Tauberian translation algebras.

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Summary. - *The approach herein investigates a topological algebra setting with translation so that Tauberian results are possible, although not necessarily accompanied by the general analytic function theorems usually associated with the L^1 group algebra. Accordingly, the exposition is preliminary for topological algebra generalizations of the known results in spectral synthesis by Malliavin, Kahane, Katznelson, Varopoulous, and Warner. The Tauberian remarks are anticipatory for an appropriate Malliavin lemma guaranteeing the existence of $a \in L^1$, $b \in L^\infty$, related by specific properties of certain support sets, and such that $\int a\bar{b} \neq 0$; correspondingly, an investigation of S sets will follow.*

Introduction.

In this expository and preliminary paper we extend the translation of convolution property of L^1 (that is, $\tau(f*g) = \tau f*g = f*\tau g$) to more general topological algebras, define the usual L^1 problems in terms of a translational setting, and determine elementary conditions under which the closure of the set of translates of an element is the whole algebra. Applications of this procedure to algebras of distributions and for approximating elements of an algebra by differential polynomials (of a given element) serve as motiviation for such a generalization; along with the observation of the importance in proving TAUBERIAN results of WIENER's theorem on the inversion of absolutely convergent FOURIER series.

In § 1 we define translation algebras and note some of their simpler properties. § 2 is devoted to proving some usual BANACH algebra results in a topological ring setting; and in § 3 we discuss the notion of TAUBERIAN condition and the use of what we define as a WIENER condition in proving our first TAUBERIAN theorem. In this latter section we also see the convenience of a translational setting for TAUBERIAN results, and develop such a setting for the statement of general TAUBERIAN problems in § 4. In the next section we give some examples and applications to differential equations. Finally, in § 6, we prove a general TAUBERIAN theorem and list some natural problems and generalizations.

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§ 1. - Translation Algebras.

Let X be a commutative topological algebra over the complex field $\mathbb{C}$. By a topological algebra we mean a topological vector space (over $\mathbb{C}$) with ring multiplication separately continuous. We do not require continuity because in the situations we investigate our prime need for continuity depends on the existence of a directed system $\{U_\alpha\} \subseteq X$ so that for each $T \in X$

$$U_\alpha * T \to T.$$

Also let X' be the space of continuous linear functionals on X. Consider the following properties:

TR. 1. - There is a system $\{U_\alpha : \alpha \in A$, a directed set$\}$ so that for all $T \in X$, $U_\alpha * T \to T$;

TR. 2. - There is a set of maps $\{\tau_h : h \in H$, an index set$\}$ so that for each $h \in H$

$$\tau_h : X \to X$$

and for all $T, S \in X$,

$$\tau_h(T*S) = T*\tau_h S = \tau_h T*S;$$

TR. 3. - Each τ_h is continuous;

TR. 4. - X' is a complex-valued function space each of whose elements is defined on H and such that for all $S, T \in X$, $\varphi \in X'$,

$$\langle \tau \cdot T, \varphi \rangle \in X' \text{ and}$$

$$\langle S*T, \varphi \rangle = \langle S, \langle \tau \cdot T, \varphi \rangle \rangle.$$

If X satisfies TR. 1–TR. 4 then X is a *translation algebra.* A translation algebra thus depends on H and $\{U_\alpha\}$, and if in a given X, there are a number of $\{U_\alpha\}$ we choose one of the family where A has lowest cardinality.

Now, if X satisfies TR. 1 and TR. 2, ring multiplication is hypocontinuous, and for each h the set $\{\tau_h U_\alpha : \alpha \in A\}$ is contained in a bounded B_h of X, then TR. 3 is satisfied. In fact, given $\{S_\alpha\}$ where $S_\alpha \to 0$. We have

$$\tau_h S_\alpha = \lim_{\beta \in A} \tau_h S_\alpha * U_\beta = \lim_{\beta \in A} S_\alpha * \tau_h U_\beta.$$

Thus by our boundedness condition and the hypocontinuity,

$$\lim_\alpha \tau_h S_\alpha = 0.$$

Actually another criterion, quite analogous to this boundedness condition, is the usual uniform space result. Thus, if X, satisfying TR. 1 and TR. 2 is T_0 (and hence completely regnlar) we consider the corresponding uniform space. Letting $S_\alpha \to 0$ we assume

$$\tau_h S_\alpha * U_\beta \xrightarrow{\alpha} 0 \text{ uniformly in } \beta \text{ or}$$

$$\tau_h S_\alpha * U_\beta \xrightarrow{\beta} \tau_h S_\alpha \text{ uniformly in } \alpha;$$

and by MOORE-SMITH, $\tau_h S_\alpha \xrightarrow{\alpha} 0$. We have not had to assume completeness since, in fact, one of the double limits is known to exist–that is

$$\lim_\beta \lim_\alpha \tau_h S_\alpha * U_\beta = 0.$$

Consider any set I of a commutative topological algebra X. We define the following orthogonal sets:

$$I^\perp = \{\varphi \in X' : \langle T, \varphi \rangle = 0 \text{ for all } T \in I\};$$

$$I^{\perp\perp} = \{T \in X : \langle T, \varphi \rangle = 0 \text{ for all } \varphi \in I^\perp\}.$$

We further define the set of translates of a given $T \in X$ to be all finite linear combinations of the form

$$\Sigma\, c_k \tau_{h_k} T,\ c_k \in \mathbb{C},$$

and we call this set $\mathcal{C}_T$.

PROPOSITION 1. - Let X be a translation algebra.

i. $I \subseteq X$ an ideal implies $\bar{I}$ is an ideal, and $M \subseteq X$ maximal implies that M is closed or $\overline{M} = X$;

ii. $T \in X$ and I a closed ideal containing T implies that $\mathcal{C}_T \subseteq I$;

iii. Let I be a closed set. I is an ideal if and only if I is an invariant subspace (that is, $\tau_h I \subseteq I$ for all $h \in H$);

iv. $T \in X$ implies that $\overline{\mathcal{C}}_T$ is a closed ideal in X.

PROOF. - *i* is obvious in any commutative topological algebra.

ii. For this part we need only assume X is a commutative topological algebra satisfying TR. 1 and TR. 2.

Let $S = \tau_h T$; we show $S \in I$ and this proves it since I is a subspace (thus, $\mathcal{C}_T \subseteq I$ and I closed inplies $\overline{\mathcal{C}}_T \subseteq I$).

$\tau_h U_\alpha * T = U_\alpha * \tau_h T = U_\alpha * S$ and $T \in I$ implies $\tau_h U_\alpha * T \in I$ so that

$$\lim_\alpha \, U_\alpha * S = S \in I$$

since I is closed.

iii. We first show that if I is a closed ideal then I is invariant. In order to do this we only need TR. 1 and TR. 2. Now, for $T \in I$ we have $\tau_h U_\alpha * T \in I$; but

$$\tau_h U_\alpha * T = U_\alpha * \tau_h T \xrightarrow{\alpha} \tau_h T$$

so that since I is closed we have $\tau_h T \in I$.

Conversely, since I is a closed subspace we need only show $S * T \in I$ for $S \in X$, $T \in I$; again, we do not use TR. 3.

Let $\varphi \in I^\perp$ implying $\langle S * T, \varphi \rangle = \langle S, \langle \tau \cdot T, \varphi \rangle \rangle$; but I invariant means that $\tau_h T \in I$ for all $h \in H$.

Thus $\langle \tau \cdot T, \varphi \rangle$ is the 0 element of X' so that $\langle S * T, \varphi \rangle = 0$.

Hence $S * T \in I^{\perp\perp} \subseteq I$ where this last inclusion is obvious by the defini- of orthogonal sets.

iv. Clearly $\overline{\mathcal{C}}_T$ is a subspace since $\mathcal{C}_T$ is.

We show that if $S \varepsilon \overline{\mathcal{C}}_T$ the $\tau_h S \varepsilon \mathcal{C}_S \subseteq \overline{\mathcal{C}}_T$ and this proves it by *iii.*

There is $\{ S_\alpha \} \subseteq \mathcal{C}_T$ such that $S_\alpha \longrightarrow S$.

Thus $\tau_h S_\alpha \xrightarrow{\alpha} \tau_h S$ by continuity of τ_h.

Hence $\tau_h S \varepsilon \overline{\mathcal{C}}_T$ since $\tau_h S_\alpha \varepsilon \mathcal{C}_T$ and $\overline{\mathcal{C}}_T$ is closed. *qed*

Again, with X a commutative topological algebra, we say that $I \subseteq X$ is *idempotent* if $I * I \subseteq I$ and *m-convex* if it is convex and idempotent. Also X is a *commutative locally convex topological algebra* if the given topology on X is locally convex. Finally, X is *locally m-convex* if there is a basis of m-convex sets for the neighborhood system at the origin.

