in L' then $(\boldsymbol{\alpha}, \boldsymbol{a})$ has even period which is at most twice the period of $(\mathbf{0}, \boldsymbol{a})$.

PROOF. – By (4) we find that

$$(\boldsymbol{\alpha}, \boldsymbol{a})^k = \left((\boldsymbol{a}^{k-1} + \boldsymbol{a}^{k-2} + \ldots + \boldsymbol{a} + \boldsymbol{I}_{2n})\boldsymbol{\alpha}, \boldsymbol{a}^k\right),$$

so the period v of $(\boldsymbol{\alpha}, \boldsymbol{a})$ is lu where u is the period of $(\mathbf{0}, \boldsymbol{a})$. Since

$$(\boldsymbol{a}^{2u-1} + \ldots + \boldsymbol{I}_{2n})\boldsymbol{\alpha} = (\boldsymbol{a}^{u} + \boldsymbol{I}_{2n})(\boldsymbol{a}^{u-1} + \boldsymbol{a}^{u-2} + \ldots + \boldsymbol{I}_{2n})\boldsymbol{\alpha} = 0\,,$$

we have $l \leqslant 2$. Were v to be odd then we should have

$$\boldsymbol{\alpha} = (\boldsymbol{a}^{v-1} + \ldots + \boldsymbol{a})\boldsymbol{\alpha} = (\boldsymbol{a} + \boldsymbol{I}_{2n})(\boldsymbol{a}^{v-2} + \boldsymbol{a}^{v-4} + \ldots + \boldsymbol{a})\boldsymbol{\alpha}\,,$$

in contradiction to Lemma 4.

We shall see below that both posibilities can occur if u is even.

$(\boldsymbol{\alpha}, \boldsymbol{a})^k$ arises from the class of $(\mathbf{0}, \boldsymbol{a}^k)$ in $\mathcal{O}_{2n}$. For later use we show that $(\boldsymbol{\alpha}, \boldsymbol{a})^2$ *cannot be conjugate to* $(\mathbf{0}, \boldsymbol{a}^2)$ *if* L' *contains* L. For, from the details of the proof of Theorem 2, their being conjugate would demand the existence of an $\hat{\boldsymbol{\alpha}}$ such that

$$(\boldsymbol{a} + \boldsymbol{I}_{2n})\boldsymbol{\alpha} = (\boldsymbol{a}^2 + \boldsymbol{I}_{2n})\,\hat{\boldsymbol{\alpha}}\,.$$

But then $\boldsymbol{\alpha} + (\boldsymbol{a} + \boldsymbol{I}_{2n})\hat{\boldsymbol{\alpha}}$ would give a point in L and thus L'. By Lemma 4 we should deduce that α was in L', a contradiction.

4. – The characters.

4.1. – We first need to give a geometric description of the characters of $\mathcal{A}_{2n}$. We write χ_0 for the unit character which takes the value 1 at all $(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})$. For each z in C_0 we define χ_z on $\mathcal{A}_{2n}$ as follows. $\chi_z\{(\mathbf{0}, \boldsymbol{I}_{2n})\}$ is 1, while on the non-identity elements of $\mathcal{A}_{2n}$ the value of $\chi_z\{(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})\}$ is 1 if α is conjugate to z and -1 otherwise. Since a line has 1 or 3 points in the polar $[2n-2]$ of z we see from (3) that χ_z is a character of $\mathcal{A}_{2n}$. Since distinct z have distinct polars we obtain all $2^{2n}-1$ non-trivial irreducible characters of $\mathcal{A}_{2n}$ this way.

4.2. – We may make $\mathcal{G}$ act on the set of characters of $\mathcal{A}_{2n}$ by defining, for each $(\mathbf{0}, \boldsymbol{a})$ in $\mathcal{G}$,

$$(\mathbf{0}, \boldsymbol{a})\,\chi_z = \chi_w\,, \qquad \text{where } \boldsymbol{w} = \boldsymbol{a}\boldsymbol{z}\,. \tag{6}$$

Since $(\mathbf{0}, \boldsymbol{a})$ preserves polarity in C_0 the value of $(\mathbf{0}, \boldsymbol{a})\chi_z$ at $(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})$ is the value of χ_z at $(\boldsymbol{a}^{-1}\boldsymbol{\alpha}, \boldsymbol{I}_{2n})$. This element is by (5) the conjugate of $(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})$ by $(\mathbf{0}, \boldsymbol{a})$. Thus we have the same action as that described by Serre (**23**, II, p. 18).

Let z_j, $j = 1, \ldots, q$, be points one from each orbit under the action of $\mathcal{G}$ in C_0, and let $\mathcal{G}_j$ be their respective stabilisers. For simplicity we now write χ_j for the character associated with z_j. Then, in the permutation representation of $\mathcal{G}$ on the characters of $\mathcal{A}_{2n}$, $\chi_0, \chi_1, \ldots, \chi_q$ form a set of orbit representatives. Further, for $j \geqslant 1$, the stabiliser of χ_j in $\mathcal{G}$ is, by (6), $\mathcal{G}_j$. The stabiliser of χ_0 is $\mathcal{G}_0 = \mathcal{G}$.

4.3. – Following Serre (**23**, II, p. 18), we extend each χ_j ($j\geq 0$) to a character $\hat{\chi}_j$ of $\mathcal{A}_{2n}\mathcal{G}_j$ by putting

$$(7) \qquad \hat{\chi}_j\{(\boldsymbol{\alpha}, \boldsymbol{a})\} = \chi_j\{(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})\}, \quad \text{for all } (\boldsymbol{0}, \boldsymbol{a}) \text{ in } \mathcal{G}_j .$$

$\mathcal{G}_j$ is the quotient of $\mathcal{A}_{2n}\mathcal{G}_j$ by $\mathcal{A}_{2n}$, so we may, in the usual manner, extend an irreducible character ϱ of $\mathcal{G}_j$ to one $\hat{\varrho}$ of $\mathcal{A}_{2n}\mathcal{G}_j$ where

$$(8) \qquad \hat{\varrho}\{(\boldsymbol{\alpha}, \boldsymbol{a})\} = \varrho\{(\boldsymbol{0}, \boldsymbol{a})\}, \quad \text{for all } (\boldsymbol{0}, \boldsymbol{a}) \text{ in } \mathcal{G}_j .$$

Now take the Kronecker product $\hat{\varrho}\times\hat{\chi}_j$ of $\hat{\varrho}$ and $\hat{\chi}_j$, and induce from this a character of $\mathcal{A}_{2n}\mathcal{G}$. By (**23**, Theorem 17) this character is irreducible, and if we take all possible pairs j, ϱ we obtain each irreducible character of $\mathcal{A}_{2n}\mathcal{G}$ just once. That part of the proof that Serre leaves as an exercise is readily verified using Mackey's criterion and its extension (**23**, II, p. 11).

4.4. – We examine how the geometry determines the values of these induced characters from those of the corresponding ϱ. Suppose that class $\mathcal{C}$ of $\mathcal{G}$ gives rise to classes $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_l$ of $\mathcal{A}_{2n}\mathcal{G}$ as in Theorem 2: we retain the notation of § 3.1.

If $j=0$ then $\mathcal{G}_0=\mathcal{G}$ and $\hat{\chi}_0$ is the unit character of $\mathcal{A}_{2n}\mathcal{G}$. So, from (8), the value of the induced character associated with ϱ in each class $\mathcal{C}_i$ is the value of ϱ in $\mathcal{C}$.

If $j\geq 1$ then the value of the character induced from $\hat{\chi}_j\times\hat{\varrho}$ on $\mathcal{C}_i$ is

$$(9) \qquad \{\sum \hat{\chi}_j\{(\boldsymbol{\alpha}, \boldsymbol{a})\}\,\hat{\varrho}\{(\boldsymbol{\alpha}, \boldsymbol{a})\}\}\,|\mathcal{A}_{2n}\mathcal{G} : \mathcal{A}_{2n}\mathcal{G}_j|/|\mathcal{C}_i| ,$$

the summation being over all $(\boldsymbol{\alpha}, \boldsymbol{a})$ in $\mathcal{C}_i \cap \mathcal{A}_{2n}\mathcal{G}_j$. These are the $(\boldsymbol{\alpha}, \boldsymbol{a})$ in $\mathcal{C}_i$ with $(\boldsymbol{0}, \boldsymbol{a})$ in $\mathcal{C}\cap\mathcal{G}_j$. If $i=0$ there are for each such $(\boldsymbol{0}, \boldsymbol{a})$ 2^{2n-r-1} such $\boldsymbol{\alpha}$ by § 3.1; namely $\boldsymbol{0}$ and those α giving points of L', the polar $[2n-r-2]$ of the fixed space L of $(\boldsymbol{0}, \boldsymbol{a})$ in C_0. Since L contains z_j we have $\chi_j(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})=1$ for these $\boldsymbol{\alpha}$. Hence from (7), (8), (9) and Theorem 2 *the value of the induced character on* $\mathcal{C}_0$ *is*

$$(10) \qquad \{\sum\varrho\{(\boldsymbol{0}, \boldsymbol{a})\}\}\,|\mathcal{G}:\mathcal{G}_j|/|\mathcal{C}| ; \quad \text{the sum being over } (\boldsymbol{0}, \boldsymbol{a}) \text{ in } \mathcal{C}\cap\mathcal{G}_j .$$

If $i\geq 1$ there are $2^{2n-r-1}\sigma_i$ such α for each $(\boldsymbol{0}, \boldsymbol{a})$. The polar $[2n-2]$ of these α meet L in the $[r-1]$ of the orbit associated with $\mathcal{C}_i$. Hence $\chi_j\{(\boldsymbol{\alpha}, \boldsymbol{I}_{2n})\}=1$ if and only if the corresponding $[r-1]$ contains z_j. Let $n_{ij}(\boldsymbol{a})$ of these σ_i $[r-1]$ contain z_j. Then *the value of the induced character on* $\mathcal{C}_i$ *is*

$$(11) \qquad \{\sum(2n_{ij}(\boldsymbol{a})-\sigma_i)\,\varrho\{(\boldsymbol{0}, \boldsymbol{a})\}\}\,|\mathcal{G}:\mathcal{G}_j|/|\mathcal{C}|\sigma_i ; \quad \text{the sum being over } (\boldsymbol{0}, \boldsymbol{a}) \text{ in } \mathcal{C}\cap\mathcal{G}_j .$$

We may briefly summarise our finding as

THEOREM 3. – *The extensions to* $\mathcal{A}_{2n}\mathcal{G}$ *of the irreducible characters of the quotient group* $\mathcal{G}$ *together with the characters determined by* (10), (11) *from all possible pairs* j, ϱ *with* $j=1, 2, \dots, q$, *form the character table of* $\mathcal{A}_{2n}\mathcal{G}$.

The members of $\mathcal{C} \cap \mathcal{G}_j$ may fall into several classes in $\mathcal{G}_j$. For all members of one such class $n_{ij}(\boldsymbol{a})$ has the same value, and in practice we may sum over these classes.

4.5. – If $\mathcal{G}$ is $\mathcal{O}_{2n}$ other than $\mathcal{O}_2^{(2)}$ then $q=2$ by the transitivity of $\mathcal{O}_{2n}$ on the m and p of C_0 (**8**, p. 37). $\mathcal{G}_1$ and $\mathcal{G}_2$ are $\mathcal{M}_{2n}$ and $\mathcal{T}_{2n}$ respectively, and we have the *characters of $\mathcal{M}_{2n+2}$ in terms of those of* $\mathcal{O}_{2n}$, $\mathcal{M}_{2n}$, $\mathcal{T}_{2n}$ and the geometry of Q. Similar statements hold for $\mathcal{M}^+_{2n+2}$. For $\mathcal{O}_2^{(2)}$ $q=1$, there being no m (§ 2.5), and the modifications for $\mathcal{M}_4^{(2)}$ are obvious.

If $\mathcal{G}$ is $\mathcal{T}_{2n}$ fixing p_0 in C_0, then $\mathcal{A}_{2n}\mathcal{G}$ is $\mathcal{C}_{2n+2}$ by § 2.6. The polar $[2n-2]$ in C_0 of p_0 meets Q in a non-singular section, and $\mathcal{T}^+_{2n}$ is the group of this section (**9**, p. 421). We need the stabiliser in the orthogonal group in the $[2n-2]$ of one of its m, and as stated in § 1 we must postpone a discussion of this. Once this matter is presented we may use Theorem 3 to give the characters of $\mathcal{C}_{2n+2}$.

5. – The groups $\mathcal{O}_4^{(1)}, \mathcal{O}_4^{(2)}, \mathcal{O}_6^{(1)}, \mathcal{O}_6^{(2)}$ and $\mathcal{T}_6^{(1)}$.

5.1. – In the remainder of this paper we apply our general results to the explicit determination of the classes and characters of those groups mentioned in § 1. The brief exposition necessary illustrates the usefulness of the method. The Theorems of this and later sections are the tables.

