On a Theorem of E. Cartan.

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Summary. - *The object of this note is to prove: - Let G be a connected, locally compact subgroup of an analytic group H modelled on a Banach space. Then G itself is a finite dimensional analytic subgroup of H.*

Introduction.

In this paper we generalise E. Cartan's famous criterium for a connected locally compact group to be an analytic group. In the classical text ([1], pp. 130–135) of C. CHEVALLEY, E. Cartan's criterium is generalised to take the form: *Let G be a connected, locally compact group. If there exists a continuous univalent homeomorphism h of G into a finite dimensional analytic group H, then G itself is a finite dimensional analytic group.*

Our generalisation takes the connected locally compact group G as a subgroup of a non-necessarily finite dimensional analytic group and derives that G must be a finite dimensional analytic group.

In [2] the reader will find a detailed exposition of all aspects used in this paper of the elementary global theory of analytic groups modelled on the open sets of a BANACH space.

The ideas behind the proof of my generalized form of E. Cartan's theorem are essentially the same as those used in [1] to prove the above cited theorem of C. CHEVALLEY. They are here translated into the language of BANACH spaces which to a certain extent, I believe, clarifies the original proof of the less general theorem.

PRELIMINARIES. - In (2) I showed that there exists an analytic mapping, exp, from the tangent space, $T_e(G)$, at the identity of an analytic group G modelled on open subsets of a BANACH space into the group G with the following important properties:

1. There exists an open neighborhood U of 0 in $T_e(G)$ and an open neighborhood $\exp(U)$ of $e \in G$ such that exp: $T \to \exp(U)$ is a homeomorphism onto.

2. $\exp((t_1 + t_2)v) = \exp(t_1 v) \exp(t_2 v)$ for all $t_i \in R$ $(\exp(U), \exp^{-1})$ will be called a *canonical chart* at the identity of G.

3. Given any analytic homomorphism $h: R \to G \; \exists \; v \in T_e(G)$ such that $h(t) = \exp(tv)$.

PROPOSITION 1. - *Let G be a locally compact subgroup of an analytic group H and let* $\exp: S_a(o) \to H$ *be a canonical coordinate system at the identity of H. Suppose further that $K = G \cap \exp(S_b(o))$, $0 < b < a$, is a compact neighborhood of the identity in G. The function defined by $\Pi(k) = \exp^{-1}(k)/\|\exp^{-1}(k)\|$ on $K - e$ is continuous and furthermore $\Pi(K \cap C(e))$ is compact.*

PROOF: The continuity of Π is evident. To show $\Pi(K - e)$ compact it clearly suffices to show that there exists a positive real number r such that

$$(*) \qquad \Pi(K \cap C(\exp(S_r(o)) = \Pi(K \cap C(e)).$$

Let $x \neq e$ be an element of $K \cap \exp(S_{b/2}(o))$ and suppose $b/(m+1) \leqq \|\exp^{-1}(x)\| < b/m$ where $m \geqq 2$. From the homomorphic property of exp in $S_a(o)$ we have $\exp^{-1}(x^m) = m \exp^{-1}(x)$ and thus $\Pi(x) = \Pi(x^m)$. Now $b/2 < (m/(m+1))b \leqq m \|\exp^{-1}(x)\| = \|\exp^{-1}(x^m)\| < b$. Q. E. D.

THEOREM. - *Let G be a connected, locally compact subgroup of an analytic group H. Then G itself is a finite dimensional analytic group; in fact, G may be given the structure of an analytic subgroup of H (see [2]).*

PROOF. - Let $(V_1 = \exp(S_1), \exp^{-1})$ be a canonical chart at the identity of H where S_1 is an open ball of $T_e(H)$ with center 0 and let $U = \overline{V}_1 \cap G$ be a connected compact neighborhood of the origin in G.

Put B equal to the boundary of U. As B is compact and $e \notin B$ there exists a positive real number a such that $\|\alpha\| > a$ for $\exp(\alpha) \in B$; we have $S_a(o) \subset S_1$.

Put $V = \exp(S_a(o))$; and designate by H_1 the subset of $L(H)$ of those infinitesimal transformation $Y \in L(H)$ such that $\exp(tY) \in G$ for all real numbers t.