The stipulation of idempotency achieves its greatest force in any study of topological algebras which desires to give a reasonable generalization of BANACH algebra theory. This follows since every complete, locally m-convex topological algebra is a projective limit of BANACH algebras (MICHAEL, p. 17). The best conditions to ensure that a topological algebra be locally m-convex are the following (MICHAEL, p. 10 and p. 15):

1. - A topological algebra is locally m-convex if and only if it is isomorphic (algebraically and topologically) to a subalgebra of a cartesian product of normed algebras;

2. – If X is a locally convex topological algebra such that

i. each m–convex barrel is a neighborhood of 0, and

ii. there is a basis $\{V_\alpha\}$ of X such that for each α and each $T \in X$ there is a constant $\lambda_{\alpha, T}$, for which

$$T * V_\alpha \subseteq \lambda_{\alpha,T} V_\alpha ,$$

then X is locally m–convex.

This latter result shows that the space usually has to be of second category or barrelled. Condition *i* is, of course, the second category or barrel condition and in a very real sense is much more restrictive than condition *ii*. With regard to this latter condition we consider the following usual situation: let X be a commutative topological algebra satisfying properties TR. 2 and TR. 4 and let Y be a topological vector space such that $Y' = X$. If the topology on X is the usual strong topology from Y then the sets

$$N(B, \varepsilon) = \{ T \in X : |\langle T, B\rangle| < \varepsilon \}$$

form a neighborhood system at $0 \in X$ where $\varepsilon > 0$ and B is bounded in Y. We form the balanced hull of each such B, then the convex hull of this, and finally the closure A of this convex hull. The result of course is that A is closed, balanced, bounded, and convex. Clearly, the sets $N(A, \varepsilon)$ form a basis for the above neighborhood system since $N(A, \varepsilon) \subseteq N(B, \varepsilon)$. With this situation, then, we have.

PROPOSITION 2. – Given $A \subseteq Y$ and $\varepsilon > 0$ as above and assume that for each $\varphi \in A$ the set $\{\tau_h T : T \in N(A, \varepsilon),\ h \in H\}$ is bounded on φ; further assume that each closed $C \subseteq Y$ is sequentially closed and sequentially complete (e.g. a complete metric space). Then for all $S \in X$ and for all $N(A, \varepsilon)$ there is λ (depending on A, ε, and such that

$$S * N(A, \varepsilon) \subseteq \lambda N(A, \varepsilon).$$

PROOF. – Since A is closed it is sequentially closed and complete, and because of the pointwise boundedness on A, we can apply the uniform boundedness theorem (e. g. KELLY, p. 105).

Thus for $S \in X$ there is $M > 0$ so that for all $\varphi \in A$

$$|\langle S, \langle \tau \cdot T, \varphi\rangle\rangle| < M.$$

Letting $\lambda = M/\varepsilon$ we have

$$S * N(A, \varepsilon) \subseteq N(A, \lambda\varepsilon) = \lambda N(A, \varepsilon). \qquad qed$$

EXAMPLE. - Clearly, the integrable distributions $\mathfrak{D}'_{L^1}$ (SCHWARTZ, 15, p. 56) on $\mathbb{R}^n$ form a commutative topological algebra.The duality properties of $\mathfrak{D}'_{L^1}$ are as follows: let $\varphi \in \dot{\mathfrak{B}}$ if $\varphi \in C^\infty$ and for each $k = 0, 1, \ldots$

$$\lim_{|x| \to \infty} \varphi^{(k)}(x) = 0;$$

and topologize $\dot{\mathfrak{B}}$ with the semi-norms p_k where

$$p_k(\varphi) = \sup \{ |\varphi^{(k)}(x)| : x \in \mathbb{R}^n \}.$$

Clearly, then, $\dot{\mathfrak{B}}$ is a complete metrizable space and $\dot{\mathfrak{B}}' = \mathfrak{D}'_{L^1}$. Also, the strong dual of $\mathfrak{D}'_{L^1}$ is $\mathfrak{B}$ where $\varphi \in \mathfrak{B}$ if $\varphi \in C^\infty$ and for each k there is $M_k > 0$ such that

$$\sup \{ |\varphi^{(k)}(x)| : x \in \mathbb{R}^n \} < M_k,$$

and where the topology on $\mathfrak{B}$ is that generated by the p_k. A compact convergence type of pseudo-topology can be put on $\mathfrak{B}$ so that $\mathfrak{B}' = \mathfrak{D}'_{L^1}$. $\mathfrak{D}'_{L^1}$ satisfies the conditions of the above proposition since $\dot{\mathfrak{B}}$ is a complete metric space and

$$\langle \tau_h T, \varphi \rangle = \langle T_x, \varphi(x + h) \rangle$$

where the right hand side is clearly a bounded set in $\dot{\mathfrak{B}}_h$ when h varies over H and T goes through $N(A, \varepsilon)$ for some closed, convex bounded, and balanced (and sequentially closed and complete, naturally) $A \subseteq \dot{\mathfrak{B}}$.

Actually the boundednees hypothesis of the above proposition can be weakened slightly if we impose the continuity hypothesis on the τ_h, reduce the convergence criterion on Y to sequential convergence, and take advantage of the commutativity of X. We finally note that essentially the only non-second category spaces which are barrelled are those infinite dimensional ones which have their strongest possible locally convex topologies (KELLY, p. 105).

The point of all the reamarks following our first proposition is to illustrate the restriction of our translation algebras if we were to require that they also be locally m-convex.

§ 2. - Gelfand-Type Results.

In algebra we have FROBENIUS' well kno theorem that any finite dimensional linear division algebra over the real field is isomorphic to the real, complex, or quaternion number systems. The usual generalization which replaces finite dimensionality by topological hypotheses is: every normed field over $\mathbb{C}$ is isometrically isomorphic to $\mathbb{C}$. The first proof of this was given by GELFAND although A. E. TAYLOR had done essentially the same thing in 1938 using a standard LIOUVILLE theorem technique. We give the result for a particular class of locally convex topological algebras.

We let our commutative topological algebra X (over $\mathbb{C}$) be locally convex such that the semi-norms defining the topology are filtrant (that is, for any finite set p_{α_i}, $i=1, \dots n$, there is a q such that $q \geq \Sigma\, p_{\alpha_i}$). Also, recall that for any ideal $I \subseteq X$, X/I is T_0 if and only if I is closed. A family of filtrant semi-norms ensures that the topology on X/I defined by the semi-norms

$$\tilde{P}(\tilde{T}) = \inf\{p(S) : S \in \tilde{T} = T + I\}$$

is the same as the topology induced by the canonical map. Thus, for a closed ideal I in X, X/I is a T_2 locally convex commutative topological algebra, and clearly the canonical map takes maximal ideals into maximal ideals.

Let $\mathfrak{M}(X)$ be the space of continuous non-zero homomorphisms on X (With a similar definition for $\mathfrak{M}(X/I)$), any commutative topological algebra. This space will correspond to the «natural transform theory» on a given X a topic which we shall discuss more fully in § 3.

Because of calculations we shall make, we state the following usual definitions: for T, $S \in X$, let

$$T \circ S = T + S + T * S;$$

the elements T for which there is S such that $T \circ S = 0$ are ***quasi-regular*** and the elements S are *quasi-inverses* of the T's. Further an ideal $I \subseteq X$ is *regular* if there is S such that for all T

$$T * S - T \in I;$$

thus if X has a unit then every ideal is regular.

LEMMA i. - If M is a regular ideal in a commutative ring X, then M is maximal if and only if X/M is a field;

ii. - If X is a commutative ring with unit then $T \in X$ has an inverse if and only if T is contained in no maximal ideal.

i and *ii* are both standard (LOOMIS, pp. 60 and 64).

We note that if X is a commutative topological algebra with continuous multiplication and such that each element T has an inverse T^{-1}, then each T is quasi-regular; further if the operation of inversion is continuous then quasi-inversion is continuous. In fact if $T \in X$ then

$$S = (-T) * (\delta + T)^{-1}$$

is its quasi-inverse; and if $T_\alpha \longrightarrow 0$ then $(\delta + T_\alpha) \longrightarrow \delta$ implying $(\delta + T_\alpha)^{-1} \longrightarrow \delta$ so that $S_\alpha \longrightarrow 0$ since multiplication is continuous.

The first part of the following proposition was proved by ARENS (ARENS, p. 625) in the non commutative case and with continuous inversion.

PROPOSITION 3 - *i.* Let X be commutative locally convex topological algebra and algebraic field over $\mathbb{C}$, and with continuous quasi-inversion; then $X = \{c\delta : c \varepsilon \mathbb{C}\}$;

ii. Let X be a commutative locally convex (with filtrant seminorms) topological algebra over $\mathbb{C}$ with continuous quasi-inversion (on quasi-regular elements) and let M be a closed regular maximal ideal of X; then X/M is algebraically and topologically isomorphic to $\mathbb{C}$.

PROOF. - *i.* Each $T \in X$ is quasi-regular.

