On the harmonic summability of Fourier series.

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Summary. - *The authors have defined the product of two summability methods and have applied it to Fourier series. The criterion are analogous to the known criterion Convergence of Fourier series.*

1. - Let $\{p_n\}$ be a sequence of numbers such that

$$P_n = p_0 + p_1 + \dots + p_n, \qquad (P_n \neq 0).$$

A series $\sum_0^\infty U_n$ with partial sums $s_n = \sum_0^n U_k$ is said to be summable by "NORLUND method [11]" defined by the sequence $\{p_n\}$ or simply summable (N, p_n), if t_n tends to a limit as $n \to \infty$, where

$$t_n = (P_n)^{-1} \sum_{k=0}^{n} p_{n-k} s_k. \tag{1.1}$$

If we choose

$$p_n = (n+1)^{-1} \tag{1.2}$$

and consequently

$$P_n = \sum_{k=0}^{n} [(k+1)^{-1}] \sim \log n \tag{1.3}$$

the transform t_n reduces to

$$(\log n)^{-1} \sum_{k=0}^{n} (k+1)^{-1} S_{n-k}. \tag{1.4}$$

This method of summability, known as Harmonic summability, was introduced by RIESZ [14] in 1924. This method is regular [2]. It is also known [14] that if a series is summable by harmonic means it is also summable, (C, δ), $\delta > 0$. We obtain another method of summation viz., $(H, 1)\ (C, 1)$ by superimposing the method $(H, 1)$ on the CESARO mean of order one.

2.1 - Let $f(t)$ be a periodic function with period 2π, and integrable in the sense of LEBESGUE over $(-\pi, \pi)$. Let its FOURIER series be given by

$$\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{1}^{\infty} A_n(t), \tag{2.1.2}$$

it being assumed, without any loss of generality that the constant term in the FOURIER series of $f(t)$ is zero.

We write

$$\varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2s\}.$$

2.2 - In 1932 HARDY and LITTLEWOOD gave the following convergence criteria for a FOURIER series at a point.

THEOREM A [3]. - If

$$\varphi(t) = o\left(\frac{1}{\log\left|\frac{1}{t}\right|}\right), \qquad \text{as } t \to 0 \tag{2.2.1}$$

and

$$A_n(t) = O(n^{-\delta}) \qquad (0 < \delta < 1) \tag{2.2.2}$$

as $n \to \infty$, then the FOURIER series of $f(t)$, at $t = x$, converges to $f(x)$.

Further LEBESGUE proved the following convergence criteria for a FOURIER series at a point.

THEOREM B [10]. - If

$$\int_0^t |\varphi(t)|\, dt = o(t) \tag{2.2.3}$$

and

$$\int_{\frac{\pi}{n}}^{\delta} \frac{\left|\varphi\left(t + \frac{\pi}{n}\right) - \varphi(t)\right|}{t}\, dt = o(1) \tag{2.2.4}$$

as $n \to \infty$ then the FOURIER series of $f(x)$ at $t = x$ converges to $f(x)$.

Gergen has proved.

THEOREM C [1]. - *If* $\varphi(t)$ satisfies

$$\int_0^t \varphi(t)dt = o(t) \tag{2.2.5}$$

and

$$\int_{\frac{k\pi}{n}}^{\delta} \frac{\left| \varphi\left(t+\frac{\pi}{n}\right) - \varphi(t) \right|}{t} dt = o(1) \tag{2.2.6}$$

then the FOURIER series is convergent.

Generalising theorem C. SUNOUCHI and IZUMI respectively proved the following convergence criteria for a FOURIER series at a point.

THEOREM D [18]. If $\varphi(t)$ satisfies

$$\int_0^t \varphi(t)dt = o(t^{\Delta}) \tag{2.2.7}$$

and

$$\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\delta} \frac{\left| \varphi\left(t+\frac{\pi}{n}\right) - \varphi(t) \right|}{t} dt = o(1) \tag{2.2.8}$$

then the FOURIER series is convergent.

THEOREM E [9]. If

$$\varphi_{\beta}(t) = o(t^{\nu}) \tag{2.2.9}$$

and

$$\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\delta} \frac{| \varphi(t+x) - \varphi(t) |}{t} dt = o(1) \tag{2.2.10}$$

as $n \to \infty$ then the FOURIER series of $f(t)$ at $t = x$, converges to $f(x)$.

2.3 - The happlication of armonic summation to FOURIER series has been discussed by JURINGIUS and HILLE and TAMARKIN [5]. Results of JURINGIUS are unpublished. He has proved that the FOURIER series is not summable by harmonic means almost everywhere.

The main interest of the harmonic method of summation lies in the TAUBERIAN theorem associated with it. IYENGAR [6] proved that if an infinite series Σa_n is summable by harmonic means to the sum S and if $\Sigma a_n = O(n^{-\delta})$, $0 < \delta < 1$, then it converges to the same sum. Using this TAUBERIAN result it is easy to see that condition (2.2.1) of Theorem A is sufficient to ensure the harmonic summability of FOURIER series. This was proved by IYENGAR [7] himself. Further SIDDIQUI [19] replaced condition (2.2.1) by a less stringent condition,

$$\int_0^t |\varphi(t)|\, dt = o\left(\frac{t}{\log \frac{1}{t}}\right) \tag{2.3.1}$$

HILLE and TAMARKIN [5] have proved a theorem for harmonic summability of FOURIER series which is analogous to Theorem B. Condition (2.2.3) was further relaxed by a weaker condition.

$$\int_0^t \varphi(t) dt = o(t) \tag{2.3.2}$$

for the harmonic summability of FOURIER series by VARSHNEY [21]. Now analogous to theorem D and Theorem E for harmonic summability are also expected. This was infact proved by SAHNEY [15].