5.2. – Suppose, henceforth, that Ω is specialised to be a ruled quadric in [7]. We recall some notation from (**7**) and (**13**) for subspaces. Planes are labelled according to their section with Ω as follows:

- d: lying on Ω,
- e: a repeated line,
- f: a line pair,
- h: a single point,
- j: a conic with $3m$ which lie in pairs on c.

Solids are similarly categorised by:

- ω: lying on Ω,
- γ: a repeated plane,
- φ: a plane pair,
- χ: a single line,
- ψ: a point cone,
- $\varkappa$: a non-singular ruled quadric,
- λ: a non-singular non-ruled quadric.

The solids have polar spaces of the same type as themselves. The polar spaces with respect to Ω of the points, lines and planes will be labelled by the corresponding capital letters. P, J, C and S have non-singular sections with Ω, those of the last two being respectively ruled and non-ruled. The C_0 and M_0 of earlier sections are of type C and M respectively. A full incidence table for the subspaces has been given (**13**, p. 16): the nature of the subspaces in a given space is often obvious.

5.3. – The 67 classes of $\mathcal{O}_8^{(1)}$, labelled I-LXVII by Hamill, are listed in (**17**, pp.76,77); and their fixed spaces with respect to Ω are given in (**13**, pp. 26, 69) except for classes LVI, LVII, LIX, LX, LXIV, LXV, LXVI, LXVII whose members fix no points. Similar results for $\mathcal{O}_8^{(1)+}$ are given in (**7**, p. 552). From these tables we may quickly determine the distribution among the classes of $\mathcal{O}_8^{(1)}$ of the elements of a subgroup fixing a subspace U pointwise: in a class of size N there are Nx/y members, where the fixed space of each member of the class has x subspaces of the same kind as U, and there are y such subspaces in [7]. If U is a c then the subgroup is $\mathcal{O}_6^{(1)}$ acting on the polar C, as in § 2. It is given in Table 1. Since Σ_8, which is isomorphic to $\mathcal{O}_6^{(1)}$ (§ 1), has 22 classes and $\mathcal{O}_6^{(1)}$ has entries in 22 classes of $\mathcal{O}_8^{(1)}$, we may label the classes of $\mathcal{O}_6^{(1)}$ by the corresponding labels in $\mathcal{O}_8^{(1)}$. A consideration of sizes, periods and power types readily identifies the cycle types in Σ_8. To find the fixed spaces in C take the polar of a fixed space in [7]—this polar is in C—and then reciprocate in C. In presenting tables we adopt the following

CONVENTION. – *For a group $\mathcal{K}$ we give first those of its classes in $\mathcal{K}^+$ and then, separated by a horizontal line those of the coset. An asterisk indicates that a class of $\mathcal{K}$ splits into two equal-sized classes in $\mathcal{K}^+$.*

TABLE 1. – *The conjugacy classes of $\mathcal{O}_6^{(1)} = \Sigma_8$ and of $\mathcal{O}_6^{(1)+}$.*

Class	Size	Period	Cycle type	Power types			Fixed space in C
				2nd	3rd	5th	
I	1	1	1^8				C
III	210	2	$1^4 2^2$				ψ
IV	112	3	$1^5 3$				λ
IX	1680	6	$1 2^2 3$	IV	III		t
X	1120	3	$1^2 3^2$				c
XI	2520	4	$1^2 2 4$	III			t
XII	1344	5	$1^3 5$				s
XIII	105	2	2^4				φ
XXX *	2688	15	35		XII	IV	–
XXXIV *	5760	7	17				–
XXXVI	1260	4	4^2	XIII			g
XXXIX	3360	6	26	X	XIII		c
II	28	2	$1^6 2$				J
V	420	2	$1^2 2^3$				e
VI	1120	6	$1^3 2 3$	IV	II		j
VII	420	4	$1^4 4$	III			h
XVII	1120	6	$2 3^2$	X	II		p
XIX	3360	12	134	IX	VII		m
XX	4032	10	125	XII		II	p
XXII	3360	6	$1^2 6$	X	V		p
XXIII	1260	4	$2^2 4$	III			f
LV	5040	8	8	XXXVI			m

TABLE 2. – *The conjugacy classes of the cubic surface group* $\mathcal{O}_6^{(2)}$ *and of* $\mathcal{O}_6^{(2)+}$.

Class	Size	Period	Power types 2nd	3rd	5th	Fixed space in S
I	1	1				S
III	270	2				ψ
IV	240	3				$\varkappa$
IX	2160	6	IV	III		t
X	480	3				s
XI	3240	4	III			t
XII	5184	5				c
XIII	45	2				χ
XIV *	1440	6	IV	XIII		g
XV	540	4	XIII			g
XXXVIII *	80	3				–
XXXIX	1440	6	X	XIII		s
XL *	720	6	XXXVIII	XIII		–
XLI *	5760	9		XXXVIII		–
XLII *	4320	12	XL	XV		–
II	36	2				J
V	540	2				e
VI	1440	6	IV	II		j
VII	1620	4	III			f
XVII	1440	6	X	II		p
XX	5184	10	XII		II	p
XXII	4320	6	X	V		p
XXIII	540	4	III			h
XXIV	4320	12	IX	XXIII		m
XXV	6480	8	XV			m

This allows character tables to be presented economically. The classes of $\mathcal{K}$ in $\mathcal{K}^+$ will be called *even*, and those in the coset *odd*.

If U is an s then the pointwise stabiliser in $\mathcal{O}_8^{(1)}$ is $\mathcal{O}_6^{(2)}$ acting in the polar S, and is given in Table 2. $\mathcal{O}_6^{(2)+}$, being generated by the squares of elements of $\mathcal{O}_6^{(2)}$ (**6**, pp. 66, 67), is the unique subgroup of index 2 in $\mathcal{O}_6^{(2)}$, and so is in $\mathcal{O}_8^{(1)+}$. Since $\mathcal{O}_6^{(2)}$ has 25 classes (**14**, p. 95) we may label these here by the corresponding labels in $\mathcal{O}_8^{(1)}$. Frame also gives the relation of these classes to those of $\mathcal{O}_6^{(2)+}$ (see also (**18**, p. 73)). Earlier, DICKSON (**5**, p. 138), FRAME (**15**, p. 483) and TODD (**24**) had independently classified $\mathcal{O}_6^{(2)+}$ in other representations.

We may repeat the procedure for $\mathcal{O}_4^{(i)}$ which is the subgroup of $\mathcal{O}_6^{(i)}$ fixing a c pointwise. The results form Table 3. $\mathcal{O}_4^{(1)}$ of order 72 acts in a $\varkappa$, and acts transitively on the $6\,g$ therein (**8**, p. 37). It must thus be that group given by LITTLEWOOD (**20**, p. 275). $\mathcal{O}_4^{(1)+}$ is (**6**, p. 68) $\Sigma_3 \times \Sigma_3$ acting on the two reguli in $\varkappa$, so any class of $\mathcal{O}_4^{(1)}$ with a 1 in its cycle pattern in Littlewood's labelling is in $\mathcal{O}_4^{(1)+}$. The

TABLE 3. – *The conjugacy classes of the groups* $\mathcal{O}_4^{(1)}$ *and* $\mathcal{O}_4^{(2)}$.

The group $\mathcal{O}_4^{(1)}$ *of order* 72.

Class	Size	Cycle type	Fixed space in $\varkappa$
I	1	1^6	$\varkappa$
III	9	$1^2 2^2$	t
IV	4	3^2	s
X *	4	$1^3 3$	–
XIII *	6	$1^4 2$	g
XXXIX *	12	123	–
II	6	2^3	j
VI	12	6	p
XXIII	18	24	m

The group $\mathcal{O}_4^{(2)} = \Sigma_5$.

Class	Size	Cycle type	Fixed space in λ
I	1	1^5	λ
III	15	12^2	t
IV	20	$1^2 3$	c
XII *	24	5	–
II	10	$1^3 2$	j
VI	20	23	p
VII	30	14	m

Periods and power types may be read off from Tables 1, 2.

identification of cycle types is now easy, and we may label classes by the corresponding class in $\mathcal{O}_6^{(1)}$. $\mathcal{O}_4^{(2)}$ acts in a λ and is Σ_5 on its $5\,m$. $\mathcal{O}_4^{(1)}$, $\mathcal{O}_4^{(2)}$ may be considered as those subgroups of $\mathcal{O}_8^{(1)}$ fixing pointwise a $\varkappa$ and a λ respectively, these being the polar spaces with respect to Ω of the [3] in which the $\mathcal{O}_4^{(i)}$ act. Alternatively, $\mathcal{O}_4^{(1)}$ and $\mathcal{O}_4^{(2)}$ are the pointwise stabilisers of an s in $\mathcal{O}_6^{(2)}$ and $\mathcal{O}_6^{(1)}$ respectively. The same labelling of the classes of $\mathcal{O}_4^{(1)}$, $\mathcal{O}_4^{(2)}$ occurs however we regard them, and by using it we keep the inter-relationship of all the groups.

$\mathcal{O}_2^{(1)}$ is of order 2 acting on a c. Its non-identity element is in class II. $\mathcal{O}_2^{(2)}$ is Σ_3 acting on the $3\,p$ of an s. It has 2 elements in IV and 3 in II.

5.4. – In order to calculate the characters of $\mathcal{M}_8^{(1)}$ we need the classification of $\mathcal{T}_6^{(1)}$ relative to the geometry of the C on which $\mathcal{O}_6^{(1)}$ acts. $\mathcal{T}_6^{(1)}$ is the stabiliser of a point p_0. The polar [4] of p_0 is a J, call it J_0. We saw in §§ 2.6, 4.5 above that $\mathcal{T}_6^{(1)}$ is $\langle\{p_0\}\rangle \times \mathcal{T}_6^{(1)+}$ where $\mathcal{T}_6^{(1)+}$ acts as the orthogonal group in J_0. Moreover, $\mathcal{T}_6^{(1)+}$ is isomorphic to $\mathrm{Sp}_4(2)$ and hence to Σ_6 (**4**, p. 99; DICKSON gives reference to JORDAN). So $\mathcal{T}_6^{(1)}$ has 11 even and 11 odd classes.

We find the distribution of $\mathcal{T}_6^{(1)}$ among the classes of $\mathcal{O}_6^{(1)}$ by the methods of the previous section; it may be read off from the second and third columns of Table 4. Entries occur in 9 even and 8 odd classes of $\mathcal{O}_6^{(1)}$. Those in III correspond to the $60\,\psi$ through p_0. Since 15 of these ψ are in J_0 (**12**, p. 634) III must split in $\mathcal{T}_6^{(1)}$. So must XI since no class of Σ_6 has size 180, though two have size 90. Thus III and XI must split as in Table 4 to give the 11 classes of $\mathcal{T}_6^{(1)+}$. One of the $16\,J$ through p_0 is J_0, so II splits in $\mathcal{T}_6^{(1)}$. The 160 elements of $\mathcal{T}_6^{(1)}$ in VI correspond in pairs to the $80\,j$ through p_0. Since 20 of these j lie in J_0 (**12**, p. 634), and so have kernel p_0, VI splits

TABLE 4. – *The conjugacy class of* $\mathcal{T}_6^{(1)} = \langle\{p_0\}\rangle \times \Sigma_6$.

Class type	Size	Class in $\mathcal{O}_6^{(1)}$	Fixed space L in C	Fixed space in J_0	Primes of L through p_0
1^6	1	I	C	J_0	$16\,J,\ 15\,F$
$1^4 2$	15	III	ψ	ψ	$3\,e,\ \ 4\,j$
$1^2 2^2$	45	III	ψ	e	$1\,e,\ \ 1\,h,\ 1\,f,\ 4\,j$
$1^3 3$	40	IV	λ	j	$3\,h,\ \ 4\,j$
123	120	IX	t	t	p_0
3^2	40	X	c	p_0	p_0
$1^2 4$	90	XI	t	t	p_0 (focus)
24	90	XI	t	t	p_0 (non-focus)
15	144	XII	s	p_0	p_0
2^3	15	XIII	φ	e	$1\,e,\ 6\,f$
6	120	XXXIX	c	p_0	p_0
$1^6\{p_0\}$	1	II	J_0	J_0	$15\,\psi$
$1^4 2\ \{p_0\}$	15	II	J	ψ	$4\,\varkappa,\ 4\,\lambda,\ 7\,\psi$
$2^3\{p_0\}$	15	V	e	e	$3\,t$ (all through focus)
$1^2 2^2\{p_0\}$	45	V	e	e	$3\,t$ (1 through focus)
$1^3 3\ \{p_0\}$	40	VI	j	j	$3\,t$
$123\ \{p_0\}$	120	VI	j	t	$1\,c,\ 1\,t,\ 1\,s$
$1^2 4\ \{p_0\}$	90	VII	h	t	$1\,t,\ 2\,s$
$3^2\{p_0\}$	40	XVII	p_0	p_0	—
$15\ \{p_0\}$	144	XX	p_0	p_0	—
$6\ \{p_0\}$	120	XXII	p_0	p_0	—
$24\ \{p_0\}$	90	XXIII	f	t	$1\,t,\ 2\,c$

in $\mathcal{T}_6^{(1)}$. An involution $\boldsymbol{A}$ of $\mathcal{O}_6^{(1)}$ in V has for its fixed space an e with $4p$. For just one of these p, the *focus* of $\boldsymbol{A}$, $\boldsymbol{A}\{p\}$ has for fixed space the polar φ of the g in e (**10**, pp. 63, 64), and so is in XIII. Since $\mathcal{O}_6^{(1)}$ is transitive on the p of C 15 of the 60 members of $\mathcal{T}_6^{(1)}$ in V have p_0 for focus. Thus II, V, VI must split as shown in Table 4 to give 11 odd classes for $\mathcal{T}_6^{(1)}$. This table may now be completed apart from the first column and the verbal entries against XI.