Assume there is non-zero $T \in X$ which is not a scalar multiple of δ. By the local convexity there is a continuous linear functional f on X such that $f(T^{-1}) \neq 0$.

Consider $(T - c\delta - \delta)$ and its quasi-inverse

$$U = (\delta + c\delta - T) * (T - c\delta)^{-1};$$

and define

$$h(c) = f([\delta + c\delta - T] * [T - c\delta]^{-1}).$$

h certainly exists for all $c \in \mathbb{C}$ by assumption, and as $c \longrightarrow 0$ then $h(c) \longrightarrow h(0)$; this is obvious for if $c \longrightarrow 0$ then $(T - c\delta - \delta) \longrightarrow (T - \delta)$ so that $(\delta + c\delta - T) * * (T - c\delta)^{-1} \longrightarrow (\delta - T) * T^{-1}$ by the continuity of quasi-inversion.

Now, with h continuous we consider

$$h(\lambda) - h(c) = f[(T - c\delta)^{-1} * (T - \lambda\delta)^{-1}(\lambda - c)] =$$

$$= (\lambda - c) f[(T - c)^{-1} * (T - \lambda\delta)^{-1}],$$

so that $h'(c)$ exists and equals

$$f[(T - c)^{-2}].$$

Thus h is entire and as $c \to \infty$ we have $h(c) \to -f(\delta)$ implying by LIOUVILLE's theorem that $h(c) \equiv -f(\delta)$.

Therefore, for $c = 0$, $f((\delta - T) * T^{-1}) = f(-\delta)$ so that

$$f(T^{-1}) + f(-\delta) = f(-\delta)$$

implying $f(T^{-1}) = 0$, a contraddiction.

ii. By part *i* we have each $\tilde{T}_\alpha \in X/M$ is of the form $\tilde{T}_\alpha = c_\alpha \tilde{\delta}$ where $c_\alpha \in \mathbb{C}$ and $\tilde{\delta}$ is the unit of X/M.

Clearly, we need only show that the map $\tilde{T}_\alpha \rightsquigarrow c_\alpha$ is bicontinuous. If $c_\alpha \to 0$ in $\mathbb{C}$ then $\tilde{T}_\alpha = c_\alpha \tilde{\delta} \to 0$ by the continuity of scalar multiplication.

Conversely, assume that if $\tilde{T}_\alpha \to 0$ then $\{c_\alpha\}$ does not necessarily converge to 0. Thus there is $\gamma > 0$ so that for any α_0 there is $\alpha \geq \alpha_0$ for which $\gamma < |c_\alpha|$.

Now, if $\tilde{p}$ is one of the semi-norms defining the topology on X/M and $\tilde{p}(\tilde{\delta}) = c > 0$ then we let

$$V = \{ \tilde{T} : \tilde{p}(\tilde{T}) \leq c\gamma \}.$$

Thus, since $T_\alpha \to 0$ there is β so that for all $\alpha \geq \beta$

$$\tilde{p}(c_\alpha \tilde{\delta}) \leq c\gamma;$$

that is, $|c_\alpha| \leq \gamma$.

Hence if we take $\alpha_0 = \beta$ we get a contradiction. *qed.*

The hypotheses for our result are automatically satisfied if X/M is the quotient algebra of a complex BANACH algebra. Further, since multiplication is continuous in (commutative) BANACH algebras we have that the continuity of inversion automatically implies the continuity of quasi-inversion. Again, in such algebras all regular maximal ideals are closed; a fact which is true in any commutative ring in which the quasi-regular elements form an open set-such rings are Q_r rings (à la JACOBSON). As is well known, and clear, if quasi-inversion is continuous then the quasi-regular elements form a topological group; and, of course, quasi-inversion is continuous in normed algebras and locally compact rings without divisors of zero. Local compactness is generally not sufficient to ensure the continuity of quasi-inversion (again via JACOBSON), and with regard to local compactness we call to mind (for the sake of perspective) the fundamental result that every T_2 locally compact topological vector space is finite dimensional Euclidean space. Also complete metric spaces do not usually have the continuity of quasi-inversion property. Finally it is easy to see that locally m-convex algebras do have such continuity (e. g. MICHAEL, p. 10).

We also assume all ensuing sets of semi-norms to be filtrant.

THEOREM 1. (GELFAND) - Let X be a commutative locally convex topological algebra over $\mathbb{C}$ and with continuous quasi-inversion. Then every closed regular maximal ideal M is the kernel of some continuous homomorphism X onto $\mathbb{C}$, and vice-versa.

PROOF. - Given M we have by the previous proposition that $X/M = \mathbb{C}$ so that the kernel of this isomorphism is M and the isomorphism is a continuous homomorphism of X onto $\mathbb{C}$.

The converse situation is straightforward and is true for any commutative topological algebra.

Let φ be the given homomorphism; obviously, ker φ is an ideal.

To show maximal let I be an ideal containing ker φ but not equal to it. We show that $I = X$.

Let $T \in I - \ker \varphi$ so that $\varphi(T) \neq 0$.

Consider any $S \in X$ and write it as

$$S = \frac{\varphi(S)}{\varphi(T)} T + U$$

where $\varphi(U) = 0$.

Thus $U \varepsilon \ker \varphi \subseteq I$, $T \in I$ so that $S \in I$; thus $X \subseteq I$, that is $X = I$.

To show regular we note that since f is onto there is $U \in X$ such that $\varphi(U) = 1$.

Hence, given any $T \in X$ we clearly have $U * T - T \varepsilon \ker \varphi$.

Finally, to show the kernel is closed we let T be a limit point of ker φ so that there is a directed system $\{T_\alpha\} \subseteq \ker \varphi$ converging to T.

The continuity of φ and the fact that $\varphi(T_\alpha) = 0$ for all α imply that $\varphi(T) = 0$ so that $T \varepsilon \ker \varphi$. *qed.*

We've presented a general form of some of GELFAND's results for the sake of perspective because so many of the TAUBERIAN properties we shall discuss don't have a BANACH algebra type setting.

§ 3. - **Tauberian Properties.**

In this section we shall prove our first TAUBERIAN result for translation algebras and give some possible ramifications of this. First, however, we review the background for TAUBERIAN theorems from an algebraic point of view; that is, we don't discuss TAUBER's original simple result and the subsequent hard analysis of HARDY, LITTLEWOOD, and WIENER in the first three decades of this century.

We begin by stating KRULL's theorem (which is trivial to prove by a standard ZORN's lemma argument): if X is a ring with unit then every proper left ideal is contained in a maximal ideal. In rings without unit this result can be modified so that with the addition of certain topological and GELFAND type conditions we have the following: every proper closed ideal is contained in a regular maximal ideal. This result is only true in special BANACH algebras and we shall recall the necessary restrictions below. As is well textbooktized, the previous theorem applies to the $L^1(G)$ group algebra, G a locally compact abelian group, and in this setting attains the usual forms originally given by WIENER in the early thirthies (for the case of the real line):

i. If $f \in L^1(G)$ has non-vanishing FOURIER transform then $\overline{\mathcal{C}}_f = L^1$;

ii. If G is not compact, $\alpha \in L^\infty(G)$, f satisfies the hypotheses of i and $f * \alpha \longrightarrow 0$, then for all $g \in L^1(G)$,

$$g * \alpha \longrightarrow 0.$$

To make the previous poragraph more precise we state the usual definitions of GELFAND theory for the specific BANACH algebra setting of the above TAUBERIAN theorems. Let X be a BANACH algebra with second dual X'' and let $\mathfrak{M}_S(X)$ be those elements of $\mathfrak{M}(X)$ which map X onto $\mathbb{C}$. If we restrict the natural map $X \longrightarrow X''$ to $\mathfrak{M}_S$ then we designate the image of X in X'' by $\widehat{X}$. Thus the mapping $T \rightsquigarrow \widehat{T} \in \widehat{X}$, where $\langle \widehat{T}, \varphi \rangle = \langle T, \varphi \rangle$ for all $\varphi \in \mathfrak{M}_S$, defines an algebraic homomorphism of X onto $\widehat{X}$. In the weak topology from $\widehat{X}$ onto $\mathfrak{M}_S$ we know that $\mathfrak{M}_S$ is locally compact, a result that is true for X any normed algebra. Now, if $X \longrightarrow \widehat{X}$ is $1-1$ then X is a *function algebra* and $\widehat{X}$ is its *Gelfand representation* in the sense that X is represented by the algebra $\widehat{X}$ of continuous complex valued functions defined on the locally compact T_2 space $\mathfrak{M}_S$. In terms of the results of § 2 consider $\mathfrak{M}_S$ as a maximal ideal space — that is, its elements are the regular, closed (automatically, since X is a BANACH algebra), maximal ideals of X. We topologize $\mathfrak{M}_S$ in the usual STONE-JACOBSON fashion: thus for any $F \subseteq \mathfrak{M}_S$, define its closure $\bar{F}$ as

$$\bar{F} = \{ M \in M_S : M \supseteq \cap M',\ M' \in F \}.$$

As is well known the weak topology on $\mathfrak{M}_S$ is the same as the STONE-JACOBSON topology if and only if $\mathfrak{M}_S$ in the weak topology is completely regular. As might seem reasonable, the complete regularity guarantees the existence of some form of local identity in $\widehat{X}$. In fact, the existence of the uniform structure which induces the STONE–JACOBSON (hull-kernel) topology on $\mathfrak{M}_S$, along with the general analytic function theorem (for BANACH algebras) mentioned at the end of § 2, is enough to prove the existence of local identities. The need for local identities for TAUBERIAN theorems will become clear in our own TAUBERIAN result. In any case, the result that any proper closed ideal is contained in a regular maximal ideal is true in any semi-simple (commutative) BANACH algebra with the properties:

i. $\mathfrak{M}_S$ is completely regnlar in the weak topology;

ii. The elements $\widehat{T} \in \widehat{X}$ (defined on the locally compact $\mathfrak{M}_S$) with compact support form a dense subset of X.