Theorem of SIDDIQUI was further generalized for NORLUND summability by RAJAGOPAL [13]. Also both the results of SIDDIQUI and HILLE and TAMARKIN were extended by SHARMA [16], [17] for double FOURIER series.

IZUMI [8] has generalized theorem B for $(C, 1)$ summability of FOURIER series. In fact he proved that

THEOREM F. - If

$$\int_0^t \varphi(u) du = o(t) \tag{2.3.3}$$

and

$$\int_{\frac{\pi}{n}}^{\pi} \frac{\left| \varphi\left(t+\frac{\pi}{n}\right) - \varphi(t) \right|}{t^2} dt = O(n) \tag{2.3.4}$$

then the FOURIER series is summable $(C, 1)$ to the sums. Now it is easy to see that theorems of HILLE and TAMARKIN and SAHNEY for the harmonic summability of FOURIER series can be generalized analogous to theorem F.

The object this paper is to prove the following theorems:

THEOREM 1. - If

$$\int_0^t \varphi(u)du = o(t^{\Delta}) \qquad (\Delta \geq 1) \tag{2.3.5}$$

and

$$\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\eta} \frac{\left| \varphi\left(t+\frac{\pi}{n}\right) - \varphi(t) \right|}{t^2} \log \frac{1}{t}\, dt = o(n \log n) \tag{2.3.6}$$

then the FOURIER series is summable $(H, 1)$ $(C, 1)$ to sum zero at the point $t=\frac{\pi}{n}$ for all $\Delta \geq 1$ and k is some positive integer tending to infinity sufficiently slowly.

THEOREM 2. - If

$$\varphi_\beta(t) = o(t^\nu) \tag{2.3.7}$$

and

$$\lim_{k,n\to\infty} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\eta} \frac{\left| \varphi\left(t+\frac{\pi}{n}\right) - \varphi(t) \right|}{t^2} \log \frac{1}{t}\, dt = o(n \log n) \tag{2.3.8}$$

then the FOURIER series is summable $(H, 1)$ $(C, 1)$ to sum zero at the point $t=\frac{\pi}{n}$ for all $0 \leq \beta \leq 1$, $\nu/\beta = \Delta \geq 1$, where $\varphi_\beta(t)$ is the β^{th} integral of $\varphi(t)$.

3.1 - We require the following lemmas

LAMMA 1. [20], - For all values of n and t, we have

$$\sum_{k=0}^{n} \frac{\sin kt}{k} = O(1). \tag{3.1.1}$$

LEMMA 2. [4] -

$$\sum_{k=0}^{n} \frac{\cos kt}{k} = O\left(\log \frac{1}{t}\right). \tag{3.1.2}$$

LEMMA 3. - For $0 < t \leq \frac{\pi}{n}$, we have

$$\sum_{k=0}^{n} \frac{\sin^2(n-k-1)t}{(n-k\,1)(k+1)t^2} = O(n \log n). \tag{3.1.3}$$

LEMMA 4. - For $0 < t \leq \frac{\pi}{n}$, we have

$$\frac{d}{dt}\left(\sum_{k=0}^{n} \frac{\sin^2(n-k-1)t}{(n-k-1)(k+1)t^2}\right) = O\left(\frac{n \log n}{t}\right). \tag{3.1.4}$$

PROOF.

$$\Sigma \frac{1}{(n-k-1)(k+1} \left\{\frac{d}{dt}\left(\frac{\sin^2(n-k-1)t}{t^2}\right)\right\}$$

$$= \Sigma \frac{1}{(n-k-1)(k+1)} \left\{\frac{t^2(n-k-1).\,2\sin(n-k-1)t\cos(n-k-1)t - \sin^2(n-k-1)t.\,2t}{t^4}\right\}$$

$$= \Sigma \frac{1}{(n-k-1)(k+1)} \left\{\frac{O(n-k-1)^2 t^3 - O(n-k-1)^2 t^3}{t^4}\right\}$$

$$= \frac{1}{t}\left[\Sigma \frac{O(n-k-1)}{(k+1)} - \frac{O(n-k-1)}{(k+1)}\right]$$

$$= \frac{1}{t}\left[O(n \log n) - O(n \log n)\right]$$

$$= O\left(\frac{n \log n}{t}\right).$$

LEMMA 5. - For $\frac{1}{n} \leq t \leq \pi$, we have

$$\sum_{k=0}^{n} \frac{\sin^2 (n-k-1) t}{(n-k-1)(k+1) t^2} = O\left(\frac{n^{-1} \log n}{t^2}\right). \tag{3.1.5}$$

PROOF. - Since $|\sin (n-k-1) t| \leq 1$ we have

$$= \sum_{k=0}^{n} \frac{1}{(n-k-1)(k+1)t^2}$$

$$= \frac{1}{t^2} \sum_{k=0}^{n} \left\{ \frac{1}{(n-k-1)} + \frac{1}{(k+1)} \right\} \cdot \frac{1}{n}$$

$$\leq \frac{1}{nt^2} \cdot \log n$$

$$= O\left(\frac{n^{-1} \log n}{t}\right).$$

4.1 - We shall now prove theorem 1.

PROOF. - Since the case $\Delta = 1$ is due to RAI [12] we shall consider here the case $\Delta > 1$ only.