Through an m of J_0 pass 2 of its $6\,\lambda$, and these $2\,\lambda$ meet in an h (**12**, p. 634). If an element of $\mathcal{T}_6^{(1)+}$ fixes each of the $6\,\lambda$ it must fix the single m common to each pair. Hence it fixes each m in J_0 and so, by the information already available in Table 4, is the identity. We conclude that $\mathcal{T}_6^{(1)+}$ is Σ_6 *acting on the* $6\,\lambda$ *of* J_0. A member of $\mathcal{T}_6^{(1)+}$ with cycle type 6 does not fix nor interchange 2 of the λ, and so can fix no m in J_0. Being in a class of size 120 it must be in XXXIX. Using information from Table 1 for periods and power types together with the known sizes of the classes of Σ_6 we may now identify the cycle types of the classes of $\mathcal{T}_6^{(1)+}$. That the geometry is used to give a definite labelling is not fortuitous: there is an alternative labelling related to ours by the outer automorphism of Σ_6.

If $\boldsymbol{A}$ is in $\mathcal{T}_6^{(1)}$ then, by Table 4, the fixed spaces of $\boldsymbol{A}$ and $\boldsymbol{A}\{p_0\}$ have different dimensions. However, since $\{p_0\}$ fixes J_0 pointwise, these fixed spaces have the same section with J_0. Hence one of $\boldsymbol{A}$, $\boldsymbol{A}\{p_0\}$ has a fixed space L not in J_0 while the other has for fixed space the intersection of L with J_0. The labelling of the odd classes of $\mathcal{T}_6^{(1)}$ follows immediately, apart from that of VII and XXIII. One of these is $1^2 4\{p_0\}$ and the other $24\{p_0\}$. Suppose, now, that $\boldsymbol{A}$ is in $1^2 4$. Then $\boldsymbol{A}$ fixes $2\,\lambda$ in J_0 and hence their polar s, s_1, s_2 say, through p_0. Since the fixed space of $\boldsymbol{A}$ is a t the $2\,p$ off J_0 on each s_i are interchanged by $\boldsymbol{A}$; so they are by $\{p_0\}$. Hence $\boldsymbol{A}\{p_0\}$ fixes s_1, s_2 pointwise, and so its fixed space is their join which must be an h. Thus $1^2 4\{p_0\}$ is VII, and so $24\{p_0\}$ is XXIII.

If $\boldsymbol{B}$ in $\mathcal{O}_6^{(1)}$ is in XI then $\boldsymbol{B}$ is conjugate in $\mathcal{O}_6^{(1)}$ to members of both $1^2 4$ and 24. Hence for one p of the fixed t of $\boldsymbol{B}$ the fixed space of $\boldsymbol{B}\{p\}$ is an h, and for the other p on t the fixed space of $\boldsymbol{B}\{p\}$ is an f. In analogy with V we call the first p the *focus* of $\boldsymbol{B}$. Table 4 is now complete.

5.5. – Each $\mathcal{T}_4$ is $\langle\{p_0\}\rangle\times\mathcal{T}_4^+$, where $\mathcal{T}_4^+$ is Σ_3. A similar, but much simpler, discussion to that in § 5.4 gives the following information. The polar plane of p_0 is denoted by j_0.

The classes of $\mathcal{T}_4^{(1)}$.

Class type	Size	Class in $\mathcal{O}_4^{(1)}$	Fixed space L in $\varkappa$	Primes of L through p_0
1^3	1	I	$\varkappa$	$4\,j$, $3\,f$
12	3	III	t	p_0
3	2	IV	s	p_0
$1^3\,\{p_0\}$	1	II	j_0	$3\,t$
$12\,\{p_0\}$	3	II	j	$1\,t$, $1\,c$, $1\,s$
$3\,\{p_0\}$	2	VI	p_0	—

The classes of $\mathcal{T}_4^{(2)}$.

Class type	Size	Class in $\mathcal{O}_4^{(2)}$	Fixed space L in λ	Primes of L through p_0
1^3	1	I	λ	$4\,j$, $3\,h$
12	3	III	t	p_0
3	2	IV	c	p_0
$1^3\,\{p_0\}$	1	II	j_0	$3\,t$
$12\,\{p_0\}$	3	II	j	$1\,t$, $1\,c$, $1\,s$
$3\,\{p_0\}$	2	VI	p_0	—

6. – The calculation of the classes and characters of $\mathcal{M}_8^{(1)}$, $\mathcal{M}_6^{(1)}$, $\mathcal{M}_6^{(2)}$, $\mathcal{M}_8^{(1)+}$, $\mathcal{M}_6^{(1)+}$, $\mathcal{M}_6^{(2)+}$.

6.1. – We use Theorem 2 to obtain the classes of $\mathcal{M}_8^{(1)}$. From each class of $\mathcal{O}_6^{(1)}$ we pick an element and determine the orbits under its centraliser of the $[r-1]$ of the fixed $[r]$. The $[r-1]$ of an orbit must all be of the same letter type. Frequent use is made of Table 1; we recall that any incidence relation of subspaces that is not obvious may be found in Table 1 of (**13**).

$\mathcal{O}_6^{(1)}$ is the centraliser of the identity element in I and acts transitively on the J and F of C (**8**, p. 37). Thus for I *the orbits are* $28\,J$, $35\,F$. The centraliser of a transvection $\{p_0\}$ in II is the corresponding $\mathcal{T}_6^{(1)}$. Since $\mathcal{T}_6^{(1)+}$ acts as the full orthogonal group in the fixed space J_0 of $\{p_0\}$ (§ 4.5) *the orbits for* II *are* $15\,\psi$, $10\varkappa$, $6\,\lambda$ (**8**, p. 40). Notice that the centraliser of an element $\boldsymbol{A}$ of $\mathcal{O}_6^{(1)}$ contains each transvection whose centre is in the fixed space of $\boldsymbol{A}$.

TABLE 5. – *The conjugacy classes of* $\mathcal{M}_6^{(1)}$ *of order* 1152 *and of* $\mathcal{M}_6^{(1)+}$ *of order* 576.

Class	Size	Period	Power types 2nd	Power types 3rd	Class in $\mathcal{O}_6^{(1)}$	Fixed space L in C	Primes of L through m_0
I	1	1			I	C	$12J$, $19F$
I j	6	2			III	ψ	$3e$, $1h$, $3f$
I f	9	2			XIII	φ	$2d$, $1e$, $4f$
III	36	2			III	ψ	$1e$, $2f$, $4j$
III m	36	4	I f		XXXVI	g	m_0 (focus)
III p	72	4	I j		XI	t	m_0
IV	16	3			IV	λ	$1h$, $6j$
IV p	48	6	IV	I j	IX	t	m_0
X *	64	3			X	c	m_0
XIII *	24	2			XIII	φ	$1d$, $6f$
XIII m*	72	4	I f		XXXVI	g	m_0 (non-focus)
XXXIX *	192	6	X	XIII	XXXIX	c	m_0
II	12	2			II	J	7ψ, $6\varkappa$, 2λ
II t	36	2			V	e	$1g$, $2t$
II c	36	4	I j		XXIII	f	$2g$, $1t$
II s	12	4	I j		VII	h	$3t$
VI	96	6	IV	II	VI	j	$1t$, $2c$
VI [−1]	96	12	IV p	II s	XIX	m	—
XXIII	144	4	III		XXIII	f	$1g$, $2c$
XXIII [−1]	144	8	III m		LV	m	—

The centraliser of a member of III has order 192 and induces a group in the fixed space ψ. The subgroup of $\mathcal{O}_6^{(1)}$ fixing ψ pointwise has order 4 since it contains 1, 1, 2 elements in I, III, IV respectively. Hence the centraliser induces in ψ a group of order at least 48. The m in ψ lie on 3 concurrent non-coplanar lines. The group in [3] fixing such a trio of lines is easily seen to have order 48; one recalls that over $GF(2)$ a simplex determines a unique «unit-point» and then uses the fundamental theorem of projective geometry. Hence the centraliser acts in ψ as the full group of its figure. Thus, by the fundamental theorem, *the orbits for* III *are* $3e$, $3f$, $1h$, $1j$. A similar argument shows that *the orbits for* XIII *are* $2d$, $1e$, $12f$.

The transvections centred on a λ generate the full group of its quadric (**6**, p 65), so *the orbits for* IV *are* $5h$, $10j$.

A transvection $\{p_0\}$ interchanges the $2m$ of a c through p_0, and interchanges the other $2p$ on an s through p_0. When we consider the transvections centred on the fixed spaces we find the following orbits under centralisers: *for* X $2m$, $1p$; *for* XII $3p$; *for* XXXIX $2m$, $1p$; *for* VI $3t$, $3c$, $1s$; *for* VII $3t$, $4s$ The centraliser of an element of XXIII has order 32 and acts on the fixed space f. By Table 4 only 16 elements of the centraliser fix both p in f. Hence there are elements of the centraliser interchanging the $2p$ in f. This fact together with the actions of the transvections centred on f shows that *the orbits for* XXIII *are* $2g$, $1t$, $4c$.

TABLE 6. – *The conjugacy classes of* $\mathcal{M}_6^{(2)}$ *of order* 1920 *and of* $\mathcal{M}_6^{(2)+}$ *of order* 960.

Class	Size	Period	Power types		Class in $\mathcal{O}_6^{(2)}$	Fixed space L in S
			2nd	3rd		
I	1	1			I	S
I j	10	2			III	ψ
I h	5	2			XIII	χ
III	60	2			III	ψ
III m	60	4	I h		XV	g
III p	120	4	I j		XI	t
IV	80	3			IV	$\varkappa$
IV p	80	6	IV	I j	IX	t
IV m *	160	6	IV	I h	XIV	g
XII *	384	5			XII	c
II	20	2			II	J
II t	60	2			V	e
II c	60	4	I j		VII	f
II s	20	4	I j		XXIII	h
VI	160	6	IV	II	VI	j
VI [—1]	160	12	IV p	II s	XXIV	m
VII	240	4	III		VII	h
VII [—1]	240	8	III m		XXV	m

The centraliser of a member $\boldsymbol{A}$ of IX contains elements of XIX since the squares of XIX are in IX, and these elements must interchange the 2 p of the fixed t of $\boldsymbol{A}$ since their only fixed point is the m of t. Hence *the orbits for* IX *are* 1 m, 2 p. The centraliser of a member $\boldsymbol{B}$ of XXXVI has order 32 and so cannot act transitively on the 3 m of the fixed space g. Since elements of LV fix only one m each and have their squares in XXXVI *the orbits for* XXXVI *are* 1 m, 2 m. We shall call the m fixed by the centraliser the *focus* of $\boldsymbol{B}$.

The centraliser of a member of V fixes the focus and acts transitively on the other 3 p in the fixed space e (**10**, p. 64), since this e is the axis of (**10**). The 3 t through the focus are thus permuted transitively since each contains one non-focal p. The third point on these t is an m. Hence the 3 m of e are permuted transitively. Thus *the orbits for* V *are* 1 g, 3 t *through the focus*, 3 t *not through the focus* *The orbits for* XI *must be* 1 m, 1 p *(focus)*, 1 p *(non-focus)*.

For classes with fixed space a point there is one orbit of length 1: this orbit is the single [—1] or empty subspace. A class of $\mathcal{O}_6^{(1)}$ with empty fixed spaces gives rise to one class of $\mathcal{M}_8^{(1)}$ by Theorem 2; there are no [—2]!