There is an approach by WILLCOX which takes a slightly different turn. He determines which BANACH algebras have the purely algebraic property that every two sided ideal contained in a regular maximal ideal. He does, in fact, find a large class of such algebras — ones in which $\mathfrak{M}_S$ with the STONE–JACOBSON topology has certain point-set theoretic properties. Instead of copying his results we only note the usual examples which have these properties:

i. The BANACH algebras considered in the previous remarks, and, in particular, $L^1(G)$, G a locally compact abelian group;

ii. Any BANACH algebra in which left and right multiplication are completely continuous; e. g. $L^1(G)$ with G compact;

iii. The group algebra of the direct product of a locally compac tabelian group and a compact group.

Now, the standard BANACH algebra TAUBERIAN result (discussed above) is a generalization of WIENER's original theorem in the usual sense; that is, instead of $L^1(\mathbb{R})$, essentially any BANACH algebra with the uniform structure and denseness properties defined above has the required summability property — expressed, of course, in terms of maximal ideals. WILLCOX, on the other hand, alters the final TAUBERIAN result so as to consider a convenient subclass of BANACH algebras for which the result holds. The standard BANACH algebra TAUBERIAN theorem extends from L^1 to more general structures with similar topological properties. In our approach we extend the translation property of L^1 to many topological algebras which are neither complete nor normed, and investigate TAUBERIAN properties of these algebras.

In any commutative topological algebra X, we restrict the natural map $X \rightarrow X''$ to $\mathfrak{M}(X)$ and designate the image of X under this restriction by $\widehat{X}$.

Each element $\widehat{T} \in \widehat{X}$ defines a map $\mathfrak{M}(X) \longrightarrow \mathbb{C}$ where $\widehat{T}(\varphi) = \varphi(T)$. The natural algebraic operations in $\widehat{X}$ are:

$$(\widehat{T} * \widehat{S})(\varphi) = \varphi(T * S),$$

$$(\widehat{T} + \widehat{S})(\varphi) = \varphi(T) + + \varphi(S),$$

$$(\alpha \widehat{T})(\varphi) = \alpha\varphi(T),$$

where $\alpha \in \mathbb{C}$ and $\varphi \in \mathfrak{M}(X)$. Clearly, then,

i. $\widehat{T} * \widehat{S} = \widehat{T * S}$,

ii. $\widehat{T} = \widehat{S}$ implies $T = S$, for X semi-simple, and for each $\varphi \in \mathfrak{M}(X)$,

iii. $(\widehat{T} * \widehat{S})(\varphi) = \widehat{T}(\varphi)\widehat{S}(\varphi)$.

We say that $\mathfrak{M}(X)$ determines the ***transform map*** of X and that the ***$\mathfrak{M}$-transform*** of $T \in X$ is $\widehat{T}$. Clearly, we use $\mathfrak{M}$ to determine an (integral) transform on X in order to preserve the classical operational property iii.

EXAMPLE. - If $X = L^1(G)$ then the $\mathfrak{M}$-transform is the usual FOURIER transform. As another example consider the continuous homomorphisms on the space $\mathcal{E}'$ of distributions with compact support. $\mathcal{E}'$ is, of course, a com. mutative locally convex algebra taken with the strong topology from $\mathcal{E}$, the space of C^∞ functions with topology defined by the usual uniform convergence (for finite numbers of derivatives) on compact sets (of $\mathbb{R}^n$ or extensions to locally compact spaces). Now, we know that any commutative algebra is a function algebra if and only if it is semi-simple. Of the many trivial characterizations of semi-simplicity it is probably best for our purposes to think of it at follows: for any non-zero element T of the algebra there is $\Phi \in \mathfrak{M}$ such that $\Phi(T) \neq 0$. This, of course, is analogous to the usual T_2- characterization of locally convex spaces in terms of non-zero semi-norms for given points. Getting back to our distibutional example, consider the map $\mathfrak{D}' \longrightarrow \mathfrak{D}'$ (the space of distributions) where

$$T \leadsto T * S$$

for fixed $S \in \mathcal{E}'$. Note that this map, say f_S, commutes with tsanslation. In fact, $\mathcal{E}'$ is algebraically isomorphic to the space of linear transformations, $\mathfrak{D}' \longrightarrow \mathfrak{D}'$, which commute with translation. The case of determining $\mathfrak{M}$ for $\mathcal{E}'$, or any topological algebra, is more difficult than the same problem for

BANACH algebras, where every homomorphism is continuous. If $\varphi \in \mathfrak{M}(\mathcal{E}')$ then for the $T \in \mathcal{E}'$

$$(\delta_j' * T)(\varphi) = T(\varphi)\delta_j'(\varphi),$$

where δ_j is the $j - th$ partial derivative of $\delta \in \mathcal{E}'$; thus, for all $T \in \mathcal{E}'$,

$$T \frac{\partial \varphi}{\partial y_j} = T(\varphi) \left. \frac{\partial \varphi}{\partial x_j} \right|_{x=0}, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Hence,

$$\frac{\partial \varphi(x)}{\partial x_j} = \left[\left. \frac{\partial \varphi(x)}{\partial x_j} \right|_{x-0} \right] \varphi(x),$$

so that

$$\varphi(x) = \exp \{ c_1 x_1 + \ldots + c_n x_n \},$$

where $c_1, \ldots, c_n \in \mathbb{C}$. Conversely, for $T, S \in \mathcal{E}'$

$$\langle T * S, \exp \{ c_1 x_1 + \ldots + c_n x_1 \} \rangle =$$

$$= \langle T, \exp \{ c_1 x_1 + \ldots + c_n x_n \} \rangle \langle S, \exp \{ c_1 x_1 + \ldots + c_n x_n \} \rangle,$$

and therefore each element φ of $\mathfrak{M}(\mathcal{E}')$ has this exponential form for every n-tuple $(c_1, \ldots, c_n) = c \in \mathbb{C}^n$. Thus, given $T \in \mathcal{E}'$, the $\mathfrak{M}$-transform

$$F_T(c) = \langle T_x, \exp \{ c_1 x_1 + \ldots + c_n x_n \} \rangle$$

is the n-dimensional bilateral LAPLACE transform whith the usual convergence in a tube, analyticity, and multiplication properties of LAPLACE transforms. Also, the representation result for the LAPLACE transform of $\mathcal{E}'$ is that this map is an algebraic isomorphism of $\mathcal{E}$ onto the algebra (under ordinary multiplication) of entire functions of exponential type tempered on vertical lines (see e.g., LIONS, BENEDETTO 2, 3 for this result and some topological remarks). It is possible at this point, by standard juggling in the duality game, to define the FOURIER transform not only on $\mathcal{E}'$ but also on the usual space of tempered distributions and hence to develop SCHWARTZ's FOURIER transform theory.

The role of the TAUBERIAN condition, so clear in the first TAUBERIAN results, has evolved almost to the point of obscurity. On the other hand, present day TAUBERIAN theorems achieve a compact form in terms of necessary and sufficient conditions; e.g., if $f \in L^1$ then $\mathcal{C}_f = L$ if and only if the FOURIER transform of f never vanishes. This particular example where the FOURIER transform doesn't vanish and the related theorem concerning the inversion of absolutely convergent FOURIER series are the motivation for the local condition we use to prove our TAUBERIAN result — a condition which we appropriately call the WIENER condition.

Let X be a translation algebra where the index set H is given a T_2 locally compact topology. Denote by $V_\alpha \subseteq H$ the smallest closed set such that for all $\varphi \in X'$ satisfying

$$C_\varphi \cap V_\alpha = \Lambda$$

(C_φ the support of φ), we have $\langle U_\alpha, \varphi \rangle = 0$. Let $T \in X$ have the property that $\widehat{T}$ is never zero.