LEMMA 1. - *Let $\sigma_k \in U$, $\sigma_k \neq e$, $k \geqq 1$, be a sequence of elements such that $\lim_k \sigma_k = e$. If $\sigma_k = \exp(\alpha_k)$ and $\lim_k \frac{\alpha_k}{\|\alpha_k\|}$ exists, then $\lim_k \frac{\alpha_k}{\|\alpha_k\|} \in H_1$.*

PROOF. - For every $k \; \exists \; a$ largest integer i_k which has the following two properties: (i) $i_k \|\alpha_k\| < a$; (ii) $\sigma_k^m \in U$ for $0 \leqq m \leqq i_k$.

Let $\sigma' = \exp(\alpha')$ be the limit point of a converging subsequence of $(\sigma_k^{i_k})$; since $i_k \|\alpha_k\| < a$. α' is a limit point of the set $\{i_k \alpha_k\}$ and therefore $\|\alpha'\| \leqq a$.

(*) Thus no cluster point of the set $\{\sigma_k^{i_k}\}$ can belong to B.

By the definition of i_k we have either $(i_k + 1)\|\alpha_k\| \geqq \alpha$ or $\alpha_k^{i_k+1} \notin U$. Let K be the set of integers i_k such that $\sigma_k^{i_k+1} \notin U$.

If K where infinite then σ' would also be a cluster point of the set $\{\sigma_k^{i_k+1}\} = \{\sigma_k^{i_k} \cdot \sigma_k\}$ since $\lim \sigma_k = e$, thus $\sigma' \in B$ contrary to (*), therefore K

must be finite. It follows that there exists $N > 0$ such that for $k > N$

$$i_k \| \alpha_k \| < a \leq (i_k + 1) \| \alpha_k \| ;$$

now since $\lim_k [(i_k + 1) \| \alpha_k \| - i_k \| \alpha_k \|] = \lim_k \| \alpha_k \| = 0$ we have the following equalities.

(1) $$\lim_k i_k \| \alpha_k \| = a,$$

(2) $$\lim_k i_k = \infty,$$

(3) suppose $0 \leq t < a$, if $m_k = [a^{-1} t i_k]$ (where «[ ]» means « the integral part of »), then $\lim_k m_k / i_k = t a^{-1}$,

and

(4) $$\lim_k (m_k \| \alpha_k \| - t) = 0.$$

It is useful to observe that

(5) $$m_k < i_k$$

and therefore

(6) $$m_k \| \alpha_k \| < a \quad \text{and} \quad \sigma_k^{m_k} \in U.$$

(4) implies

(iii) $$\lim_k m_k \alpha_k = t \lim_k \frac{\alpha_k}{\| \alpha_k \|}$$

which implies that

$$\lim_k \sigma_k^{m_k} = \exp\left((t, \lim_k \frac{\alpha_k}{\| \alpha_k \|} \right), \; 0 \leq t < a, \quad \text{exists.}$$

Put $\sigma(t) = \exp\left(t, \lim_k \frac{\alpha_k}{\| \alpha_k \|}\right) \sigma(t) \in \bar{U}$ since $\sigma_k^{m_k} \in U$.

Replacing σ_k by $\sigma_k^{-1}(\sigma_k^{-1} \in U$ for k sufficiently large) we have $\sigma_k^{-1} = \exp(-\alpha_k)$, thus $\sigma(-t) = \exp(-tY)$, where

$$Y = \lim_k \frac{\alpha_k}{\| \alpha_k \|} \; ; \quad \text{for} \quad | t_2 | \, , \; | t_1 | \, , \; | t_1 + t_2 | < a$$

we have $\sigma(t_1 + t_2) = \sigma(t_1)\sigma(t_2)$. Designate by $Hom_c(R, G)$ the set of continuous homomorphisms of the additive group R into G.

As R is simply connected, there exists a continuous extensions $\Theta \in Hom_c(\mathrm{R}, G)$ of $\sigma(t)$ whose corresponding element in $L(H)$ is Y.

COROLLARY. - Let $\Theta \in Hom_c(R, G)$; we have

$$\Theta(t) \in U \quad \text{for} \quad t < a(\mid Y(\Theta) \mid)^{-1}.$$

LEMMA 2. - H_1 is a finite dimensional subspace of $L(H)$.

PROOF. - Note that if $Y \in H_1$ then $aY \in H_1$ since $\exp(t(aY)) = (\exp(at)Y) \in (G)$.