Theorem 2 and its Corollary give the classes of $\mathcal{M}_8^{(1)}$ with their sizes and types of fixed spaces in [7]. This information forms part of Table 7, where classes are labelled as follows. If a class of $\mathcal{O}_6^{(1)}$ has numerical label K the class of $\mathcal{M}_8^{(1)}$ containing it is labelled K; by § 5.6 it is in class K of $\mathcal{O}_8^{(1)}$. The other classes of $\mathcal{M}_8^{(1)}$ arising

TABLE 7. – *The conjugacy classes of* $\mathcal{M}_8^{(1)}$ *of order* 2580480 *and of* $\mathcal{M}_8^{(1)+}$ *of order* 1290240.

Class	Size	Period	Power types 2nd	Power types 3rd	Power types 5th	Class in $O_8^{(1)}$	Fixed space
I	1	1				I	[7]
I J	28	2				III	T
I F	35	2				XIII	G
III	840	2				III	T
III e	2520	2				VIII	γ
III h	840	4	I F			XV	χ
III f	2520	4	I F			XXXVI	φ
III j	6720	4	I J			XI	ψ
IV	448	3				IV	S
IV h	2240	6	IV	I F		XIV	χ
IV j	4480	6	IV	I J		IX	ψ
IX	26880	6	IV	III		IX	ψ
IX m	26880	12	IV h	III h		XXXII	g
IX p	53760	12	IV j	III j		XXVII	t
X	17920	3				X	$\varkappa$
X m *	35840	6	X	I F		XXXI	g
X p	17920	6	X	I J		XXVI	t
XI	40320	4	III			XI	ψ
XI m	40320	4	III e			XXVIII	g
XI p_1	40320	8	III h			XXXV	t
XI p_2	40320	8	III f			XXXVII	t
XII	21504	5				XII	λ
XII p	64512	10	XII		I J	XXIX	t
XIII	420	2				XIII	G
XIII d *	840	2				LXI	ω
XIII e	420	2				VIII	γ
XIII f	5040	4	I F			XXXVI	φ
XXX *	172032	15		XII	IV	XXX	c
XXXIV *	368640	7				XXXIV	c
XXXVI	20160	4	XIII			XXXVI	φ
XXXVI m_1	20160	4	XIII e			XXVIII	g
XXXVI m_2 *	40320	4	XIII d			LXII	g
XXXIX	53760	6	X	XIII		XXXIX	$\varkappa$
XXXIX m *	107520	6	X	XIII d		LXIII	g
XXXIX p	53760	6	X	XIII e		XXXIII	t
II	56	2				II	P
II ψ	840	2				V	E
II $\varkappa$	560	4	I J			XXIII	F
II λ	336	4	I J			VII	H
V	3360	2				V	E
V g	3360	4	I J			LIII	d
V t_1	10080	4	I F			XXI	e
V t_2	10080	4	I J			XVIII	e
VI	8960	6	IV	II		VI	J
VI s	8960	12	IV j	II $\varkappa$		XXIV	h

Table 7 (*continued*).

Class	Size	Period	Power types			Class in $\mathcal{O}_8^{(1)}$	Fixed space
			2nd	3rd	5th		
VI t	26880	6	IV	II ψ		XVI	e
VI c	26880	12	IV j	II λ		XIX	f
VII	3360	4	III			VII	H
VII t	10080	4	III			XVIII	e
VII s	13440	8	III h			XXV	h
XVII	35840	6	X	II		XVII	j
XVII [−1]	35840	12	X p	II $\varkappa$		XLVI	m
XIX	107520	12	IX	VII		XIX	f
XIX [−1]	107520	24	IX m	VII s		XLVII	m
XX	129024	10	XII		II	XX	j
XX [−1]	129024	20	XII p		II λ	XLIV	m
XXII	107520	6	X	V		XXII	j
XXII [−1]	107520	12	X p	V g		LIV	m
XXIII	10080	4	III			XXIII	F
XXIII t	10080	4	III			XVIII	e
XXIII g	20160	4	III			LIII	d
XXIII c	40320	8	III f			LV	f
LV	161280	8	XXXVI			LV	f
LV [−1]	161280	8	XXXVI m_1			XLVIII	m

from K in $\mathcal{O}_6^{(1)}$ are labelled Ka where a is the letter type of the subspaces of the associated orbit: for the three cases where there are two orbits of the same kind of subspace a suffix 1 is added to indicate that the subspaces of the orbit contain the focus, and a suffix 2 for the other orbit. The other information in Table 7 for class K of $\mathcal{M}_8^{(1)}$ comes directly from the corresponding information in Table 1.

The other entries in Table 7 are compiled in order as follows. For a class Ka of $\mathcal{M}_8^{(1)}$ one finds, using Tables 2, 4 of (**13**), the possible classes of $\mathcal{O}_8^{(1)}$ in which it can lie, and then uses Table 3 of (**18**) to find their corresponding periods and power types in $\mathcal{O}_8^{(1)}$. Then one uses Table 1 and the information in § 3.5 to give possible periods for Ka and its possible power types in $\mathcal{M}_8^{(1)}$; these possible power type classes occur earlier in Table 7 and so, we may assume, have their entries complete. Making these two strands of information compatible uniquely gives the entries for all Ka except XI p_1, XI p_2, XXXVI m_1, XXXVI m_2, XXXIX m, V t_1 V t_2 and the entry XXXVI m_1 for the squares of LV[−1]. Suppose that, apart from the asterisks, Table 7 is otherwise complete. We tabulate for the 7 exceptional classes the ambuguities with which our procedure leaves us.

Class in $\mathcal{M}_8^{(1)}$	Possible classes (and their corresponding squares) in $\mathcal{O}_8^{(1)}$
XI p_1, XI p_2	XXXV (XV), XXXVII (XXXVI)
XXXVI m_1, XXXVI m_2	XXVIII (VIII), LXII (LXI)
XXXIX m	XXXI (X), LXIII (X)
V t_1, V t_2	XVIII (III), XXI (XIII)

$\mathcal{M}_8^{(1)}$ has 35 840 members in XXXI in $\mathcal{O}_8^{(1)}$ (**13**, p. 68), and these, by the information already at hand in Table 7, are all in X m. Hence XXXIX m is in LXIII. There are 60 480, 40 320 members of $\mathcal{M}_8^{(1)}$ in XXVIII, LXII of $\mathcal{O}_8^{(1)}$ respectively. 40 320 of those in XXVIII are in XI m so, by a consideration of sizes, XXXVI m_1, XXXVI m_2 are in XXVIII, LXII respectively.

An $(\boldsymbol{\alpha}, \boldsymbol{a})$ of V t_1 has its α polar to the focus p of $\boldsymbol{a}$. In view of the discussion in § 3.5 $(\boldsymbol{\alpha}, \boldsymbol{a})^2$ is in I F or I J. But $\{p\}$ fixes α and commutes with $\boldsymbol{a}$ so, by (4),

$$(\boldsymbol{\alpha}, \boldsymbol{a})^2 = \left((\boldsymbol{a} + \boldsymbol{I}_6)\boldsymbol{\alpha}, \boldsymbol{a}^2\right) = \left((\boldsymbol{a}\{p\} + \boldsymbol{I}_6)\boldsymbol{\alpha}, (\boldsymbol{a}\{p\})^2\right) = \left(\boldsymbol{\alpha}, \boldsymbol{a}\{p\}\right)^2.$$

Since $\boldsymbol{a}\{p\}$ is in class XIII of $\mathcal{O}_6^{(1)}$ by § 5.4, $(\boldsymbol{\alpha}, \boldsymbol{a})^2$ must be in I F which is in XIII in $\mathcal{O}_8^{(1)}$. Hence V t_1 is in XXI. The 10080 members of $\mathcal{M}_8^{(1)}$ in that class (**13**, p. 68) are thus accounted for, so V t_2 is in XVIII. An $(\boldsymbol{\alpha}, \boldsymbol{a})$ of XI p_1 has its α polar to the focus p of $\boldsymbol{a}$ and, again, $(\boldsymbol{\alpha}, \boldsymbol{a})^2 = \left((\boldsymbol{a}\{p\} + \boldsymbol{I}_6)\boldsymbol{\alpha}, (\boldsymbol{a}\{p\})^2\right)$. The fixed space of $\boldsymbol{a}\{p\}$ is (§ 5.4) an h which, by Lemma 4, is polar to $(\boldsymbol{a}\{p\} + \boldsymbol{I}_6)\boldsymbol{\alpha}$. Hence, by § 2.4, the fixed space of $(\boldsymbol{\alpha}, \boldsymbol{a})^2$ contains the χ joining h to the point m_0 stabilised by $\mathcal{M}_8^{(1)}$. Thus XI p_1 must, by § 3.5, have its squares in III h which is in XV in $\mathcal{O}_8^{(1)}$. Consequently XI p_1 is in XXXV. Similarly XI p_2 has its squares in III f and is in XXXVII. We may now complete the information in Table 7 for $\mathcal{M}_8^{(1)}$.

6.2. – A similar discussion, using products of transvections in $\mathcal{O}_6^{(1)+}$, yields the classes of $\mathcal{M}_8^{(1)+}$. We present a more speedy alternative procedure. An even class of $\mathcal{M}_8^{(1)}$ either forms a single class of $\mathcal{M}_8^{(1)+}$ or is the union of two classes of $\mathcal{M}_8^{(1)+}$ of the same size. The former possibility occurs if and only if the centralisers in $\mathcal{M}_8^{(1)}$ of elements of this class of $\mathcal{M}_8^{(1)}$ contain odd elements. If an element $\boldsymbol{A}$ in an even class Ka of $\mathcal{M}_8^{(1)}$ has p in its fixed space then, by Lemma 5, these p are polar to the m_0 stabilised by $\mathcal{M}_8^{(1)}$. Consequently the centraliser in $\mathcal{M}_8^{(1)}$ of $\boldsymbol{A}$ contains the associated transvections which are odd elements, and so Ka is a single class of $\mathcal{M}_8^{(1)+}$.

If an even class K of $\mathcal{O}_6^{(1)}$ splits in $\mathcal{O}_6^{(1)+}$ then, by Theorem 2, the corresponding classes of $\mathcal{M}_8^{(1)}$ split in $\mathcal{M}_8^{(1)+}$: the centralisers in $\mathcal{O}_6^{(1)}$ and $\mathcal{O}_6^{(1)+}$ of a member of K coincide and have the same orbits in the fixed space. Further, by Theorem 2, if an even class K of $\mathcal{O}_6^{(1)}$ does not split in $\mathcal{O}_6^{(1)+}$ then the class K of $\mathcal{M}_8^{(1)}$ does not split in $\mathcal{M}_8^{(1)+}$, nor does any Ka corresponding to an orbit of length one. Using these criteria we deduce from Tables 1, 7 that classes XXX, XXXIV of $\mathcal{M}_8^{(1)}$ split in $\mathcal{M}_8^{(1)+}$, and that the only other classes that can split are X m, XIII d, XXXVI m_2, XXXIX m. These classes lie in XXXI, LXI, LXII, LXIII respectively in $\mathcal{O}_8^{(1)}$ and, by Table 7, are the only classes of $\mathcal{M}_8^{(1)}$ so to do. Further (**7**, p. 522) XXXI, LXI, LXII, LXIII all split in $\mathcal{O}_8^{(1)+}$, and the resulting classes all contain members of $\mathcal{M}_8^{(1)+}$ (**7**, p. 523). Thus X m, XIII d, XXXVI m_2, XXXIX m split in $\mathcal{M}_8^{(1)+}$. Alternatively we may use the result that the number of splitting classes is the excess of the number of even classes over the number of odd classes (**1**, p. 338; Burnside gives an elementary ofpro of in Note E on p. 472), and this is $35 - 29 = 6$.

We may infer the orbits under centralisers in $\mathcal{O}_8^{(1)+}$. For X there are two orbits of 1 m, similarly for XXXIX. For XXXVI there are three orbits of 1 m, and for XIII two orbits of 1 d in the fixed φ. This last fact accords with the fixing by $\mathcal{O}_6^{(1)+}$ of the 2 families of d on a Klein quadric, and the corresponding splitting of XIII d accords with the fixing by $\mathcal{O}_8^{(1)+}$ of the 2 families of ω on a Study quadric. All other orbits are as for $\mathcal{O}_6^{(1)}$.

6.3. – The same techniques yield the classes of $\mathcal{M}_6^{(i)}$ from those of $\mathcal{O}_4^{(i)}$ described in Table 3. We omit the details and give $\mathcal{M}_6^{(1)}$, $\mathcal{M}_6^{(2)}$ in Tables 5, 6; the class labelling indicates the corresponding orbits of the centralisers of $\mathcal{O}_4^{(i)}$. To obtain periods and power types in $\mathcal{M}_6^{(i)}$ and splitting in $\mathcal{M}_6^{(i)+}$ we make repeated use of similar information in Tables 1, 2 for $\mathcal{O}_6^{(i)}$. The last column of Table 5 is readily compiled. One recalls that the point m_0 stabilised by $\mathcal{M}_6^{(1)}$ is a vertex of the quadric in a fixed space of a class of type Ka. $\mathcal{O}_6^{(1)}$ is transitive on the m of C so 36 of the 108 members of $\mathcal{M}_6^{(1)}$ in XXXVI in $\mathcal{O}_6^{(1)}$ have m_0 for focus; these must be in III d.