WIENER CONDITION. - Given V_α. There is $S \in X$ such that for all $\varphi \in \mathfrak{M}(X)$ with $C_\varphi \cap V_\alpha \neq \Lambda$,

$$\frac{1}{T(\varphi)} = S(\varphi).$$

Clearly if X has unit δ then $\delta(\varphi) = 1$ only; and the WIENER condition is automatically satisfied with $S = \delta$.

The importance of just how vital a well defined system of approximate identities is begins to become evident in this condition and will be clear in our TAUBERIAN result. The fact that our translation alegbras have approximate identities permits us to have a purely local WIENER condition.

Classically, TAUBERIAN conditions have taken the form of growth restrictions on coefficients in series. In the original version of WIENER'S theorem (stated at the beginning of this section) the hypothesis that $\alpha \in L^\infty$ is the TAUBERIAN condition. On the other hand, the notion of an inversion criterion (as our WIENER condition) has always played just as vital a role in TAUBERIAN theorems; a fact most clearly observed in BEURLING's uniqueness theorem (*Acta Mathematica*, 1945) of which WIENER's result is a simple corollary. It is standard now, of course, to prove WIENER's inversion of FOURIER series theorem as a corollary of a general analytic function theorem and then to prove WIENER's TAUBERIAN result in terms of this corollary. Our particular approach allows us to consider algebras in which general analytic function theorems may not hold but which do have a local WIENER condition so that TAUBERIAN results can still be proved. Thus, our translation algebras may not even have an inversion of FOURIER series theorem but will have TAUBERIAN properties. In fact —

THEOREM 3. - Let X be a semi-simple translation algebra with locally compact index set H, and let $T \in X$ with the property that $\widehat{T}$ is never 0. If T satisfies the WIENER condition then $\overline{\mathfrak{T}}_T = X$.

PROOF. - It is only necessary to show that each $U_\alpha \in \overline{\mathfrak{T}}_T$; in fact, if $W \in X$ and each $U_\alpha \in \overline{\mathfrak{T}}_T$ then $W * U_\alpha \in \overline{\mathfrak{T}}_T$ so that

$$\lim W * U_\alpha = W \in \overline{\mathfrak{T}}_T.$$

By the WIENER condition there exists $S \in X$ so that for all φ with $C_\varphi \cap V_\alpha \neq \Lambda$ we have

$$1 = T(\varphi) S(\varphi).$$

Thus, for all $\varphi \in \mathfrak{M}$,

$$U_\alpha(\varphi) = T(\varphi) S(\varphi) U_\alpha(\varphi).$$

Hence,

$$\widehat{U}_\alpha(\varphi) = \widehat{T}(\varphi)[\widehat{S}\widehat{U}_\alpha(\varphi)] = \widehat{T}(\varphi) \widehat{S * U}_\alpha(\varphi)$$

so that for all $\varphi \in \mathfrak{M}$

$$U_\alpha(\varphi) = (T * S * U_\alpha)(\varphi).$$

Therefore, $U_\alpha = T*(S*U_\alpha)$, and since $\overline{\mathfrak{T}}_T$ is an ideal we have $U_\alpha \in \overline{\mathfrak{T}}_T$.

REMARK 1. - In this result, the local compactness of H is more a convenience than anything else. For most examples it is a reasonable assumption — in particular, for all algebras of distribution. Also, a weaker WIENER condition would be: given V_α; there is a compact neighborhood C of V_α and an $S \in X$ so that for all $\varphi \in \mathfrak{M}(X)$ with $C_\varphi \subseteq C$, we have

$$1 = T(\varphi) S(\varphi).$$

The theorem would be true for most translation algebras of distributions satisfying this condition since the proof would only be altered by finding a $C^\infty \beta : H \longrightarrow \mathbb{C}$ such that $\beta(V_\alpha) = 1$ and $\beta(H - C) = 0$; then for all $\varphi \in \mathfrak{M}$ we would define $U_\alpha(\varphi) = U_\alpha(\beta\varphi)$ so that

$$U_\alpha(\varphi) = T(\varphi) S(\varphi) U_\alpha(\varphi)$$

since $U_\alpha(\beta\varphi)$ is clearly well defined.

REMARK 2. - The classical converse for THEOREM 2 takes the form: if $\overline{\mathfrak{T}}_T = X$ then $\widehat{T}$ is never zero. This result, though trivial, depends heavily on the FOURIER transform properties of the space usually considered, viz. L^1. Without a WIENER condition such a converse is not to be expected in a general translation algebra; again, with a WIENER condition as in the previous remark we trivially show that $T(\varphi) \neq 0$ for all φ with support contained in C, but it is impossible to prove that $\widehat{T}$ is not zero for any φ. On the other hand, if this latter WIENER condition is changed to read 'for all compact neighborhoods C of V_α' (instead of, 'there is a compact neighborhood C of V_α'), then it follows immediately from this condition that $\widehat{T}$ is never zero. Finally, with the WIENER condition of THEOREM 2, if for at least one α, $V\alpha = H$ then it is again obvious that $\widehat{T}$ is never zero.

REMARK 3. - THEOREM 2 depends heavily on the existence of local identities either in the space itself or in quotient spaces. In the following section we shall write a general TAUBERIAN problem without reference to such indentities. For the moment, though, we examine the possibility of extending BANACH algebra techniques for our purposes. Recall that in BANACH algebras both general analytic function theorems and complete regularity were used to show the existence of some form of indentities. The complete regularity alone guarantees the existence of

$$T : X' \longrightarrow [0,\ 1]$$

which separates closed sets and points, but such a map need not be linear, or if proved to be linear, it need not be a member of X. Also, even if X' is locally convex, the HAHN-BANACH theorem only assures the existence of a continuous linear functional (extended from a subspace $\mathcal{M}_C \subseteq X'$) equal to 1 at a finite number of points of $\mathcal{M}_C$; and a quick look at theproof shows that not much better than this can be accomplished.

Besides topological considerations (as complete regularity) the other major problem in showing the existence of local identities via BANACH algebra techniques is a reasonable extensions of the multiplicative BANACH algebra inequality to more general algebras. As mentioned above, complete spaces are necessary for general analytic function theorems; but even with completeness, it is not necessarily true that

$$|\langle T{*}S,\ \varphi\rangle| \leq |T(\varphi)|\,|S(\varphi)|,\ \ T,\ S \in X,\ \ \varphi \in X',$$

an inequality which is needed in order to prove such theorems.

REMARK 4. - We close this section with a couple of brief comments on algebras X with unit. Consider $\mathcal{E}'$ with unit δ; then the $\mathcal{M}$ (bilateral LAPLACE) transform of δ is the analytic function $\widehat{\delta} \equiv 1$ and so is never zero. On the other hand, it is easy (SCHWARTZ, 15, p. 17) to see that $\overline{\mathcal{T}}_\delta = \mathcal{E}'$. Because of the inherent algebraic structure of $\mathcal{E}'$ there is no need for a WIENER condition to prove this; although it is also a trivial corollary of our first proposition in § 1. If we consider a non-unit element T in our algebra X so that $\overline{\mathcal{T}}_T = X$ then the unit $\delta \in \overline{\mathcal{T}}_T$ so that there is a directed system $U_\alpha \in \mathcal{T}_T$ with the property $U_\alpha \longrightarrow \delta$. When V_α is compact and

$$U_\alpha = \overset{\alpha}{\Sigma}\ c\tau T$$

then the support of each τT is compact. When H has a group structure, for example in the case of distributions, then the fact that $C_{\tau T}$ is compact implies that C_T is compact.

§ 4. - Tauberian Problems.

We are now in a position, after the motivation of the three previous paragraphs, to formulate explicitely the TAUBERIAN problem for topological algebras. Throughout this section, X will be a (complex, for convenience) commutative topological algebra, $\mathfrak{M}(X)$, the set of non-zero homomorphisms $X \to \mathbb{C}$, and $\widehat{X}$, the restriction to $\mathfrak{M}(X)$ of X''. Further, we assume there is a set $\mathcal{T}$ of maps $\tau_h : X \to X$, h in an index set H, such that for any S, $T \in X$ and any $h \in H$,

$$\tau_h(S * T) = \tau_h S * T = S * \tau_h T.$$

The pair $\langle X, \mathcal{T} \rangle$ is a *translation algebra*. We shall discuss some examples in § 5. If for some $T \in X$, $\widehat{T} \in \widehat{X}$ is never zero and $\overline{\mathcal{T}}_T = X$, then $\langle X, \mathcal{T} \rangle$ is a *Tauberian translation algebra*. We note in passing that for algebras of distributions the operation of differentiation satisfies the translation condition. Also, if, given an arbitrary translation algebra the addition of a boundedness or local inversion condition yields the result that $\overline{\mathcal{T}}_T = X$ then these conditions are referred to as *Tauberian* or *Wiener*, respectively.