The CESARO mean of the first order of FOURIER series of $f(t)$ is given by

$$\sigma_n = \frac{1}{2\pi (n+1)} \int_0^\pi \varphi(t) \frac{\sin^2 (n+1) t}{\sin^2 t/2} dt$$

$$= \frac{1}{2\pi(n+1)} \int_0^\pi \varphi(t) \cdot \frac{\sin^2 (n+1) t}{t^2} dt + o(1).$$

On account of the regularity of Harmonic summation, we need only prove that,

$$\frac{1}{\log n} \sum_{k=0}^{n} \frac{1}{(k+1) \, . \, 2\pi(n-k-1)} \int_0^\pi \varphi(t) \frac{\sin^2 (n-k-1) t}{t^2} dt$$

$$= o(1) \quad \text{as} \quad n \to \infty.$$

We have

$$=\frac{1}{2\pi\log n}\int_0^{n}\varphi(t)\sum_{k=0}^{n}\frac{\sin^2(n-k-1)t}{(n-k-1)(k+1)t^2}\,dt$$

$$=\frac{1}{2\pi\log n}\left\{\int_0^{\frac{k\pi}{n}}+\int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}+\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\pi}\right\}\varphi(t)\sum_{k=0}^{n}\frac{\sin^2(n-k-1)t}{(n-k-1)(k+1)t^2}\,dt$$

$$=I_1+I_2+I_3,\quad\text{say.}\tag{4.1.1}$$

Now for $0<t\leq\frac{k\pi}{n}$

$$\frac{1}{2\pi\log n}\sum_{k=0}^{n}\frac{\sin^2(n-k-1)t}{(n-k-1)(k+1)t^2}=O(n)\ \text{by virtue of Lemma 3.}$$

$$I_1=O\left(\int_0^{\frac{k\pi}{n}}\varphi(t)\cdot\frac{1}{2\pi\log n}\sum_{k=0}^{n}\frac{\sin^2(n-k-1)t}{(n-k-1)(k+1)t^2}\,dt\right)$$

$$=O\left(n\int_0^{\frac{k\pi}{n}}\varphi(t)dt\right)$$

$$=O\left\{n\cdot o\left(\frac{k\pi}{n}\right)^{\Delta}\right\}$$

$$=o(1)\quad\text{as}\quad n\to\infty.\tag{4.1.2}$$

$$I_2=\frac{1}{2\pi\log n}\int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}\varphi(t)\sum_{k=0}^{n}\frac{\sin^2(n-k-1)t}{(n-k-1)(k+1)t^2}\,dt$$

$$=\frac{1}{2\pi\log n}\int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}\frac{\varphi(t)}{t^2}\sum_{k=0}^{n}\frac{1-\cos 2(n-k-1)t}{(n-k-1)(k+1)}\,dt$$

$$= \frac{1}{2\pi \log n} \int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\varphi(t)}{t^2} \sum_{k=0}^{n} \frac{1}{(n-k-1)(k+1)} dt$$

$$- \frac{1}{2\pi \log n} \int_{\frac{kn}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\varphi(t)}{t^2} \sum_{k=0}^{n} \frac{\cos 2(n-k-1)t}{(n-k-1)(k+1)} dt$$

$$= I_{2,1} - I_{2,2}, \qquad \text{say}.$$

Now $$I_{2,1} = \frac{1}{2\pi \log n} \int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\varphi(t)}{t^2} \sum_{k=0}^{n} \frac{1}{(n-k-1)(k+1)} \cdot dt$$

$$= \frac{1}{2\pi \log n} \int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\varphi(t)}{t^2} \sum_{k=0}^{n} \left\{ \frac{1}{(n-k-1)} + \frac{1}{(k+1)} \right\} \cdot \frac{1}{n} dt$$

$$\leq \frac{1}{n.2\pi \log n} \int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\varphi(t)}{t^2} \log n dt$$

$$= O\left(\frac{1}{n}\right) \int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\varphi(t)}{t^2} dt$$

$$= O\left(\frac{1}{n}\right) \left\{ \left[\frac{\Phi(t)}{t^2} \right]_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} + 2 \int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\Phi(t)}{t^3} dt \right\}$$

$$= O\left(\frac{1}{n}\right) \left\{ \left[\frac{o(t^{\Delta})}{t^2} \right]_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} + 2 \int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{o(t^{\Delta})}{t^3} dt \right\}$$

$$= \frac{o(1)}{n}\left\{\left[t^{\Delta-2}\right]_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} + \left[t^{\Delta-2}\right]_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}\right\}$$

$$= \frac{o(1)}{n}\left\{\left(\frac{k\pi}{n}\right)^{\frac{\Delta-2}{\Delta}} - \left(\frac{k\pi}{n}\right)^{\Delta-2}\right\}$$

$$= o(1).$$

On the other hand $I_{2.2} = \frac{1}{2\pi\log n}\int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\varphi(t)}{t^2}\sum_{k=0}^{n}\frac{\cos 2(n-k-1)t}{(n-k-1)(k+1)}dt$.