If we regard $\mathcal{O}_6^{(1)}$, $\mathcal{O}_6^{(2)}$ as the pointwise stabilisers in $\mathcal{O}_8^{(1)}$ of c, s respectively (§ 5.3), then $\mathcal{M}_6^{(1)}$, $\mathcal{M}_6^{(2)}$ are the pointwise stabilisers in $\mathcal{O}_8^{(1)}$ of an f and an h respectively. We may interpret the 6th columns of Tables 5, 6 as giving their distribution in $\mathcal{O}_8^{(1)}$.

Classes of $\mathcal{M}_8^{(1)}$ and $\mathcal{M}_6^{(i)}$ with the same label lie in the same class of $\mathcal{O}_8^{(1)}$, and are related to the containment of $\mathcal{O}_4^{(i)}$ by $\mathcal{O}_6^{(1)}$. A few classes of $\mathcal{M}_6^{(1)}$, $\mathcal{M}_6^{(2)}$ with the same label are not in the same class of $\mathcal{O}_8^{(1)}$: the 6th columns of Tables 5, 6 ensure no confusion can arise.

6.4. – $\mathcal{O}_2^{(1)}$ and $\mathcal{O}_2^{(2)}$ are described in § 5.3. For $\mathcal{M}_4^{(1)}$ we obtain the classification below.

Class number	Class of $\mathcal{M}_4^{(1)}$	Size	Period	Class in $\mathcal{O}_4^{(1)}$	Fixed space L in $\varkappa$	Primes of L through m_0
1	I	1	1	I	$\varkappa$	$2j$, $5f$
1	I p	1	2	III	t	m_0
3*	I m*	2	2	XIII	g	m_0
4	II	2	2	II	j	$1t$, $2c$
5	II [−1]	2	4	XXIII	m_0	—

The first column is added in anticipation of future use. Having more than one involution $\mathcal{M}_4^{(1)}$ is a copy of the dihedral group D_8. $\mathcal{M}_4^{(1)+}$ is a Klein 4-group. The splitting of I m* accords with the 2 fixed g belonging one to each regulus in $\varkappa$.

$\mathcal{O}_4^{(1)}$ is Σ_5 on the 5 m of its λ, so $\mathcal{M}_4^{(2)}$ is a Σ_4. We obtain the following tabulation: cycle types are inferred immediately from those of $\mathcal{O}_4^{(2)}$ in Table 3.

Cycle type	Class of $\mathcal{M}_4^{(2)}$	Size	Period	Class in $\mathcal{O}_4^{(2)}$	Fixed space L in λ	Primes of L through m_0
1^4	I	1	1	I	λ	$6j$, $1h$
2^2	I p	3	2	III	t	m_0
13	IV *	6	3	IV	c	m_0
$1^2 2$	II	6	2	II	j	$1t$, $2c$
4	II [−1]	6	4	VII	m_0	—

TABLE 8

(*a*) *The irreducible characters of* $\mathcal{M}_6^{(1)}$.

Class				W_2	W_3	W_1						W_4
I	1	1	4	4	4	2	6	6	12	9	9	18
I j	1	1	4	4	4	2	2	2	4	−3	−3	−6
I f	1	1	4	4	4	2	−2	−2	−4	1	1	2
III	1	1	·	·	·	−2	2	−2	·	1	1	−2
III m	1	1	·	·	·	−2	−2	2	·	1	1	−2
III p	1	1	·	·	·	−2	·	·	·	−1	−1	2
IV	1	1	1	−2	−2	2	3	3	−3	·	·	·
IV p	1	1	1	−2	−2	2	−1	−1	1	·	·	·
X *	1	1	−2	1	1	2	·	·	·	·	·	·
XIII *	1	−1	·	2	−2	·	·	·	·	3	−3	·
XIII m*	1	−1	·	2	−2	·	·	·	·	−1	1	·
XXXIX *	1	−1	·	−1	1	·	·	·	·	·	·	·
II	1	−1	2	·	·	·	4	−2	2	3	3	·
II t	1	−1	2	·	·	·	·	2	2	−1	−1	·
II c	1	−1	2	·	·	·	−2	·	−2	1	1	·
II s	1	−1	2	·	·	·	2	−4	−2	−3	−3	·
VI	1	−1	−1	·	·	·	1	1	−1	·	·	·
VI [−1]	1	−1	−1	·	·	·	−1	−1	1	·	·	·
XXIII	1	1	·	·	·	·	·	·	·	1	−1	·
XXIII [−1]	1	1	·	·	·	·	·	·	·	−1	1	·

One of each associated pair is given.

(*b*) *The splitting in* $\mathcal{M}_6^{(1)+}$ *of the self-associated characters of* $\mathcal{M}_6^{(1)}$.

Class	W_1 (i)	W_1 (ii)	W_2 (i)	W_2 (ii)	W_3 (i)	W_3 (ii)	W_4 (i)	W_4 (ii)
X (i)	1	1	−1	2	2	−1	·	·
X (ii)	1	1	2	−1	−1	2	·	·
XIII (i)	−1	1	·	2	−2	·	3	−3
XIII (ii)	1	−1	2	·	·	−2	−3	3
XIII m (i)	−1	1	·	2	−2	·	−1	1
XIII m (ii)	1	−1	2	·	·	−2	1	−1
XXXIX (i)	1	−1	−1	·	·	1	·	·
XXXIX (ii)	−1	1	·	−1	1	·	·	·

See " 6.6 for reading off full table for $\mathcal{M}_6^{(1)+}$.

TABLE 9

(*a*) *The irreducible characters of* $\mathcal{M}_6^{(2)}$.

Class				Y_1			Y_2			
I	1	4	5	6	5	15	10	10	20	10
I *j*	1	4	5	6	1	3	2	−2	−4	−2
I *h*	1	4	5	6	−3	−9	−6	2	4	2
III	1	·	1	−2	1	−1	2	2	·	−2
III *m*	1	·	1	−2	1	−1	2	−2	·	2
III *p*	1	·	1	−2	−1	1	−2	·	·	·
IV	1	1	−1	·	2	·	−2	1	−1	1
IV *p*	1	1	−1	·	−2	·	2	1	−1	1
IV *m* *	1	1	−1	·	·	·	·	−1	1	−1
XII *	1	−1	·	1	·	·	·	·	·	·
II	1	2	1	·	3	3	·	4	2	−2
II *t*	1	2	1	·	−1	−1	·	·	2	2
II *c*	1	2	1	·	1	1	·	−2	−2	·
II *s*	1	2	1	·	−3	−3	·	2	−2	−4
VI	1	−1	1	·	·	·	·	1	−1	1
VI [−1]	1	−1	1	·	·	·	·	−1	1	−1
VII	1	·	−1	·	1	−1	·	·	·	·
VII [−1]	1	·	−1	·	−1	1	·	·	·	·

One of each associated pair is given.

(*b*) *The splitting in* $\mathcal{M}_6^{(2)+}$ *of the self-associated characters of* $\mathcal{M}_6^{(2)}$.

Class	Y_1 (i)	Y_1 (ii)	Y_2 (i)	Y_2 (ii)
IV *m* (i)	·	·	$i\sqrt{3}$	$-i\sqrt{3}$
IV *m* (ii)	·	·	$-i\sqrt{3}$	$i\sqrt{3}$
XII (i)	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$	·	·
XII (ii)	$\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$	·	·

See § 6.6 for reading off full table for $\mathcal{M}_6^{(2)+}$.

6.5. – The usual characters of $\mathcal{M}_4^{(1)}$, $\mathcal{M}_4^{(2)}$, given explicitly in (**20**, pp. 265, 273), may be confirmed by the methods of Chapter 4.

The characters of the $\mathcal{M}_6$ may then be written down using Theorem 3. The values of the $n_{ij}(\boldsymbol{a})$ required in formulae (10), (11) are given in the last columns of the tables for the $\mathcal{T}_4$ and $\mathcal{M}_4$ (§§ 5.5, 6.4). Further, since cycle types are given in Table 3 the required characters of the $\mathcal{O}_4$ may be read off from (**20**, pp. 265, 275). Then, we may proceed to the characters of $\mathcal{M}_8^{(1)}$. The $n_{ij}(\boldsymbol{a})$ are now given by the last columns of Tables 4, 5, and, since cycle types are given in Tables 1, 4, the characters of $\mathcal{T}_6^{(1)}$ and $\mathcal{O}_6^{(1)}$ may be read off those of Σ_6 and Σ_8, which are conveniently tabulated in (**20**, pp. 266, 267). The irreducible characters of $\mathcal{M}_6^{(1)}$, $\mathcal{M}_6^{(2)}$, $\mathcal{M}_8^{(1)}$ are presented in Tables 8, 9, 10.

$\mathcal{M}_8^{(1)}$ has 29 pairs of associated characters and we give one from each pair; the other is obtained by changing the signs of the entries in the odd classes. $\mathcal{M}_8^{(1)}$ has also 6 self-associated characters which vanish on the odd classes, and we label these. We present other character tables similarly.

6.6. – It is a straightforward matter to repeat the process and obtain the characters of $\mathcal{M}_6^{(1)+}$, $\mathcal{M}_6^{(2)+}$, $\mathcal{M}_8^{(1)+}$. Those characters of alternating groups that are required may be found from (**20**, p. 272). In accordance with general theory (**1**, Note E) 29 of the irreducible characters of $\mathcal{M}_8^{(1)+}$ are the restrictions of the 29 pairs of associated pairs of characters of $\mathcal{M}_8^{(1)}$. Further, each self-associated character X of $\mathcal{M}_8^{(1)}$ is, on restriction to $\mathcal{M}_8^{(1)+}$, the sum of two irreducible characters X(i), X(ii), and the other 12 characters of $\mathcal{M}_8^{(1)+}$ so arise. The values of X(i), X(ii) in a class of $\mathcal{M}_8^{(1)+}$ which is a full class of $\mathcal{M}_8^{(1)}$ are each half the value of X in that class. Hence it is only necessary to present, and in practice calculate, the values of X(i), X(ii) on the other classes of $\mathcal{M}_8^{(1)+}$: if a class K (Ka) of $\mathcal{M}_8^{(1)}$ becomes two of $\mathcal{M}_8^{(1)+}$ we label these K(i), K(ii) (Ka(i), Ka(ii)). The same principles are adopted when giving other character tables.

7. – The classes and characters of the $\mathcal{S}_6$ and $\mathcal{S}_6^+$.

7.1. – $\mathcal{M}_4^{(1)} = \mathcal{A}_2^{(1)}\,\mathcal{O}_2^{(1)}$ must be an $\mathcal{S}_4^{(1)}$. $\mathcal{O}_2^{(1)}$ is the group of a c and the action of $\mathcal{A}_2^{(1)}$ is described in §§ 2.2, 2.5. On composing these actions in $\varkappa$ we find 5 orbits of points under the action of $\mathcal{M}_4^{(1)} = \mathcal{S}_4^{(1)}$. These are the point m_0 stabilised by $\mathcal{S}_4^{(1)}$, the other 4 m in the polar plane f_0 of m_0, the 2 p in f_0, the 4 m off f_0, the 4 p off f_0. $\mathcal{M}_4^{(1)}$ is tabulated in § 6.4. The geometry is so simple that the calculation, by our techniques, of the classes and characters of $\mathcal{S}_6^{(1)}$ and $\mathcal{S}_6^{(1)+}$ is almost trivial. Usually the algebraic construction of the (complex) characters of 2-groups is difficult. Except for the attachment of suffices we label the classes of $\mathcal{S}_6^{(1)}$ by the same principles used for $\mathcal{M}_8^{(1)}$ (see § 6.1). Suffices, usually 0, are attached to those classes corresponding to an orbit of subspaces through m_0 under a centraliser in $\mathcal{S}_4^{(1)}$, and not to classes associated with an orbit of subspaces not through m_0. Where two suffices occur $1f_0$ corresponds to f_0 and $1f_1$ to the other f through m_0.