In order to examine a more general situation, define the *spectrum*

$$Z_T = \{ \varphi \in \mathfrak{M} : T(\varphi) = 0 \}$$

of $T \in X$. $\langle X, \mathcal{T} \rangle$ is a *general Tauberian translation algebra* if there is some $T \in X$ such that for all $S \in X$ satisfying

$$Z_T \subseteq Z_S$$

we can prove $S \in \overline{\mathcal{T}}_T$. We shall discuss such a situation in § 6. This general area has, of course, been well trodden in a BANACH algebra setting.

It is trivial to see that in translation algebras which are fields and for which some $\overline{\mathcal{T}}_T$ is a closed ideal that $\overline{\mathcal{T}}_T = X$ (in fact, the equation $S * T = \delta$ has a solution $S = T^{-1}$ so that $\delta \in \overline{\mathcal{T}}_T$).

Now, there are two classes of translation algebras which are important not only because of their relation with the TAUBERIAN problem bu also because of their connection with other phases of harmonic analysis. First, X is a *spectral algebra* if for each $T \in X$, $\overline{\mathcal{T}}_T$ is the intersection of all maximal ideals containing it; as is well known, L^1 is not a spectral algebra, but is very close to it. Related problems include finding those translation algebras which have elements with this intersection property or, even more, dense subsets of such elements. These spectral algebras are in fact the motivation for studying general TAUBERIAN translation algebras (as we shall see in § 6). Second, we consider those translation algebras in wich each closed ideal is of the form $\bar{\bar{\mathfrak{T}}}_T$. The determination of such algebras is not an easy task, and, in fact, it is not known if even L^1 has this property.

Particular cases of such algebras have been studied although from a BANACH algebra setting as opposed to the translational approach. For example, RUDIN has characterized all closed ideals of the BANACH algebra of continuous complex valued functions on the closed unit disk of $\mathbb{C}$ and analytic in the interior. Further, and with regard to spectral algebras this characterization gives necessary and sufficient conditions (RUDIN, THEOREM 3, p. 433) for any such closed ideal to be intersection of all maximal ideals in the given algebra.

§ 5. - Examples.

In this section we first give examples of various BANACH algebras and algebras of distributions with corresponding natural translation maps and/or approximate identities; and then we look at the TAUBERIAN problem when the set of translation maps is a set (or even algebra) of differential polynomials with constant coefficients. Finally, we consider, in some detail the TAUBERIAN properties of $\mathfrak{D}'_{L^1}$.

A. - Let X be a complex commutative B^* algebra. As is well known (e.g. RICKART) X is a semi-simple and every closed ideal is an intersection of maximal ideals. More important for our purposes is the fact that X has a system of approximate identities (RICKART, p. 245) so that the setting is appropriate for TAUBERIAN considerations if there is a family of translation or differentiation maps defined on an algebra and with values in X.

We now let X be a commutative BANACH algebra of functions defined on an index set H and with values in an algebra with no divisors of zero. If the set of translation maps is defined on an index set H' and if there is $h_0 \in H$ and non-zero $T \in X$ for which $T(h_0) = 0$ then for all $S \in \bar{\bar{\mathfrak{T}}}_T$ $S(h_0) = 0$; clearly, by the pointwise mutiplication for X, we have for each $h \in H'$ that $(\tau_h T)(h_0)\, S(h_0) = T(h_0)\, (\tau_h S\,(h_0)$ so that at the very least $S(h_0) = 0$ or even possibly $(\tau_h T)(h_0) = 0$ for all $h \in H'$. It is, of course, possible to «translationalize» in a natural way many of the standard BANACH algebras (see e.g. RICKART) although there is little point to pushing things too far for the the cases of operator algebras because of the general lack of commutativity. On the other hand a TAUBERIAN investigation of group algebras seems a more reasonable pursuit, and besides the usual results for $L^1(G)$ it is interesting to find TAUBERIAN properties of various subalgebras (e g. $\mathfrak{D}$ SCHWARTZ's test functions), convolution algebras of measures, and BEURLING's weight function spaces L^ω (e.g. LOOMIS, pp. 180–181). This last case leads to the bilateral LAPLACE transform in much the same way as $\mathcal{E}'$ does.

It is, of course possible to find TAUBERIAN results on range spaces (by integral transforms, for example) in terms of domains which do in fact have a TAUBERIAN theorem. Thus, if $\mathcal{F}$ is the FOURIER transform on L^1 then we define the convolution $T*S$ in $\mathcal{F}(L^1)$ as $\mathcal{F}^{-1}(T)*\mathcal{F}^{-1}(S)$.

In the case of algebras of distributions we already know that $\mathcal{T}_\delta = \mathcal{E}'$. Closely related to $\mathcal{E}'$ we have the algebra $\mathcal{H}'$ of analyitic functionals. We let $\mathcal{H}$ be the space of entire functions on $\mathbb{C}^n$ with the usual topology of compact convergence; then $\mathcal{H}(\mathbb{C}^n)$ is a closed subspace of $\mathcal{E}(\mathbb{R}^{2n}$ and its dual is $\mathcal{H}'$. Convolution is then defined in $\mathcal{H}'$ in terms of $\mathcal{E}'$, and the LAPLACE transform $\langle T_s, e^{-sz}\rangle$ (of $T \in \mathcal{H}'$) is an algebraic isomorphism of $\mathcal{H}'$ onto the space of entire function of exponential type. There is, of course, the expected set of translation maps and differentiations on $\mathcal{H}'$.

Of the many other algebras with unit and standard translation and differentiation maps we mention:

i. All distributions on a circle Γ; these form a convenient space to develop the theory or FOURIER series.

ii. All distributions on $\mathbb{R}^{n+1}$ with support in the cone $t \geq 0$, $t^2 - - x_1^2 - \dots - x_n^2 \geq 0$.

iii. All distributions on $\mathbb{R}$ with snpport in the right half line $[0, \infty)$.

This last example is particularly useful for an operational calculus, and has prompted the definition of many similar algebras (e.g. SCHWARTZ, *b*, 28–33). Related to *ii* we have the algebra of (MARCEL) RIESZ'S distributions (SCHWARTZ, 14, pp. 49–50) with the set of translation maps given by various « powers » of the d'ALEMBERT differential operator. This particular algebra, with ramifications and generalizations, thereof, has been discussed in detail by METHÉE with an excellent account in TRÈVES' notes (TRÈVES, pp. 75–96) on LORENTZ invariant differential polynomials.

As an example of a translation algebra X without unit in which it is easy to find approximate identities, we consider all those distributions on $\mathbb{R}$ of the form

$$\sum_{m,n} a_{m,n} \tau_n \delta^{(m)},$$

where the sum is finite, $a_{m,n} \in \mathbb{R}$, and $\sum a_{m,n} = 0$. Clearly, $\delta \notin X$. Also, for

$$T = \sum_{m,n} a_{m,n} \tau_n \delta^{(m)}, \qquad S \sum_{i,j} a_{i,j} \tau_j \delta^{(i)}$$

we have

$$T*S = \sum_{m,n} \sum_{i,j} a_{i,j} a_{m,n} \tau_{n+1} \delta^{(m+1)}$$

which can be re-subscripted to be of the form $\Sigma b_{p,q} \tau_q \delta^{(p)}$, and where it is easily seen that $\Sigma b_{p,q} = 0$ by the right re-grouping. The various orders of differentiation form a set of translation maps, as do the usual translation maps τ_k, k an integer, and the union of these two sets. Again, for $T \in X$, $\mathcal{C}_T \subseteq X$.

Now, consider a sequence of the form

$$\overset{n}{\delta} = \delta - \tau_n \delta^{(m)}$$

wher $n \longrightarrow \infty$. It is easily seen that this sequence does not converge in the topology of $\mathcal{E}'$ but does converge to δ in the somewhat coarser topology on $\mathcal{D}'$. As is well known the mapping $(S, T) \rightsquigarrow S * T$, $S \in \mathcal{E}'$, $T \in \mathcal{D}'$, is a separately continuous map $\mathcal{E}' \times \mathcal{D}' \longrightarrow \mathcal{D}'$, so that for $T \in X \subseteq \mathcal{E}'$, $T * \overset{n}{\delta} \longrightarrow T$. A simple calculation also shows that $T * \overset{n}{\delta} \in X$ and hence $\{\overset{n}{\delta}\}$ is a sequence of approximate identities in X. Also $\tau_k : X \longrightarrow X$ ip continuous.

B. - Let X be any algebra of distribntions and let the family of translation maps be all differential polynomials D_α with constant coefficients such that

$$D_\alpha : X \longrightarrow X;$$

for example, let $X = \mathcal{E}'$. The TAUBERIAN problem in this setting is to find $\overline{\mathcal{C}}_T$ for a given $T \in X$; that is, for a given T to find which elements of X can be approximated by differential polynomials $\Sigma c_\alpha D_\alpha T$. In this setting, the order of the differential operator is arbitrary, and it is also interesting to find those elements of X which can be approximated by differential polynomials $\Sigma c_\alpha D_\alpha T$ where the order is always less than some fixed integer.