$$= \frac{1}{2\pi\log n}\int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\varphi(t)}{t^2}\sum_{k=0}^{n}\frac{1}{n}\left\{\frac{\cos 2(n-k-1)t}{(n-k-1)} + \frac{\cos 2(n-k-1)t}{(k+1)}\right\}dt$$

$$= I_{2.2.1} + I_{2.2.2}, \text{ say}.$$

Considering $I_{2.2.1}$, we have

$$I_{2.2.1} = \frac{1}{\pi\log n}\int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\varphi(t)}{nt^2}\sum_{k=0}^{n}\frac{\cos 2(n-k-1)t}{2(n-k-1)}dt$$

$$= \frac{1}{\pi\log n}\int_{\left(\frac{k\pi}{n}\right)^{*}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\varphi(t)}{nt^2}\cdot O\left(\log\frac{1}{t}\right)dt$$

$$= O\left(\frac{1}{n}\right)\int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{u}\right)^{\frac{1}{\Delta}}} \frac{\varphi(t)}{t^2}\,dt$$

$$= O\left(\frac{1}{n}\right)\left\{\left[\frac{\Phi(t)}{t^2}\right]_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} + 2\int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\Phi(t)}{t^3}\,dt\right\}$$

$$= o(1).$$

Again

$$I_{2.2.2} = \frac{1}{\pi \log n} \int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\varphi(t)}{nt^2} \sum_{k=0}^{n} \frac{\cos 2(n-k-1)t}{2(k+1)} dt$$

$$= \frac{1}{\pi \log n} \int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\varphi(t)}{nt^2} \left\{ \sum_{k=0}^{n} \frac{\cos 2nt \cos 2(k+1)t}{2(k+1)} + \sum_{k=0}^{n} \frac{\sin 2nt \sin 2(k+1)t}{2(k+1)} \right\} dt$$

$$= J_1 + J_2, \text{ say}.$$

With the help of Lemma 2, we have

$$J_1 = \frac{1}{\pi \log n} \int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\varphi(t)}{nt^2} \cos 2nt \sum_{k=0}^{n} \frac{\cos 2(k+1)t}{2(k+1)} dt$$

$$= \frac{1}{\pi \log n} \int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\varphi(t)}{nt^2} \cos 2nt \cdot O\left(\log \frac{1}{t}\right) dt$$

$$= \frac{1}{\pi n} \int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\varphi(t)}{t^2} \cos 2nt\, dt$$

$$= \frac{1}{n} \left\{ \left[\frac{\Phi(t)}{t^2} \cdot \cos 2nt \right]_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} + 2 \int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\Phi(t)}{t^3} \cos 2nt\, dt \right.$$

$$+\int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\Phi(t)}{t^2}\cdot 2\,n \sin 2\,nt\,dt\Big\}$$

$$=\frac{1}{n}\Big\{\left[o\left(\frac{t^{\Delta}}{t^2}\right)\cos 2\,nt\right]_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}+\int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} o\left(\frac{t^{\Delta}}{t^3}\right)\cos 2\,nt\,dt$$

$$+\int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} o\left(\frac{t^{\Delta}}{t^2}\right) 2\,n\,sin\,2\,nt\,dt\Big\}$$

$$=o(1)+o\left(\frac{1}{n}\right)\Big\{\int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} t^{\Delta-3}\,dt+n^2\int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} t^{\Delta-2}\,dt\Big\}$$

$$=o(1) \quad \text{as} \quad n\to\infty.$$

Similarly $J_2=o(1)$.

(4.1.3) Therefore $I_2=o(1)$.

Finally

$$\frac{1}{2\pi\log n}\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\pi} \frac{\varphi(t)}{t^2}\sum_{k=0}^{n}\frac{\sin^2(n-k-1)\,t}{(n-k-1)\,(k+1)}\,dt$$

$$=\frac{1}{2\pi\log n}\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\pi} \frac{\varphi(t)}{t^2}\sum_{k=0}^{n}\frac{1-\cos 2\,(n-k-1)\,t}{(n-k-1)\,(k+1)}\,dt$$

$$=\frac{1}{2\pi\log n}\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\pi} \frac{\varphi(t)}{t^2}\sum_{k=0}^{n}\frac{1}{(n-k-1)\,(k+1)}\,dt\,-$$

$$-\frac{1}{\pi \log n} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\pi} \frac{\varphi(t)}{t^2} \sum_{k=0}^{n} \frac{\cos 2\,(n-k-1)\,t}{2\,(n-k-1)\,(k+1)}\,dt$$

$$= o(1) - \frac{1}{\pi \log n} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\pi} \frac{\varphi(t)}{t^2} \sum_{k=0}^{n} \frac{\cos 2\,(n-k-1)\,t}{2\,(n-k-1)\,(k+1)}\,dt$$

$$= L_1 - L_2, \text{ say.}$$

$$L_2 = \frac{1}{\pi \log n} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\pi} \frac{\varphi(t)}{t^2} \sum_{k=0}^{n} \frac{1}{n} \left\{ \frac{\cos 2\,(n-k-1)t}{2\,(n-k-1)} + \frac{\cos 2\,(n-k-1)t}{2\,(k+1)} \right\} dt$$

$$= L_{2.1} + L_{2.2}, \text{ say.}$$

$$L_{2.1} = \frac{1}{n\pi \log n} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\pi} \frac{\varphi(t)}{t^2} \sum_{k=0}^{n} \frac{\cos 2\,(n-k-1)\,t}{2(n-k-1)}\,dt$$

$$= \frac{1}{n \cdot \pi \log n} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\pi} \frac{\varphi(t)}{t^2} \cdot O\left(\log \frac{1}{t}\right) dt$$

$$= O\left(\frac{1}{n}\right) \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\pi} \frac{\varphi(t)}{i^2}\,dt$$