7.2. – $\mathcal{O}_2^{(2)}$ is Σ_3 acting on an s, so $\mathcal{S}_2^{(2)}$ is the stabiliser in $\mathcal{O}_2^{(2)}$ of a p on s. Thus $\mathcal{S}_4^{(2)} = \mathcal{A}_2^{(2)}\mathcal{S}_2^{(2)}$ is the subgroup $\mathcal{C}_4^{(2)}$ of $\mathcal{M}_4^{(2)}$ fixing a t, call it t_0, through the m_0 stabilised by $\mathcal{M}_4^{(2)}$. The orbits in λ of $\mathcal{S}_4^{(2)}$ are m_0, the 2 p in t_0, the other 4 p in the polar plane h_0 of m_0, the 4 m off h_0, the 4 p off h_0. $\mathcal{S}_4^{(2)}$, being a Sylow 2-subgroup of $\mathcal{M}_4^{(2)} = \Sigma_4$ is a D_8, and so is isomorphic to $\mathcal{S}_4^{(1)}$. $\mathcal{S}_2^{(1)}$ is $\left\langle\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\rangle$, and this group is also an $\mathcal{S}_2^{(2)}$. The map τ given by

$$\tau\colon (\boldsymbol{\alpha}, \boldsymbol{a})^{(1)} \to (\boldsymbol{\alpha}, \boldsymbol{a})^{(2)},$$

Table 10. – (*a*) *The irreducible characters of* $\mathcal{M}_8^{(1)}$; *values on the even classes.*

Class										X_1	X_2				X_3
I	1	7	20	21	28	64	35	14	70	56	90	42	35	35	140	140
I J	1	7	20	21	28	64	35	14	70	56	90	42	−5	−5	−20	−20
I F	1	7	20	21	28	64	35	14	70	56	90	42	3	3	12	12
III	1	3	4	1	4	·	−5	2	2	·	−6	2	7	7	4	4
III e	1	3	4	1	4	·	−5	2	2	·	−6	2	−1	−1	4	4
III h	1	3	4	1	4	·	−5	2	2	·	−6	2	−5	−5	4	4
III f	1	3	4	1	4	·	−5	2	2	·	−6	2	3	3	4	4
III j	1	3	4	1	4	·	−5	2	2	·	−6	2	−1	−1	−4	−4
IV	1	4	5	6	1	4	5	−1	−5	−4	·	−6	5	5	5	−10
IV h	1	4	5	6	1	4	5	−1	−5	−4	·	−6	−3	−3	−3	6
IV j	1	4	5	6	1	4	5	−1	−5	−4	·	−6	1	1	1	−2
IX	1	·	1	−2	1	·	1	−1	−1	·	·	2	1	1	1	−2
IX m	1	·	1	−2	1	·	1	−1	−1	·	·	2	1	1	1	−2
IX p	1	·	1	−2	1	·	1	−1	−1	·	·	2	−1	−1	−1	2
X	1	1	−1	·	1	−2	2	2	1	−1	·	·	2	2	−4	2
X m *	1	1	−1	·	1	−2	2	2	1	−1	·	·	·	·	·	·
X p	1	1	−1	·	1	−2	2	2	1	−1	·	·	−2	−2	4	−2
XI	1	1	·	−1	·	·	−1	·	·	·	2	−2	1	1	·	·
XI m	1	1	·	−1	·	·	−1	·	·	·	2	−2	1	1	·	·
XI p_1	1	1	·	−1	·	·	−1	·	·	·	2	−2	−1	−1	·	·
XI p_2	1	1	·	−1	·	·	−1	·	·	·	2	−2	−1	−1	·	·
XII	1	2	·	1	−2	−1	·	−1	·	1	·	2	·	·	·	·
XII p	1	2	·	1	−2	−1	·	−1	·	1	·	2	·	·	·	·
XIII	1	−1	4	−3	−4	·	3	6	−2	8	−6	−6	11	−5	12	28
XIII d *	1	−1	4	−3	−4	·	3	6	−2	8	−6	−6	3	3	12	12
XIII e	1	−1	4	−3	−4	·	3	6	−2	8	−6	−6	−5	11	12	−4
XIII f	1	−1	4	−3	−4	·	3	6	−2	8	−6	−6	−1	−1	−4	−4
XXX *	1	−1	·	1	1	−1	·	−1	·	1	·	−1	·	·	·	·
XXXIV *	1	·	−1	·	·	1	·	·	·	·	−1	·	·	·	·	·
XXXVI	1	−1	·	1	·	·	−1	2	−2	·	2	2	3	−1	·	4
XXXVI m_1	1	−1	·	1	·	·	−1	2	−2	·	2	2	−1	3	·	—
XXXVI m_2 *	1	−1	·	1	·	·	−1	2	−2	·	2	2	−1	−1	·	
XXXIX	1	−1	1	·	−1	·	·	·	1	−1	·	·	2	−2	·	−2
XXXIX m *	1	−1	1	·	−1	·	·	·	1	−1	·	·	·	·	·	·
XXXIX p	1	−1	1	·	−1	·	·	·	1	−1	·	·	−2	2	·	2

One of each associated pair is given.

X_4	X_5	X_6																
140	70	210	210	420	315	315	630	28	140	252	280	140	448	140	280	252	140	28
−20	−10	−30	−30	−60	−45	−45	−90	4	20	36	40	20	64	20	40	36	20	4
12	6	18	18	36	27	27	54	−4	−20	−36	−40	−20	−64	−20	−40	−36	−20	−4
4	−10	14	−10	4	3	3	−18	8	12	12	−8	8	·	4	−16	·	·	4
4	6	−2	6	4	−5	−5	−2	·	4	4	8	·	·	−4	·	−8	−8	−4
4	14	−10	14	4	−9	−9	6	4	·	·	−16	4	·	8	−8	12	12	8
4	−2	6	−2	4	−1	−1	−10	−4	−8	−8	·	−4	·	·	8	4	4	·
−4	−2	−2	−2	−4	3	3	6	·	·	·	·	·	·	·	·	·	·	·
−10	10	15	15	−15	·	·	·	10	20	·	10	−10	−20	−10	10	·	20	10
6	−6	−9	−9	9	·	·	·	2	4	·	2	−2	−4	−2	2	·	4	2
−2	2	3	3	−3	·	·	·	−2	−4	·	−2	2	4	2	−2	·	−4	−2
−2	2	−1	−1	1	·	·	·	2	·	·	−2	2	·	−2	2	·	·	−2
−2	2	−1	−1	1	·	·	·	−2	·	·	2	−2	·	2	−2	·	·	2
2	−2	1	1	−1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
2	4	·	·	·	·	·	·	1	−1	·	1	2	−2	2	1	·	−1	1
·	·	·	·	·	·	·	·	−1	1	·	−1	−2	2	−2	−1	·	1	−1
−2	−4	·	·	·	·	·	·	1	−1	·	1	2	−2	2	1	·	−1	1
·	−2	·	·	·	−1	−1	2	2	·	·	·	−2	·	·	·	2	−2	·
·	−2	·	·	·	−1	−1	2	−2	·	·	·	2	·	·	·	−2	2	·
·	2	·	·	·	1	1	−2	·	2	−2	·	·	·	2	·	·	·	−2
·	2	·	·	·	1	1	−2	·	−2	2	·	·	·	−2	·	·	·	2
·	·	·	·	·	·	·	·	3	·	−3	·	·	3	·	·	−3	·	3
·	·	·	·	·	·	·	·	−1	·	1	·	·	−1	·	·	1	·	−1
−4	6	−6	−6	−12	27	−21	6	4	−4	12	−8	−12	·	12	8	−12	4	−4
12	6	−6	−6	−12	3	3	6	−4	4	−12	8	12	·	−12	−8	12	−4	4
28	6	−6	−6	−12	−21	27	6	4	−4	12	−8	−12	·	12	8	−12	4	−4
−4	−2	2	2	4	−1	−1	−2	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
−4	−2	−2	2	·	−1	3	−2	·	·	·	·	·	·	·	·	·	·	·
4	−2	−2	2	·	3	−1	−2	·	·	·	·	·	·	·	·	·	·	·
·	2	2	−2	·	−1	−1	2	·	·	·	·	·	·	·	·	·	·	·
2	·	·	·	·	·	·	·	1	−1	·	1	·	·	·	−1	·	1	−1
·	·	·	·	·	·	·	·	−1	1	·	−1	·	·	·	1	·	−1	1
−2	·	·	·	·	·	·	·	1	−1	·	1	·	·	·	−1	·	1	−1

TABLE 10 (continued). – (*b*) *The splitting in* $\mathcal{M}_8^{(1)+}$ *of the self-associated characters of* $\mathcal{M}_8^{(1)}$.

Class	X_1 (i)	X_1 (ii)	X_2 (i)	X_2 (ii)	X_3 (i)	X_3 (ii)	X_4 (i)	X_4 (ii)	X_5 (i)	X_5 (ii)	X_6 (i)	X_6(ii)
X *m* (i)	·	·	·	·	3	−3	−3	3	·	·	·	·
X *m* (ii)	·	·	·	·	−3	3	3	−3	·	·	·	·
XIII *d* (i)	−3	−3	−3	−3	−2	14	−2	14	−5	11	27	−21
XIII *d* (ii)	−3	−3	−3	−3	14	−2	14	−2	11	−5	−21	27
XXX (i)	·	·	$\frac{1}{2}(-1+i\sqrt{15})$	$\frac{1}{2}(-1-i\sqrt{15})$	·	·	·	·	·	·	·	·
XXX (ii)	·	·	$\frac{1}{2}(-1-i\sqrt{15})$	$\frac{1}{2}(-1+i\sqrt{15})$	·	·	·	·	·	·	·	·

§ 6.6 describes how the full table for $\mathcal{M}_8^{(1)+}$ can be read off.

(*c*) *The irreducible characters of* $\mathcal{M}_8^{(1)}$; *values on the odd classes.*

(The characters are in the same order as in the table for the even classes).

Class											X_1	X_2			
II	1	5	10	9	10	16	5	4	10	4	·	·	15	−15	30
II ψ	1	5	10	9	10	16	5	4	10	4	·	·	−1	1	−2
II $\varkappa$	1	5	10	9	10	16	5	4	10	4	·	·	3	−3	6
II λ	1	5	10	9	10	16	5	4	10	4	·	·	−5	5	−10
V	1	1	2	−3	2	·	−3	·	−2	4	·	·	3	−3	6
V *g*	1	1	2	−3	2	·	−3	·	−2	4	·	·	3	−3	6
V t_1	1	1	2	−3	2	·	−3	·	−2	4	·	·	−1	1	−2
V t_2	1	1	2	−3	2	·	−3	·	−2	4	·	·	−1	1	−2
VI	1	2	1	·	1	−2	−1	1	1	−2	·	·	3	−3	−3
VI *s*	1	2	1	·	1	−2	−1	1	1	−2	·	·	−3	3	3
VI *t*	1	2	1	·	1	−2	−1	1	1	−2	·	·	−1	1	1
VI *c*	1	2	1	·	1	−2	−1	1	1	−2	·	·	1	−1	−1
VII	1	3	2	3	−2	·	1	−2	−4	·	·	·	1	−1	2
VII *t*	1	3	2	3	−2	·	1	−2	−4	·	·	·	1	−1	2
VII *s*	1	3	2	3	−2	·	1	−2	−4	·	·	·	−1	1	−2
XVII	1	−1	1	·	1	−2	2	−2	1	1	·	·	·	·	·
XVII [−1]	1	−1	1	·	1	−2	2	−2	1	1	·	·	·	·	·
XIX	1	·	−1	·	1	·	1	1	−1	·	·	·	1	−1	−1
XIX [−1]	1	·	−1	·	1	·	1	1	−1	·	·	·	−1	1	1
XX	1	·	·	−1	·	1	·	−1	·	−1	·	·	·	·	·
XX [−1]	1	·	·	−1	·	1	·	−1	·	−1	·	·	·	·	·
XXII	1	1	−1	·	−1	·	·	·	1	1	·	·	·	·	·
XXII [−1]	1	1	−1	·	−1	·	·	·	1	1	·	·	·	·	·
XXIII	1	−1	2	−1	−2	·	1	2	·	·	·	·	5	3	2
XXIII *t*	1	−1	2	−1	−2	·	1	2	·	·	·	·	−3	−5	2
XXIII *g*	1	−1	2	−1	−2	·	1	2	·	·	·	·	1	−1	2
XXIII *c*	1	−1	2	−1	−2	·	1	2	·	·	·	·	−1	1	−2
LV	1	−1	·	1	·	·	−1	·	·	·	·	·	1	1	·
LV [−1]	1	−1	·	1	·	·	−1	·	·	·	·	·	−1	−1	·

Class	X_1 (i)	X_1 (ii)	X_2 (i)	X_2 (ii)	X_3 (i)	X_3 (ii)	X_4 (i)	X_4 (ii)	X_5 (i)	X_5 (ii)	X_6 (i)	X_6 (i)
XXXIV (i)	$\frac{1}{2}(-1+i\sqrt{7})$	$\frac{1}{2}(-1-i\sqrt{7})$	·	·	·	·	·	·	·	·	·	·
XXXIV (ii)	$\frac{1}{2}(-1-i\sqrt{7})$	$\frac{1}{2}(-1+i\sqrt{7})$	·	·	·	·	·	·	·	·	·	·
XXXVI m_2 (i)	1	1	1	1	−2	2	−2	2	−1	3	−1	3
XXXVI m_2 (ii)	1	1	1	1	2	−2	2	−2	3	−1	3	−1
XXXIX m (i)	·	·	·	·	−1	1	−1	1	2	−2	·	·
XXXIX m (ii)	·	·	·	·	1	−1	1	−1	−2	2	·	·