Thus, given T the TAUBERIAN theorem actually determines those elements S such that $P(D)T = S$, where $P(D)$ is some differential polynomial.

In a slightly more general TAUBERIAN setting consider thesame problem for two different algebras X and Y; that is

$$D_\alpha : X \longrightarrow X.$$

We also note here as another direction of some interest that the problem of the previous paragraph is closely related to the fundamental representation theorems of Distribution theory; for example, it is well known that $T \in \mathcal{D}'_{L^1}(\mathbb{R})$ if and only if

$$T = \Sigma f_i^{(n_i)}, \quad f_i \in L^1.$$

Thus, for our particular case we would be examining those elements of $\mathcal{D}'_{L^1}$ where the f_i is a single fixed element of L^1.

Looking at this situation is a slightly different light we see that we are, in fact, dealiug with a generalization of one of the standard problems of differential equations. For example, in the one-dimensional case, the operator

$$L = \sum_{k-0}^{m} a_k D^k, \quad a_k \in \mathcal{E}, \quad a_k(x) \text{ never zero,}$$

maps $\mathfrak{D}'$ onto $\mathfrak{D}'$ and has m-dimensional null space each of whose elements $\varphi \in \mathcal{E}$ and satisfies $L(\varphi) = 0$. More generally, then, given a closed ideal I in an algebra X the problem is to find the largest set of differential polynomials (with constant sufficients) such that the range of a given T for all these polynomials is maximal in I. This problem and its obvious perturbations are therefore not only usual queries in differential equations but are also naturally posed in a TAUBERIAN setting.

When considering such families of differential operators $X \to Y$ we can topologize them, as expected, as subalgebras of the usual space of continuous linear maps $\mathfrak{L}(X, Y)$ — provided, of course, that we only consider continuous differential operators. Finally, in algebras of distributions it is no problem at all to consider sets of translation maps which include both differential maps and true translation operators.

C. - We finally consider the TAUBERIAN properties of $\mathfrak{D}'_{L^1}(\mathbb{R})$. First note that $\delta \in \mathfrak{D}'_{L^1}$ although this is not clear by the representation theorem for $\mathfrak{D}'_{L^1}$ mentioned above. On the other hand, the *sequence*

$$\overset{n}{\delta}(x) = \begin{cases} 0, \; x \in (-\infty, -1/n] \cup [1/n, \infty) \\ -n^2 |x| + n, \; x \in [-1/n, 1/n] \end{cases}$$

forms a system of approximate identities in $\mathfrak{D}'_{L^1}$. To see this, we first note that $\overset{n}{\delta} \to \delta$ in $\mathcal{E}'$; in fact, for $\varphi \in \mathcal{E}$,

$$|\langle \overset{n}{\delta} - \delta, \; \varphi \rangle| = 2 \left| \delta(\xi) \frac{\varphi(\xi) - \varphi(0)}{n} \right| < 2 \left| \frac{\varphi(\xi) - \varphi(0)}{n\xi} \right|$$

for some $\xi \in (-1/n, 1/n)$ — this follows by the mean value theorem for integration and since $1 > \xi n$, $\overset{n}{\delta}(\xi) \leq n$ for all $\xi \in (-1/n, 1/n)$. Thus, for a given bounded set in $\mathcal{E}$ we need only show that

$$\left| \frac{\varphi(\xi) - \varphi(0)}{\xi} \right|$$

is uniformly bounded (for all φ in this set, noting, of course, the ξ depends on n); but

$$\left|\frac{\varphi(\xi)-\varphi(0)}{\xi}\right| \leq \left|\frac{\varphi(\xi)-\varphi(0)}{\xi}-\varphi'(0)\right| + |\varphi'(0)|,$$

and

$$\left|\frac{\varphi(\xi)-\varphi(0)}{\xi}-\varphi'(0\right| \varphi''(\lambda)$$

for some $\lambda \in (0, \xi)$ (or $(\xi, 0)$ if $\xi < 0$), so that by the definition of bounded sets in $\mathcal{E}$ we have required uniform boundedness.

Now, it is a straightforward calculation that $T * S \in \mathcal{D}'_{L^1}$ for $T \in \mathcal{D}'_{L^1}$ and $S \in \mathcal{E}'$; in fact, we need only use the representation theorems to express elements of $\mathcal{D}'_{L^1}$ and $\mathcal{E}'$ in terms of sums of derivatives of L^1 and continuous functions (with supports in a fixed compact set), respectively.

Further, we note that the map $\langle T, S \rangle \rightsquigarrow T*S$, $T \in \mathcal{D}'_{L^1}$, $S \in \mathcal{E}'$ is separately continuous in the sense that if $S_\alpha \rightsquigarrow 0$ in $\mathcal{E}'$ then $T*S_\alpha \rightsquigarrow 0$ in $\mathcal{D}'_{L^1}$; similarly, if the $S_\alpha \in \mathcal{D}'_{L^1}$ we also have separate continuity. Both these facts result from trivial manipulations with bounded sets in $\dot{\mathcal{B}}$ and $\mathcal{E}$. In any case, the $\overset{n}{\delta}$ form a sequence of approximate identities for $\mathcal{D}'_{L^1}$. Also, $\mathcal{D}'_{L^1}$ is a translation algebra in the sense of § 1 since the usual distributional translation obviously takes $\mathcal{D}'_{L^1}$ into $\mathcal{D}'_{L^1}$, and such translation is continuous because it takes bounded sets of $\dot{\mathcal{B}}$ into bounded sets of $\dot{\mathcal{B}}$. Hence, for each $T \in \mathcal{D}'_{L^1}$, $\overline{\mathcal{C}}_T$ is a closed ideal.

Another means of defining ideals is as follows: let

$$T \in I_{jp}$$

if

$$T = \Sigma f_i^{(k_i)},$$

a finite sum, where for each i, $f_i, \ldots, f_i^{(p)} \in L^1$ and

$$\min \{k_i : \text{for all } i\} \geq j;$$

I_{jp} is an ideal and for all p, $\bar{I}_{op} = \mathcal{D}'_{L^1}$ since $\mathcal{D} \subseteq I_{op}$.

For $f \in L^1$ and with non-vanishing FOURIER transform we have that $\overline{\mathcal{C}}_f = \mathcal{D}'_{L^1}$; from which it can be shown that if $\alpha \in \mathcal{B}$ and

$$f*\alpha(x) \longrightarrow A \int f, \quad \text{as } x \longrightarrow \infty,$$

then for all $T \in \mathcal{D}'_{L^1}$

$$T*\alpha(x) \longrightarrow A \langle T, 1 \rangle \text{ as } x \longrightarrow \infty.$$

This classical form of the TAUBERIAN theorem for $\mathfrak{D}'_{L^1}$ can also be proved making judicious use of classical proofs for L^1 (e.g. WIENER'S or BEURLING'S juxtaposed with appropriate lemmas to deal with the derivatives of L^1 functions in $\mathfrak{D}'_{L^1}$ (e.g. BENEDETTO, 1). Also our remarks are valid in $\mathbb{R}_n$, although we operate in $\mathbb{R}$ as a matter of convenience for the remainder of this section.

Let $\mathcal{O}$ be that subset of $\mathfrak{D}'_{L^1}$ whose elements T have the property that $\langle T, 1\rangle = 0$. Thus, if $T = f^{(n)}$, $n \geq 1$, and $f \in L^1$, then $T \in \mathcal{O}$. As a trivial application of our TAUBERIAN result we have for all $T \in \mathcal{O}$ and $\alpha \in \mathfrak{B}$ that $T * \alpha \in \dot{\mathfrak{B}}$. To see this we note that $T*\alpha(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$ and

$$(T*\alpha)^{(j)}(x) = (T*\alpha^{(j)})(x) \longrightarrow 0$$

for all j since $\alpha^{(j)} \in \mathfrak{B}$. Clearly not all elements of $\dot{\mathfrak{B}}$ are of the form $T*\alpha$, $T \in \mathcal{O}$, $\alpha \in \mathfrak{B}$; for example,

$$\beta(x) = e^{-e^{-x}} e^{-x}$$

has the properties that $\beta \in L^1 \cap \dot{\mathfrak{B}}$ and $\mathfrak{F}(\beta)(x) = \Gamma(1 - ix)$, so that $\mathfrak{F}(\beta)$ is never 0. On the other hand, $\mathfrak{F}(T)(0) = 0$ for all $T \in \mathcal{O}$ so that $\mathfrak{F}(T*\alpha)(0) = 0$.

We can, in fact, show that the complement in $\dot{\mathfrak{B}}$ of the set of elements represented by $T*\alpha$, $T \in \mathcal{O}$, $\alpha \in \mathfrak{B}$, is dense in $\dot{\mathfrak{B}}$; and that those elements (in $\dot{\mathfrak{B}}$) of the form $T*\alpha$ are not dense. The proofs of these observations involve short, semi-interesting calculations. Finally, if we weaken the topology on $\dot{\mathfrak{B}}$ to the compact convergence criterion (for each derivative) it is true that elements of the form $T*\alpha$, $T \in \mathcal{O}$, $\alpha \in \mathfrak{B}$, are a dense subset of $\dot{\mathfrak{B}}$.