$$= o(1).$$

Again

$$L_{2.2} = \frac{1}{n \cdot \pi \log n} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\pi} \frac{\varphi(t)}{t^2} \Sigma \left\{ \frac{\cos 2\,nt \cos 2\,(k+1)\,t}{2\,(k+1)} - \right.$$

$$- \frac{\sin 2\,nt \sin 2\,(k+1)}{2(k+1)} \Big\} dt$$

$$= L_{2.2.1} - L_{2.2.2}\,, \text{ say.}$$

With the help of Lemma 2 we have

$$L_{2.2.1} = \frac{1}{n \cdot \pi \log n} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\pi} \frac{\varphi(t)}{t^2} \cos 2\,nt \log \frac{1}{t}\, dt$$

$$= - \frac{1}{n \cdot \pi \log n} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}} + \frac{\pi}{2n}}^{\pi + \frac{\pi}{2n}} \frac{\varphi\left(t - \frac{\pi}{2n}\right)}{\left(t - \frac{\pi}{2n}\right)^2} \cos 2\,nt \log \frac{1}{\left(t - \frac{\pi}{2n}\right)}\, dt$$

$$= \frac{1}{2n\,\pi \log n} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\pi} \frac{\varphi(t)}{t^2} \cos 2\,nt \log \frac{1}{t}\, dt$$

$$= - \frac{1}{2\pi \log n} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}} + \frac{\pi}{2n}}^{\pi + \frac{\pi}{2n}} \frac{\varphi\left(t - \frac{\pi}{2n}\right)}{\left(t - \frac{\pi}{2n}\right)^2} \cos 2\,nt \log \frac{1}{t}\, dt$$

$$= \frac{1}{2\,\pi\, n \log n} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}} + \frac{\pi}{2n}} \frac{\varphi(t)}{t^2} \cos 2\,nt \log \frac{1}{t}\, dt$$

$$- \frac{1}{2\pi\, n \log n} \int_{\pi}^{\pi + \frac{\pi}{2n}} \frac{\varphi\left(t - \frac{\pi}{2n}\right)}{\left(t - \frac{\pi}{2n}\right)^2} \cos 2\,nt \log \frac{1}{\left(t - \frac{\pi}{2n}\right)}\, dt$$

$$+\frac{1}{2\pi n\log n}\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}+\frac{\pi}{2n}}^{\pi}\left\{\frac{\varphi(t)}{t^2}-\frac{\varphi\left(t-\frac{\pi}{2n}\right)}{\left(t-\frac{\pi}{2n}\right)^2}\right\}\cos 2\,nt\log\frac{1}{t}\,dt$$

$$+\frac{1}{2\,\pi\,n\log n}\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}+\frac{\pi}{2n}}^{\pi}\frac{\varphi\left(t-\frac{\pi}{2n}\right)}{\left(t-\frac{\pi}{2n}\right)^2}\cos 2\,nt\left\{\log\frac{1}{t}-\log\frac{1}{\left(t-\frac{\pi}{2n}\right)}\right\}dt$$

$$=P_1+P_2+P_3+P_4,\ \text{say.}$$

$$P_1=\frac{1}{2\,\pi\,n\log n}\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\left(\frac{h\pi}{n}\right)^{\frac{1}{\Delta}}+\frac{\pi}{2n}}\frac{\varphi(t)}{t^2}\cos 2\,nt\log\frac{1}{t}\,dt$$

$$=\frac{1}{n}\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}+\frac{\pi}{2n}}\frac{\varphi(t)}{t^2}\cos 2\,nt\,dt$$

$$=\frac{1}{n}\left\{\left[\frac{\Phi(t)}{t^2}\cos 2\,nt\right]_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}+\frac{\pi}{2n}}+2\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}+\frac{\pi}{2n}}\frac{\Phi(t)}{t^3}\cos 2\,nt\,dt\right.$$

$$\left.+\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}+\frac{\pi}{2n}}\frac{\Phi(t)}{t^3}\,2\,n\,\mathrm{Sin}\,2\,nt\,dt\right\}$$

$$=o(1)+o\left(\frac{1}{n}\right)\left\{\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}+\frac{\pi}{2n}}t^{\Delta-3}\,dt+n^2\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}+\frac{\pi}{2n}}t^{\Delta-2}\,dt\right\}$$

$$=o(1)\ \text{as}\ n\to\infty.$$

Similarly $P_2 = o(1)$.
Now

$$P_4 = \frac{1}{2\pi\, n \log n} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}} + \frac{\pi}{2n}}^{\pi} \frac{\varphi\left(t - \frac{\pi}{2n}\right)}{\left(t - \frac{\pi}{2n}\right)^2} \cos 2\,nt \left\{ \log \frac{1}{t} - \log \frac{1}{\left(t - \frac{\pi}{2n}\right)} \right\} dt$$

$$= \frac{1}{2\pi\, n \log n} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}} + \frac{\pi}{2n}}^{\pi} \frac{\varphi\left(t - \frac{\pi}{2\pi}\right)}{\left(t - \frac{\pi}{2n}\right)^2} \cos 2\,nt \log\left(1 - \frac{\pi}{2\,nt}\right) dt$$

$$= \frac{1}{2\pi\, n \log n} \left\{ \left[\frac{\Phi\left(t - \frac{\pi}{2n}\right)}{\left(t - \frac{\pi}{2n}\right)^2} \cos 2\,nt \log\left(1 - \frac{\pi}{2\,nt}\right) \right]_{\left(\frac{k\pi}{\pi}\right)^{\frac{1}{\Delta}} + \frac{\pi}{2n}}^{\pi} \right.$$