X_3	X_4	X_5						X_6											
·	·	·	60	−30	30	45	45	·	16	50	54	40	20	16	−10	−20	−36	−40	−14
·	·	·	−4	2	−2	−3	−3	·	·	2	6	8	4	16	6	12	12	8	2
·	·	·	12	−6	6	9	9	·	−4	−14	−18	−16	−8	−16	−2	−4	·	4	2
·	·	·	−20	10	−10	−15	−15	·	4	10	6	·	·	−16	−10	−20	−24	−20	−6
·	·	·	·	6	6	−3	−3	·	4	2	6	−8	·	·	6	−4	·	4	2
·	·	·	·	6	6	−3	−3	·	−4	−2	−6	8	·	·	−6	4	·	−4	−2
·	·	·	·	−2	−2	1	1	·	·	−2	2	·	−4	·	2	4	−4	·	−2
·	·	·	·	−2	−2	1	1	·	·	2	−2	·	4	·	−2	−4	4	·	2
·	·	·	3	3	−3	·	·	·	4	2	·	−2	2	−2	−4	4	·	2	−2
·	·	·	−3	−3	3	·	·	·	2	−2	·	−4	4	2	−2	2	·	−2	−4
·	·	·	−1	−1	1	·	·	·	·	2	·	2	−2	−2	·	·	·	2	2
·	·	·	1	1	−1	·	·	·	−2	−2	·	·	·	2	2	−2	·	−2	·
·	·	·	2	−4	−2	−3	−3	·	6	6	−6	·	−6	·	6	·	6	−6	--6
·	·	·	2	−4	−2	−3	−3	·	−2	−2	2	·	2	·	−2	·	−2	2	2
·	·	·	−2	4	2	3	3	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	1	−1	·	1	2	−2	2	1	·	−1	1
·	·	·	·	·	·	·	·	·	−1	1	·	−1	−2	2	−2	−1	·	1	−1
·	·	·	−1	−1	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	1	1	−1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	1	·	−1	·	·	1	·	·	−1	·	1
·	·	·	·	·	·	·	·	·	−1	·	1	·	·	−1	·	·	1	·	−1
·	·	·	·	·	·	·	·	·	1	−1	·	1	·	·	·	−1	·	1	−1
·	·	·	·	·	·	·	·	·	−1	1	·	−1	·	·	·	1	·	−1	1
·	·	·	−2	·	−2	5	−3	·	2	−2	2	·	−2	·	−2	·	2	−2	2
·	·	·	−2	·	−2	−3	5	·	2	−2	2	·	−2	·	−2	·	2	−2	2
·	·	·	−2	·	−2	1	1	·	−2	2	−2	·	2	·	2	·	−2	2	−2
·	·	·	2	·	2	−1	−1	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	−1	1	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	1	−1	·	·	·	·	·	·	·	·	·	·	·	·

is, by (4), an explicit isomorphism from $\mathcal{S}_4^{(1)}$ to $\mathcal{S}_4^{(2)}$: superfices are attached to distinguish these two groups. If we number a class of $\mathcal{S}_4^{(2)}$ by the same number as its image under τ^{-1}, then, in $\mathcal{S}_4^{(2)}$ classes 1, 2, 3, 4, 5 have for fixed spaces λ, t_0, the other $2t$ in h_0, j, m_0 respectively. Other information may be inferred from Table 3.

A replica of the discussion for $\mathcal{S}_6^{(1)}$ gives the classes and characters of $\mathcal{S}_6^{(2)}$. Apart from the labelling of classes we obtain the same tables as for $\mathcal{S}_6^{(1)}$. This is evidence for an isomorphism. In fact we may give a simple proof that a *Sylow 2-subgroup of the cubic surface group is isomorphic to a Sylow 2-subgroup of* Σ_8. *For suppose* that $\boldsymbol{a}$ is now in $\mathcal{S}_4^{(1)}$, that $\boldsymbol{\alpha}' = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, and that $\boldsymbol{m}_0' = (1, 0, 0, 0)$ is the vector of both m_0 above. Then simple matrix calculations show that

$$\theta: (\boldsymbol{\alpha}, \boldsymbol{a}) \to (\boldsymbol{\alpha} + (\alpha_2 + \alpha_3)\, \boldsymbol{m}_0, \tau(\boldsymbol{a}))$$

is an isomorphism from $\mathcal{S}_6^{(1)}$ to $\mathcal{S}_6^{(2)}$.

This result may be induced from known group-theoretic results. $\mathcal{O}_6^{(1)}$ is a copy of $SO_5(3)$ and so (**2**, pp. 145, 146) its Sylow 2-subgroups are each isomorphic to the Wreath product $D_8 \wr Z_2$, where Z_2 is cyclic of order 2, and so is a Sylow 2-subgroup of Σ_8 (**16**, pp. 81, 82). However this proof gives no information concerning the relation of the classes of $\mathcal{S}_6^{(1)}$ to those of $\mathcal{S}_6^{(2)}$; our θ gives an explicit correspondence.

TABLE 11. – *The classes of the isomorphic groups* $\mathcal{S}_6^{(1)}$, $\mathcal{S}_6^{(2)}$ *of order* 2^7 *and of the isomorphic* $\mathcal{S}_6^{(1)+}$, $\mathcal{S}_6^{(2)+}$

Group as $\mathcal{S}_6^{(1)}$		Size	Period	Group as $\mathcal{S}_6^{(2)}$	
Class in $\mathcal{M}_6^{(1)}$	Class			Class	Class in $\mathcal{M}_6^{(2)}$
I	1	1	1	1	I
I f	1 f_0	1	2	1 h_0	I h
I f	1 f_1 *	4	2	1 j_1 *	I j
I f	1 f	4	2	1 h	I h
I j	1 j_0	2	2	1 j_0	I j
I j	1 j	4	2	1 j	I j
III	2	4	2	2	III
III m	2 m_0	4	4	2 m_0	III m
III p	2 p	8	4	2 p	III p
XIII	3 *	8	2	3 *	III
XIII m	3 m_0 *	8	4	3 m_0 *	III m
XIII m	3 m *	16	4	3 p *	III p
II	4	4	2	4	II
II t	4 t_0	4	2	4 t_0	II t
II t	4 t	8	2	4 t	II t
II c	4 c_0	8	4	4 c_0	II c
II c	4 c	4	4	4 s	II s
II s	4 s	4	4	4 c	II c
XXIII	5	16	4	5	VII
XXIII [—1]	5 [—1]	16	8	5 [—1]	VII [—1]

Moreover, τ takes $S_4^{(1)+}$ to $S_4^{(2)+}$, so θ induces an isomorphism from $S_6^{(1)+}$ to $S_6^{(2)+}$: *the Sylow 2-subgroups of the simple cubic surface group and the alternating group of degree* 8 *are isomorphic.*

TABLE 12

(*a*) *The irreducible characters of* $S_6^{(1)}$ *and* $S_6^{(2)}$.

Class in $S_6^{(1)}$			Z_1			Z_2	Z_3	Z_4					Class in $S_6^{(2)}$
1	1	1	2	1	1	2	4	4	4	4	2	2	1
$1\,f_0$	1	1	2	1	1	2	4	4	−4	−4	2	2	$1\,h_0$
$1\,f_1$ *	1	1	2	1	1	2	·	·	·	·	−2	−2	$1\,j_1$ *
$1\,f$	1	1	2	−1	−1	−2	·	·	2	−2	·	·	$1\,h$
$1\,j_0$	1	1	2	1	1	2	−4	−4	·	·	2	2	$1\,j_0$
$1\,j$	1	1	2	−1	−1	−2	·	·	−2	2	·	·	$1\,j$
2	1	1	−2	1	1	−2	·	·	·	·	2	−2	2
$2\,m_0$	1	1	−2	1	1	−2	·	·	·	·	−2	2	$2\,m_0$
$2\,p$	1	1	−2	−1	−1	2	·	·	·	·	·	·	$2\,p$
3 *	1	−1	·	1	−1	·	2	−2	·	·	·	·	3 *
$3\,m_0$ *	1	−1	·	1	−1	·	−2	2	·	·	·	·	$3\,m_0$ *
$3\,m$ *	1	−1	·	−1	1	·	·	·	·	·	·	·	$3\,p$ *
4	1	1	·	1	1	·	·	·	2	2	2	·	4
$4\,t_0$	1	1	·	1	1	·	·	·	−2	−2	2	·	$4\,t_0$
$4\,t$	1	1	·	−1	−1	·	·	·	·	·	·	−2	$4\,t$
$4\,c_0$	1	1	·	1	1	·	·	·	·	·	−2	·	$4\,c_0$
$4\,c$	1	1	·	−1	−1	·	·	·	2	−2	·	2	$4\,s$
$4\,s$	1	1	·	−1	−1	·	·	·	−2	2	·	2	$4\,c$
5	1	−1	·	1	−1	·	·	·	·	·	·	·	5
5 [−1]	1	−1	·	−1	1	·	·	·	·	·	·	·	5 [−1]

One of each associated pair is given.

(*b*) *The splitting in* $S_6^{(i)+}$ *of the self-associated characters of* $S_6^{(i)}$.

Class in $S_6^{(1)+}$	Z_1(i)	Z_1(ii)	Z_2(i)	Z_2(ii)	Z_3(i)	Z_3(ii)	Z_4(i)	Z_4(ii)	Class in $S_6^{(2)+}$
$1\,f_1$ (i)	1	1	1	1	2	−2	2	−2	$1\,j_1$ (i)
$1\,f_1$ (ii)	1	1	1	1	−2	2	−2	2	$1\,j_1$ (ii)
3 (i)	1	−1	1	−1	2	·	−2	·	3 (i)
3 (ii)	−1	1	−1	1	·	2	·	−2	3 (ii)
$3\,m_0$ (i)	1	−1	1	−1	−2	·	2	·	$3\,m_0$ (i)
$3\,m_0$ (ii)	−1	1	−1	1	·	−2	·	2	$3\,m_0$ (ii)
$3\,m$ (i)	1	−1	−1	1	·	·	·	·	$3\,p$ (i)
$3\,m$ (ii)	−1	1	1	−1	·	·	·	·	$3\,p$ (ii)

See ¶ 6.6 for reading off full table for $S_6^{(i)+}$.

Table 11 gives the classes of the isomorphic $\mathcal{S}_6^{(i)}$. Classes of $\mathcal{S}_6^{(2)}$ are labelled in an analogous fashion to those of $\mathcal{S}_6^{(1)}$: where 2 suffices are required 0 corresponds to an orbit of subspaces through t_0. The pairing of the classes of $\mathcal{S}_6^{(1)}$, $\mathcal{S}_6^{(2)}$ is that induced by θ.

We should perhaps point out that $\mathcal{S}_8^{(1)}$ and $\mathcal{S}_8^{(2)}$ are not isomorphic. Although we shall not pursue it here, a discussion of these groups by our present methods shows that $\mathcal{S}_8^{(1)}$, $\mathcal{S}_8^{(2)}$ have respectively 2304, 3072 elements with period 8.

8. – The classes of $\mathcal{C}_8^{(1)}$ and $\mathcal{C}_8^{(1)+}$.

8.1. – We must, since $\mathcal{C}_8^{(1)} = \mathcal{A}_6^{(1)}\mathcal{T}_6^{(1)}$, discuss orbits under centralisers in $\mathcal{T}_6^{(1)}$. From a class of $\mathcal{T}_6^{(1)}$ we choose an element and find the orbits under its centralizer of the $[r-1]$ of the fixed $[r]$ in C. We use Table 4 and revert to the notation of § 5.4.

The centraliser for class 1^6 is $\mathcal{T}_6^{(1)}$ and this acts transitively on the 15 m and 15 non-kernel p in J_0. Further, it acts transitively on the 6 λ and 10 $\varkappa$ in J_0, and so acts transitively on their polar s and c through p_0. Since $\{p_0\}$ interchanges the 2 points off J_0 on one of these s or c, $\mathcal{T}_6^{(1)}$ acts transitively on the m off J_0 and the p off J_0. Reciprocating we find that *the orbits for* 1^6 *are* J_0, 15 J, 15 F, *all through* p_0; 12 J, 20 F, *not through* p_0.

The kernel p of the fixed space J of a member of $1^4 2\{p_0\}$ is not p_0. There are in J and through p_0, 7 ψ of which one is the intersection of J_0 and J, 4 $\varkappa$, 4 λ (**12**, p. 634), and in J and not through p_0, 8 ψ, 6 $\varkappa$, 2 λ. Consequently the centraliser must have at least 7 orbits: we shall deduce later that there are exactly 7.