§ 6. - A General Tauberian Theorem and Some Remarks.

In this section we prove the general TAUBERIAN theorem mentioned in § 4 where $Z_T \subseteq Z_S$. In order to demonstrate this ve make the natural changes for the appropriate WIENER condition in this situation. It might be well at this point to underscore the theme of this note; thus, in L^1 we show that $\overline{\mathcal{C}_f} = L^1$ when the FOURIER transform of f never vanishes–to do this we use the facts that L^1 has approximate identities and that the inverse of a non-vanishing FOURIER transform is the FOURIER transform of some $L^1(G)$ function on compact sets. Hence, we have considered general algebras X with translation properties (similar to those of L^1) so that $\overline{\mathcal{C}_T} = X$ when the $\mathfrak{M}$–transform of T never vanishes; these algebras are equipped with approximate identities and a local WIENER condition corresponding to the above inversion of non-vanishing FOURIER transforms.

For the proposition and theorem of this section we let X be a semi-simple translation algebra (as in § 1).

PROPOSITION 4. - Let $Z_T \subseteq \operatorname{int} Z_S$ (with the induced strong T_2 topology on $\mathfrak{M}(X)$). Define $\widehat{W} : \mathfrak{M}(X) \longrightarrow \mathbb{C}$ as

$$\widehat{W}(\varphi) = \begin{cases} \dfrac{S(\varphi)}{T(\varphi)}, & \varphi \notin Z_T \\ 0, & \varphi \in Z_T . \end{cases}$$

Then Z_T is closed, $\widehat{W}$ is continuous, and for all non-zero $c \in \mathbb{C}$. $\widehat{W}(c\varphi) = \widehat{W}(\varphi)$.

PROOF. - We first note that $\mathcal{C}(Z_T)$ is open; in fact for $\varphi \in \mathcal{C}(Z_T)$,

$T(\varphi) = c \neq 0$, so that we need only take any neighborhood $\tilde{N}$ of c which does not include 0 and note that $T^{-1}(\tilde{N})$ is a neighborhood of φ.

It is also clear that $\widehat{W}(c\varphi) = \widehat{W}(\varphi)$ for all non-zero $c \in \mathbb{C}$; for $\widehat{W}(c\varphi) = S(c\varphi)/T(c\varphi) = S(\varphi)/T(\varphi)$ if $T(c\varphi) \neq 0$, that is if $T(\varphi) \neq 0$; and $\widehat{W}(c\varphi) = 0$ if $T(c\varphi) = 0$, that is if $T(\varphi) = 0$.

Now, let $\varphi_\alpha \longrightarrow \varphi$ so that $S(\varphi_\alpha) \longrightarrow S(\varphi)$ and $T(\varphi_\alpha) \longrightarrow T(\varphi)$.

If $T(\varphi) \neq 0$ then, since Z_T is closed, there is a neighborhood N of φ so that for all $\Psi \in N$, $T(\Psi) \neq 0$. Hence for all $\alpha \geq$ some α_0, $\varphi_\alpha \in N$ and thus $\widehat{W}(\varphi_\alpha) \longrightarrow \widehat{W}(\varphi)$.

On the other hand, if $T(\varphi) = 0$ then $\varphi \in Z_T \subseteq \operatorname{int} Z_S$.

Therefore there is a neighborhood N of φ so that for all $\Psi \in N$, $S(\Psi) = 0$; thus for all $\alpha \geq$ some α_0, $\varphi_\alpha \in N$ implying $S(\varphi_\alpha) = 0$ and hence $\widehat{W}(\varphi_\alpha) \longrightarrow \widehat{W}(\varphi)$. *qed.*

GENERALIZED WIENER CONDITION: Given V_α and $T \in X$. There is $R \in X$ so that for all $\varphi \in \mathfrak{M}(X)$ with the properties

$$C_\varphi \cap V_\alpha \neq \Lambda,$$

and

$$\varphi \in \mathcal{C}(Z_T)$$

we have

$$1 = T(\varphi) R(\varphi).$$

THEOREM 3. - Let X be a semi-simple translation algebra with T_2 locally compact index set H and let $T, S \in X$ have the property that $Z_T \subseteq Z_S$. If T satisfies the generalized WIENER Condition then $S \in \overline{\mathcal{T}_T}$.

PROOF. – As before, we need only show that $S * U_\alpha \in \overline{\mathcal{T}_T}$ for each α.

By the WIENER condition

$$1 = R(\varphi) T(\varphi)$$

for all $\varphi \in \mathcal{C}(Z_T)$ such that $C_\alpha \cap V_\alpha \neq \Lambda$.

Thus, $S(\varphi) = R(\varphi) T(\varphi) S(\varphi)$ for all φ so that $C_\alpha \cap V_\alpha \neq \Lambda$; and therefore

$$U_\alpha(\varphi) S(\varphi) = U_\alpha(\varphi) R(\varphi) T(\varphi) S(\varphi)$$

for all $\varphi \in \mathfrak{M}(X)$.

Hence, as in Theorem 2.

$$U_\alpha * S = T * (U_\alpha * R * S)$$

and $U_\alpha * S \in \overline{\mathcal{T}_T}$. *qed.*

For the remainder of this final section we discuss two problem areas related to our translation considerations. First, we note those natural queries whose positive resolution would streamline our TAUBERIAN approach. In the second area we first consider a means of constructing reasonable translation algebras and then we sum up the problem of spectral algebras more precisely than remarked in § 4.

A 1. – It is advantageous to determine $\mathfrak{M}(X)$ so that we can actually compute $\langle T, \varphi \rangle$ $(T \in X, \varphi \in \mathfrak{M}(X))$. Granted, if X is a commutative locally m-convex algebra, then $\mathfrak{M}(X)$ can be characterized (MICHAEL, p. 11) in terms of maximal ideals of X in much the same manner as the BANACH algebra setting. This approach does not give much aid in actually calculating the $\mathfrak{M}$-transform of $T \in X$; for this purpose it is useful to find $\mathfrak{M}(X)$ «internally» as we did in § 3 for ξ'.

A 2. – Since our TAUBERIAN approach relies so heavily on a WIENER condition and the existence of approximate identities it is then reasonable to find conditions in which a given algebra has one or both of these properties.

A 3. – We are also interested in effectively eliminating Tr. 3 and/or Tr. 4 as hypotheses for large classes of algebras. There are two major reasons for this desired generalization. First, it is natural to determine those algebras X and elements $T \in X$ where $\overline{\mathcal{T}_T}$ is a closed ideal; and, more important for applications, it is desirable to explicitly approximate elements of a given algebra by differential polynomials of a fixed element–as considered in § 5. Presently, Tr. 4 tells us that we have enough elements in H so that with the conditions of our TAUBERIAN theorems we can have $\overline{\mathcal{T}_T} = X$; one of the ways it does this is to contribute quite significantly in proving that $\overline{\mathcal{C}_T}$ is ideal.

A 4. - Finally, it is natural to extend the definition of translation so that our translation maps $\tau_h : X \longrightarrow Y$ (Y not necessarily equal to X) have the property that for some $T \in X$, $\overline{\mathcal{C}_T}$ is a closed ideal in Y. Here Y is an algebra and $X \subseteq Y$.

B 1. - We have noted above that our translation approach is somehow most effective in spaces like L^1 or $\mathcal{D}'_{L^1}$. As a means of generating similar spaces and at the same time producing a partial integration theory for distributions we consider the following situation. Let μ be a RADON measure. Then the integrable functions (on $\mathbb{R}^n$) $\mathcal{Z}(\mu)$ with respect to μ are identified with finite measures absolutely continuous with respect to μ. Thus, we have essentially a subspace of distributions. The μ-integrable distributions would then be all finite linear combinations of derivatives of these measures-much the same as $\mathcal{D}'_{L^1}$ is constructed. A natural followup would be to manufacture a RADON-NIKODYM theorem so that the subset S_μ of absolutely continuous measures (with respect to μ) which generates locally integrable functions (with respect to μ) can be determined. Thus, the crux of the matter is to construct distribution theory for locally integrable functions with respect to an arbitrary RADON measure instead of just LEBESGUE measure; and, then, to show that the space corresponding to $\mathcal{D}'_{L^1}$ does, in fact, have meaning.

B 2. - As mentioned in § 4, L^1 is not a spectral algebra; but we do know that $\overline{\mathcal{C}_f}$ is the intersection of all closed maximal ideals containing it if and only if $Z_f \subseteq Z_g$ implies $g \in \overline{\mathcal{C}_f}$. Hence, it is natural to attempt to find those translation algebras where a local WIENER condition can be replaced by (or replaces) such an intersection property.

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