$$+ \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}} + \frac{\pi}{2n}}^{\pi} \Phi\left(t - \frac{\pi}{2n}\right) \cdot \frac{2\,n \sin 2\,nt}{\left(t - \frac{\pi}{2n}\right)^2} \log\left(1 - \frac{\pi}{2\,nt}\right) dt$$

$$+ \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}} + \frac{\pi}{2n}}^{\pi} \Phi\left(t - \frac{\pi}{2n}\right) \frac{\cos 2\,nt}{\left(t - \frac{\pi}{2n}\right)^2} \log\left(1 - \frac{\pi}{2\,nt}\right) dt$$

$$\left. + \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}} + \frac{\pi}{2n}}^{\pi} \Phi\left(t - \frac{\pi}{2n}\right) \frac{\cos 2\,nt}{\left(t - \frac{\pi}{2n}\right)^3} \cdot \frac{\pi}{2\,nt} \cdot dt \right\}$$

$$= o(1) + \frac{1}{2\pi n \log n} \left\{ \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}} + \frac{\pi}{2n}}^{\pi} o\left(t - \frac{\pi}{2n}\right)^{\Delta} \cdot \frac{2\,n}{\left(t - \frac{\pi}{2n}\right)^2} \cdot O\left(\frac{1}{nt}\right) dt \right.$$

$$+\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}+\frac{\pi}{2n}}^{\pi} o\left(t-\frac{\pi}{2n}\right)^{\Delta}\cdot\frac{1}{\left(t-\frac{\pi}{2n}\right)^{3}}\cdot O\left(\frac{1}{nt}\right)dt$$

$$=o(1)+\frac{o(1)}{2\pi\, n\log n}\left\{\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}+\frac{\pi}{2n}}^{\pi}\left(t-\frac{\pi}{2n}\right)^{\Delta-2}dt\right.$$

$$\left.+\frac{1}{n}\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}+\frac{\pi}{2n}}^{\pi}\left(t-\frac{\pi}{2n}\right)^{\Delta-3}td\right.$$

$$=o(1).$$

Next

$$P_3=\frac{1}{2\pi n\log n}\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}+\frac{\pi}{2n}}^{\pi}\left\{\frac{\varphi(t)}{t^2}-\frac{\varphi\left(t-\frac{\pi}{2n}\right)}{\left(t-\frac{\pi}{2n}\right)^2}\right\}\cos 2nt\log\frac{1}{t}\,dt$$

$$=\frac{1}{\pi n\log n}\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}+\frac{\pi}{2n}}^{\pi}\frac{\varphi(t)-\varphi\left(t-\frac{\pi}{2n}\right)}{t^2}\cos 2nt\log\frac{1}{t}\,dt$$

$$+\frac{1}{2\pi n^2\log n}\int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}\frac{\pi}{2n}}^{\pi}\varphi\left(t-\frac{\pi}{2n}\right)\frac{\left(t-\frac{\pi}{2n}\right)}{t^2\left(t-\frac{\pi}{2n}\right)^2}\cos 2\,nt\log\frac{1}{t}\,dt$$

$$=P_{3.1}+P_{3.2},\ \text{say}.$$

With the help of the second mean value theorem, hypothesis (2.3.5) and integration by parts

$$P_{3.2}=o(1).$$

Further using hypothesis (2.3.6) it is easy to see

$$P_{3.1}=o(1).$$

$$L_{2.2.2} = \frac{1}{\pi\, n \log n} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\pi} \frac{\varphi(t)}{t^2} \sum_{k=0}^{n} \frac{\sin 2\, nt \sin 2(k+1)t}{2(k+1)}\, dt$$

$$= \frac{1}{\pi\, n \log n} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\pi} \frac{\varphi(t)}{t^2} \sin 2\, nt \cdot O(1)\, dt$$

$$= \frac{1}{\pi n \log n} \int_{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}^{\pi} \frac{\varphi(t)}{t^2} \sin 2\, nt\, dt$$

$$= o(1) \quad \text{as in} \quad J_1 .$$

(4.1.4) Thus $I_3 = o(1)$.

Collection of (4.1.1), (4.1.2), (4.1.3) and (4.1.4) completes the proof of the theorem.

(5.1) - To prove Theorem 2 by virtue of Theorem 1, we shall consider the following two cases separately

Case I	$0 < \beta < 1$	and	$\Delta = 1$
Case II	$0 \leq \beta \leq 1$	and	$\Delta > 1$.

PROOF OF CASE I. - By virtue of Theorem 1, il is sufficient to show here that if (2.3.7) holds, then

(5.1.1) $$I_1 = o(1) \quad \text{as} \quad n \to \infty .$$

Since

$$I_1 = \frac{1}{2\pi \log n} \int_0^{\frac{k\pi}{n}} \varphi(t) \sum_{k=0}^{n} \frac{\sin^2 (n-k-1)t}{(n-k-1)(k+1)t^2}\, dt$$

$$= \frac{1}{2\pi \log n} \left\{ \left[\Phi(t) \sum_{k=0}^{n} \frac{\sin^2 (n-k-1)t}{(n-k-1)(k+1)t^2} \right]_0^{\frac{k\pi}{n}} - \int_0^{\frac{k\pi}{n}} \Phi(t) \frac{d}{dt} \left(\sum_{k=0}^{n} \frac{\sin^2 (n-k-1)t}{(n-k-1)(k+1)t^2} \right) dt \right.$$

(5.1.2) $$= I_{1.1} - I_{1.2} \text{ say}.$$

Also by (2.3.7) we have

(5.1.3) $$\Phi(t) = o(t^{\gamma-\beta+1}) = o(t)$$

(5.1.4) Hence $I_{1.1} = o(1)$, as $n \to \infty$ by Lemma 3.