All other orbits may be found by the methods used for $\mathcal{O}_6^{(1)}$. It is helpful to consider classes in their related pairs in $\langle\{p_0\}\rangle \times \Sigma_6$, and, of course, we may only use transvections with centres in J_0. Except for classes whose fixed spaces are p_0 we list these orbits: those before the semi-colon are subspaces containing p_0, and those after it are subspaces not containing p_0.

$1^6\{p_0\}$:	15 ψ; 10 $\varkappa$, 6 λ
$1^4 2$:	3 e, 4 j; 3 f, 1 h, 4 j
$1^2 2^2$:	1 f, 1 e, 1 h, 4 j; 2 f, 2 e, 4 j
$1^2 2^2\{p_0\}$:	1 t (through focus), 2 t; 1 g, 2 t (through focus), 1 t
$1^3 3$:	1 j (in J_0), 3 h, 3 j; 2 h, 6 j
$1^3 3\{p_0\}$:	3 t; 3 c, 1 s
123, $1^2 4$, 24:	1 p; 1 p, 1 m
$123\{p_0\}$:	1 t, 1 c, 1 s; 2 t, 2 c
$1^2 4\{p_0\}$:	1 t, 2 s; 2 t, 2 s
$24\{p_0\}$:	1 t, 2 c; 2 g, 2 c
3^2, 6:	1 p; 2m*
15:	1 p; 2p*
2^3:	1 e, 6 f; 2d*, 6 f
$2^3\{p_0\}$:	3 t; 1 g, 3 t.

TABLE 13. – *The conjugacy classes of* $\mathfrak{J}_8^{(1)}$ *of order* 92160 *and of* $\mathfrak{J}_8^{(1)+}$ *of order* 46080.

Even classes

Class	Size	Class in $\mathcal{M}_8^{(1)}$
1^6	1	I
$1^6 J_0$	1	I J
$1^6 J_1$	15	I J
$1^6 J$	12	I J
$1^6 F_0$	15	I F
$1^6 F$	20	I F
$1^4 2$	60	III
$1^4 2\, e_0$	180	III e
$1^4 2\, j_0$	240	III j
$1^4 2\, j$	240	III j
$1^4 2\, f$	180	III f
$1^4 2\, h$	60	III h
$1^2 2^2$	180	III
$1^2 2^2 f_0$	180	III f
$1^2 2^2 f$	360	III f
$1^2 2^2 e_0$	180	III e
$1^2 2^2 e$	360	III e
$1^2 2^2 h_0$	180	III h
$1^2 2^2 j_0$	720	III j
$1^2 2^2 j$	720	III j
$1^3 3$	160	IV
$1^3 3\, j_0$	160	IV j
$1^3 3\, j_1$	480	IV j
$1^3 3\, j$	960	IV j
$1^3 3\, h_0$	480	IV h
$1^3 3\, h$	320	IV h
123	1920	IX
$123\, p_0$	1920	IX p
$123\, p$	1920	IX p
$123\, m$	1920	IX m
3^2	640	X
$3^2 p_0$	640	X p
$3^2 m$ *	1280	X m
$1^2 4$	1440	XI
$1^2 4\, p_0$	1440	XI p_1
$1^2 4\, p$	1440	XI p_2
$1^2 4\, m$	1440	XI m
24	1440	XI
$24\, p_0$	1440	XI p_2
$24\, p$	1440	XI p_1
$24\, m$	1440	XI m
15	2304	XII
$15\, p_0$	2304	XII p
$15\, p$ *	4608	XII p
2^3	60	XIII
$2^3 e_0$	60	XIII e
$2^3 f_0$	360	XIII f
$2^3 f$	360	XIII f
$2^3 d$ *	120	XIII d
6	1920	XXXIX
$6\, p_0$	1920	XXXIX p
$6\, m$ *	3840	XXXIX m

Odd classes

Class	Size	Class in $\mathcal{M}_8^{(1)}$
$1^6\ \{p_0\}$	2	II
$1^6\ \{p_0\}\, \psi_0$	30	II ψ
$1^6\ \{p_0\}\, \varkappa$	20	II $\varkappa$
$1^6\ \{p_0\}\, \lambda$	12	II λ
$1^4 2\ \{p_0\}$	30	II
$1^4 2\ \{p_0\}\, \psi_0$	30	II ψ
$1^4 2\ \{p_0\}\, \psi_1$	180	II ψ
$1^4 2\ \{p_0\}\, \psi$	240	II ψ
$1^4 2\ \{p_0\}\, \varkappa_0$	120	II $\varkappa$
$1^4 2\ \{p_0\}\, \varkappa$	180	II $\varkappa$
$1^4 2\ \{p_0\}\, \lambda_0$	120	II λ
$1^4 2\ \{p_0\}\, \lambda$	60	II λ
$2^3\ \{p_0\}$	120	V
$2^3\ \{p_0\}\, t_0$	360	V t_1
$2^3\ \{p_0\}\, t$	360	V t_2
$2^3\ \{p_0\}\, g$	120	V g
$1^2 2^2 \{p_0\}$	360	V
$1^2 2^2 \{p_0\}\, t'_0$	360	V t_1
$1^2 2^2 \{p_0\}\, t_0$	720	V t_2
$1^2 2^2 \{p_0\}\, t'$	720	V t_1
$1^2 2^2 \{p_0\}\, t$	360	V t_2
$1^2 2^2 \{p_0\}\, g$	360	V g
$1^3 3\ \{p_0\}$	320	VI
$1^3 3\ \{p_0\}\, t_0$	960	VI t
$1^3 3\ \{p_0\}\, c$	960	VI c
$1^3 3\ \{p_0\}\, s$	320	VI s
$123\ \{p_0\}$	960	VI
$123\ \{p_0\}\, t_0$	960	VI t
$123\ \{p_0\}\, t$	1920	VI t
$123\ \{p_0\}\, c_0$	960	VI c
$123\ \{p_0\}\, c$	1920	VI c
$123\ \{p_0\}\, s_0$	960	VI s
$1^2 4\ \{p_0\}$	720	VII
$1^2 4\ \{p_0\}\, t_0$	720	VII t
$1^2 4\ \{p_0\}\, t$	1440	VII t
$1^2 4\ \{p_0\}\, s_0$	1440	VII s
$1^2 4\ \{p_0\}\, s$	1440	VII s
$3^2\ \{p_0\}$	1280	XVII
$3^2\ \{p_0\}\, [-1]$	1280	XVII $[-1]$
$15\ \{p_0\}$	4608	XX
$15\ \{p_0\}\, [-1]$	4608	XX $[-1]$
$6\ \{p_0\}$	3840	XXII
$6\ \{p_0\}\, [-1]$	3840	XXII $[-1]$
$24\ \{p_0\}$	720	XXIII
$24\ \{p_0\}\, t_0$	720	XXIII t
$24\ \{p_0\}\, c_0$	1440	XXIII c
$24\ \{p_0\}\, c$	1440	XXIII c
$24\ \{p_0\}\, g$	1440	XXIII g

Periods, fixed spaces and other information may be read off from Table 7.

A star indicates an orbit which becomes two under the corresponding centraliser in $\mathcal{T}_6^{(1)+}$. Those for 3^2, 6, 2^3 are inferred from the action of centralisers in $\mathcal{O}_6^{(1)+}$ which is described in the last paragraph of § 6.2. The centraliser in $\mathcal{T}_6^{(1)+}$ of an A in class 15 has order 5 and so is $\langle A \rangle$. Consequently we have the star.

$\mathcal{C}_8^{(1)}$ has 52 even and at least 48 odd classes. Since at least 4 classes split in $\mathcal{C}_8^{(1)+}$ the excess of the number of even over odd classes is at least 4 (**1**, Note E). Hence $\mathcal{C}_8^{(1)}$ has 48 odd classes, and the orbits for $1^2 2\{p_0\}$ are as stated. Further the only orbits that split under $\mathcal{T}_6^{(1)+}$ are those starred above, and they correspond to the classes of $\mathcal{C}_8^{(1)}$ which form two classes in $\mathcal{C}_8^{(1)+}$.

We give the classes in Table 13. Except for suffices we label the classes by the principles used for $\mathcal{M}_8^{(1)}$ (see § 6.1). In analogy with $\mathcal{S}_6^{(1)}$ *suffices are only attached to classes associated with an orbit of subspaces through* p_0; where two suffices are used 0 corresponds to the intersection of the fixed space with J_0. Dashes are attached to $1^2 2^2\{p_0\} t_0'$ and $1^2 2^2\{p_0^*\} t'$ to indicate that the subspaces of the associated orbit contain the focus of the fixed space.

8.2. – In conclusion we may remark that our techniques have been applied to give the classes and characters of other groups, including $\mathcal{M}_8^{(2)}$ and $\mathcal{M}_8^{(2)+}$. These, in turn, have been used in a geometrical study and classification of $\mathcal{O}_8^{(2)}$ and $\mathcal{O}_8^{(2)+}$, which it is hoped to present soon. We may mention, too, that at a recent conference in Ganisville, and in a related preprint, J. S. FRAME and A. RUDVALIS have announced some progress in the problem of determining the characters of the orthogonal and symplectic groups over $GF(2)$.

REFERENCES

(**1**) W. BURNSIDE, *Theory of groups of finite order*, 2nd ed., Cambridge, 1911.
(**2**) R. CARTER - P. FONG, *The Sylow 2-subgroups of the finite classical groups*, J. Algebra, **1** (1964), pp. 139-151.
(**3**) G. M. CONWELL, *The 3-space PG(3, 2) and its group*, Annals of Math., (2) **11** (1910), pp. 60-76.
(**4**) L. E. DICKSON, *Linear groups*, Leipzig, 1901.
(**5**) L. E. DICKSON, *Canonical forms of quaternary Abelian substitutions in an arbitrary Galois field*, Trans. Amer. Math. Soc., **2** (1901), pp. 103-138.
(**6**) J. DIEUDONNE, *La géometrie des groupes classiques*, 3rd ed., Berlin, 1971.
(**7**) R. H. DYE, *The simple group FH(8, 2) of order $2^{12}\cdot 3^5\cdot 5^2\cdot 7$ and the associated geometry of triality*, Proc. London Math. Soc., (3) **18** (1968), pp. 521-562.
(**8**) R. H. DYE, *On the transitivity of the orthogonal and symplectic groups in projective space*, Proc. Cambridge Philos. Soc., **68** (1970), pp. 33-43.
(**9**) R. H. DYE, *On the conjugacy classes of involutions of the simple orthogonal groups over perfect fields of characteristic two*, J. Algebra, **18** (1971), pp. 414-425.
(**10**) R. H. DYE, *On the conjugacy classes of involutions of the orthogonal groups over perfect fields of characteristic* 2, Bull. London Math. Soc., **3** (1971), pp. 61-66.

(**11**) W. L. EDGE, *The conjugate classes of the cubic surface group in an orthogonal representation*, Proc. Roy. Soc. London, (A) **233** (1955), pp. 126-46.

(**12**) W. L. EDGE, *Quadrics over* $GF(2)$ *and their relevance for the cubic surface group*, Canadian J. of Math., (3) **10** (1960), pp. 583-603.

(**13**) W. L. EDGE, *An orthogonal group of order* $2^{13}\cdot 3^5\cdot 5^2\cdot 7$, Annali di Mat., (4) **61** (1963), pp. 1-96.

(**14**) J. S. FRAME, *The classes and representations of the groups of* 27 *lines and* 28 *bitangents*, ibid., (4) **32** (1951), pp. 83-119.

(**15**) J. S. FRAME, *The simple group of order* 25920, Duke Math. J., **2** (1936), pp. 477-484.

(**16**) M. HALL, *The theory of groups*, New York, 1959.

(**17**) C. M. HAMILL, *On a finite group of order* 576, Proc. Cambridge Philos. Soc., **44** (1948), pp. 26-36.

(**18**) C. M. HAMILL, *A collineation group of order* $2^{13}\cdot 3^5\cdot 5^2\cdot 7$, Proc. London Math. Soc., (3) **3** (1953), pp. 54-79.

(**19**) D. E. LITTLEWOOD, *Group characters and the structure of groups*, ibid., (2) **39** (1935), pp. 150-199.

(**20**) D. E. LITTLEWOOD, *The theory of groups characters*, Oxford, 1940.

(**21**) H. H. MITCHELL, *Determination of all primitive collineation groups in more than four variables which contain homologies*, Amer. J. of Math., **36** (1914), pp. 1-12.

(**22**) E. J. F. PRIMROSE, *Quadrics in finite geometries*, Proc. Cambridge Philos. Soc., **47** (1951), pp. 299-304.

(**23**) J.-P. SERRE, *Représentations linéaires des groupes finis*, Paris, 1967.

(**24**) J. A. TODD, *On the simple group of order* 25920, Proc. Roy. Soc. London, (A) **189** (1947), pp. 326-358.