Also

$$I_{1.2} = \frac{1}{2\pi \log n} \int_0^{\frac{k\pi}{n}} \Phi(t) \frac{d}{dt}\left(\Sigma \frac{\sin^2(n-k-1)t}{(n-k-1)(k+1)t^2}\right) dt$$

$$= \frac{1}{2\pi \log n} \int_0^{\frac{k\pi}{n}} \frac{d}{dt}\left(\sum_{k=0}^{n} \frac{\sin^2(n-k-1)t}{(n-k-1)(k+1)t^2}\right) dt \int_0^t \varphi_\beta(u)(t-u)^{-\beta} du$$

$$= \frac{1}{2\pi \log n} \int_0^{\frac{k\pi}{n}} \varphi_\beta(u)\, du \int_0^{\frac{k\pi}{n}} (t-u)^{-\beta} \frac{d}{dt}\left(\sum_{k=0}^{n} \frac{\sin^2(n-k-1)t}{(n-k-1)(k+1)t^2}\right) dt$$

by changing the order of integration

$$= \frac{1}{2\pi \log n} \int_0^{\frac{k\pi}{n}} \varphi_\beta(u) du \left\{ \int_u^{u+\frac{1}{n}} + \int_{u+\frac{1}{n}}^{\frac{k\pi}{n}} \right\} (t-u)^{-\beta} \frac{d}{dt}\left(\sum_{k-0}^{n} \frac{\sin^2(n-k-1)t}{(n-k-1)(k+1)t^2}\right) dt$$

$$= \frac{1}{2\pi \log n} \int_0^{\frac{k\pi}{n}} \varphi_\beta(u)\, du \left\{ \int_0^{u+\frac{1}{n}} (t-u)^{-\beta} \frac{d}{dt}\left(\sum_{k=0}^{n} \frac{\sin^2(n-k-1)t}{(n-k-1)(k+1)t^2}\right) dt \right.$$

$$\left. + \int_{u+\frac{1}{u}}^{\frac{k\pi}{n}} (t-u)^{-\beta} \frac{d}{dt}\left(\sum_{k=0}^{n} \frac{\sin^2(n-k-1)t}{(n-k-1)(k-1)t^2}\right) dt \right\}$$

$$= \frac{1}{2\pi \log n} \int_0^{\frac{k\pi}{n}} \varphi_\beta(u)\, du \left\{ \int_u^{u+\frac{1}{n}} (t-u)^{-\beta} \cdot O\left(\frac{n \log n}{t}\right) dt \right.$$

$$+ n^\beta \int_{u+\frac{1}{n}}^{\xi} \frac{d}{dt}\left(\sum_{k=0}^{n} \frac{\sin^2(n-k-1)t}{(n-k-1)(k+1)t^2}\right) td$$

by Lemma (4), where $u+\frac{1}{n}<\xi<\frac{k\pi}{n}$

$$=\frac{1}{2\pi\log n}\int_0^{\frac{k\pi}{n}}\varphi_\beta(u)du\left\{\left[(t-u)^{1-\beta}\cdot\frac{1}{u}\right]_u^{u+\frac{1}{n}}\cdot O(n\log n)\right.$$

$$\left.+n^\beta\left[\sum_{k=0}^{n}\frac{\sin^2(n-k-1)t}{(n-k-1)(k+1)t^2}\right]_0^{\frac{k\pi}{n}}\right\}$$

$$=\frac{1}{2\pi\log n}\int_0^{\frac{k\pi}{n}}o(u^\nu)\,du\left\{o(1)+\frac{1}{u\cdot n^{1-\beta}}\cdot O(n\log n)\right.$$

$$\left.+n^\beta\cdot O(n\log n)\right\}\qquad\text{by (2.3.7) \& (3.1.3)}$$

$$=o(n^\beta)\int_0^{\frac{k\pi}{n}}u^{\nu-1}du+o(n^{\beta+1})\int_0^{\frac{k\pi}{n}}u^\nu du$$

$$=o(n^\beta)\left[u^\nu\right]_0^{\frac{k\pi}{n}}+o(n^{\beta+1})\left[u^{\nu+1}\right]_0^{\frac{k\pi}{n}}$$

$$=o(1),\tag{5.1.5}$$

as $n\to\infty$ such that $o(k^{\nu+1})$ is $o(1)$.

From (5.1.2), (5.1.3), (5.1.4) and (5.1.5) we see that (5.1.1) holds true and this completes the proof of case I.

PROOF OF CASE II. - By virtue of Theorem 1 it is sufficient to show here that

$$I_1+I_2=o(1)\quad\text{as}\quad n\to\infty\tag{5.1.6}$$

$$\text{for}\quad 0<\beta<1\quad\text{and}\quad\Delta>1.$$

Since by virtue of (2.3.7)

$$\Phi(t)=0(t^{\nu-\beta+1})\tag{5.1.7}$$

We have similarly as in case I

$$I_1=0(1)\quad\text{as}\quad n\to\infty\tag{5.1.8}$$

Next

$$I_2 = \frac{1}{2\pi \log n} \int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \varphi(t) \sum_{k=0}^{n} \frac{\sin^2 (n-k-1)t}{(n-k-1)(k+1)t^2}\, dt$$

$$= \frac{1}{2\pi \log n} \left[\Phi(t) \cdot O\left(\frac{n^{-1} \log n}{t^2}\right)\right]_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}$$

$$- \frac{1}{2\pi \log n} \int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \Phi(t) \frac{d}{dt}\left(\Sigma \frac{\sin^2 (n-k-1)t}{(n-k-1)(k+1)t^2}\right) dt$$

$$= \frac{1}{2\pi \log n} \left[o\,(t^{\Delta-2}) \cdot O\,(n^{-1} \log n)\right]_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}$$

$$- \frac{1}{2\pi \log n} \int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{d}{dt}\left(\sum_{k=0}^{n} \frac{\sin^2 (n-k-1)t}{(n-k-1)(k+1)t^2}\right) dt \int_0^t \Phi_\beta(u)(t-u)^{-\beta}\, du$$

$$= o(1) + O\left(\frac{1}{\log n}\right) \int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \varphi_\beta(u)\, du \int_u^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} (t-u)^{-\beta} \frac{d}{dt}\left(\sum_{k=0}^{n} \frac{\sin^2 (n-k-1)t}{(n-k-1)(k+1)t^2}\right) dt$$

$$+ O\left(\frac{1}{\log n}\right) \int_0^{\left(\frac{k\pi}{n}\right)} \varphi_\beta(u)\, du \int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} (t-u)^{-\beta} \frac{d}{dt}\left(\Sigma \frac{\sin^2 (n-k-1)t}{(n-k-1)(k+1)t^2}\right) dt$$

by changing the order of integration

(5.1.9) $$= o(1) + I_{2.1} + I_{2.2}, \text{ say.}$$

Here

$$I_{2.1} = O\left(\frac{1}{\log n}\right) \int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \Phi_\beta(u)\, du \left\{ \int_u^{u+\frac{1}{n}} + \int_{u+\frac{1}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \right\} \cdot$$

$$(t-u)^{-\beta} \frac{d}{dt}\left(\Sigma \frac{\sin^2(n-k-1)t}{(n-k-1)(k+1)t^2}\right) dt$$

(5.1.10) $$= I_{2.1} + I_{2.1.2}; \quad \text{say.}$$

$$I_{2.1.1} = O\left(\frac{1}{\log n}\right) \int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \varphi_\beta(u) du \int_u^{u+\frac{1}{n}} (t-u)^{-\beta} \cdot O\left(\frac{n \log n}{t}\right) dt \qquad \text{by (3.1.4)}$$

$$= O(n) \int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{\varphi_\beta(u)}{u} \Big[(t-u)^{1-\beta} \Big]_u^{u+\frac{1}{n}} du, \qquad \text{by mean value theorem}$$

$$= O(n^\beta) \int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \frac{o(u^\nu)}{u} du \qquad \text{by (2.3.7)}$$

$$= o(n^\beta) \Big[u^\nu \Big]_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}$$

$$= o\left(\frac{1}{n^{\nu-\beta}}\right) + o(k^\nu)$$

(5.1.11) $$= o(1)$$

as $n \to \infty$ and k is defined such that $o(k^\nu)$ is $o(1)$ by (5.1.5).

Also

$$I_{2.1.2} = O\left(\frac{1}{\log n}\right) \int_{\left(\frac{k\pi}{n}\right)}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} n^\beta \varphi_\beta(u) du \left[\sum_{k=0}^{n} \frac{\sin^2(n-k-1)t}{(n-k-1)(k+1)t} \right]_{u+\frac{1}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}$$

$$= O\left(\frac{n^{\beta}}{\log n}\right)\int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} \varphi_{\beta}(u)du \left\{ O\,(n^{-1}\log n)\left[\frac{1}{t^2}\right]_{u-\frac{1}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}\right.$$

$$= O\,(n^{\beta+\frac{2\beta}{\nu}-1})\int_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} o\,(u^{\nu})\,du + O(n^{\beta-1})\int_{\frac{k\pi}{u}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} o\left(\frac{u^{\nu}}{u^{2}}\right)du$$

by (2.3.7) and the fact that $\frac{1}{un} < 1$, since $u > \frac{1}{n}$,

$$= o\,(n^{\beta+\frac{2\beta}{\nu}})\Big[\,u^{\nu+1}\,\Big]_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}} + O\,(n^{\beta-1})\Big[\,u^{\nu+1}\,\Big]_{\frac{k\pi}{n}}^{\left(\frac{k\pi}{n}\right)^{\frac{1}{\Delta}}}$$

$$= o\,(k^{\beta+\frac{2\beta}{\nu}}) + o\left(\frac{1}{n^{\nu-\beta+2-2\beta/\nu}}\right) + o\,(k^{\nu+1}) + o\left(\frac{1}{n^{\nu-\beta+1}}\right)$$

(5.1.12) $$= o\,(1)$$

as $n\to\infty$, and k is defined such that $o(k^{\beta+\frac{2\beta}{\nu}-1})$ is $O(1)$ as $n\to\infty$ by (5.1.5).

We have from (5.1.10), (5.1.11) and (5.1.12)

(5.1.13) $$I_{2.1} = o(1) \quad \text{as} \quad n\to\infty.$$

Similarly we can show that

(5.1.14) $$I_{2.2} = o(1) \quad \text{as} \quad n\to\infty.$$

Hence from (5.1.9), (5.1.13) and (5.1.14), it follows that

(5.1.15) $$I_2 = o(1) \quad \text{as} \quad n\to\infty.$$

This completes the proof of case II.

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