To prove that H_1 is a vector-space we need only prove that $Z_1, Z_2 \in H_1$ implies $Z_1 + Z_2 \in H_1$. Put $\Theta_i(t) = \exp(tZ_i)$ and put $\sigma_k = \Theta_1(k^{-1})\Theta_2(k^{-1})(1 \leqq k < \infty)$. For k sufficiently large, $\sigma_k \in U$; moreover $\exp^{-1}(\Theta_i(k^{-1})) = k^{-1}Z_i$. From (2) $\exp^{-1}(\sigma_k) = k^{-1}(Z_1 + Z_2) + k^{-2}A(k)$ where $\| A(k) \|$ remains bounded for $R \geqq 1$.

We may assume without loss of generality that $Z_1 + Z_2 \neq 0$; thus $\sigma_k \neq e$ for k sufficiently large. Now

$$\left| k^{-1} \| Z_1 + Z_2 \| - k^{-2} \| A(k) \| \right| \leqq \| \exp^{-1}(\sigma_k) \| \leqq k^{-1} \| Z_1 + Z_2 \| + k^{-2} \| A(k) \|$$

and

$$\lim_{k \to \infty} \frac{\exp^{-1}(\sigma_k)}{k^{-1} \| Z_1 + Z_2 \| - k^{-2} \| A(k) \|} = \frac{Z_1 + Z_2}{\| Z_1 + Z_2 \|} = \lim_{k \to \infty} \frac{\exp^{-}(\sigma_k)}{k^{-1} \| Z_1 + Z_2 \| + k^{-2} \| A(k) \|};$$

therefore

$$\lim_{k \to \infty} \frac{\exp^{-1}(\sigma_k)}{\| \exp^{-1}(\sigma_k) \|} = \frac{Z_1 + Z_2}{\| Z_1 + Z_2 \|} \in K_1$$

by lemma 1. Thus $Z_1 + Z_2 \in H_1$.

H_1 is a closed subspace of $L(H)$. In effect, let α_n be a CAUCHY sequence in H_1 converging to $\alpha \in H, \neq 0$. Put $\beta_n = \| \alpha - \alpha_n \| \alpha_n \in H_1$ put $\sigma_n = \exp(\beta_n)$. Clearly $\lim_n \sigma_n = e$ and for n sufficiently large we have $\exp(\beta_n) \in G$, thus $\lim_n \frac{\beta_n}{\| \beta_n \|} = \frac{\alpha_n}{\| \alpha_n \|} = \frac{\alpha}{\| \alpha \|} \in H_1$ therefore $\alpha \in H_1$.

As exp is a homeomorphism of a closed neighborhood of the origin of H_1 onto a compact subset of $h(G)$ we have that H_1 must be finite dimensional.

Q. E. D.

The lemma implies that if $Y \in H_1$ is such that $\| Y \| < a$ then there exists an element $g \in U$ such that $g = \exp(Y)$. This proves that $\exp / H_1 \cap S_a(o)$ is a homeomorphism onto a subset $U_1 \subset U$ containing the origin.

LEMMA 3. - U_1 is a neighborhood of $e \in G$.

PROOF. - H_1 being finite dimensional there exists a direct sum decomposition of $L(H) = H_1 + H_2$, where H_2 is a closed subspace of $L(H)$; let $\pi_i L(H) \to H_i$ be the canonical projection. The element $\exp(\alpha_1) . \exp(\alpha_2)$, $\alpha_i \in H_i$, is in V when $\| \alpha_1 \|$ and $\| \alpha_2 \|$ are sufficiently small. Put $\exp^{-1}(\exp(\alpha_1)\cdot\exp(\alpha_2)) = f(\alpha_1, \alpha_2) = (\pi_{1,o} f)(\alpha_1, \alpha_2) + (\pi_{2,o} f)(\alpha_1, \alpha_2)$; $f_i(\alpha_1, \alpha_2) = (\pi_{i,o} f)(\alpha_1, \alpha_2)$ is an analytic function such that $(\partial f_i / \partial \alpha_i)_{(o,o)} = i_{H_i}$ and $(\partial f_i / \partial \alpha_j)_{(o,o)} = 0$ for $i \neq j$. From the inverse function theorem (2, Appendix I) we deduce the existence of a chart (U, Ψ) at $e \in H$ such that $\Psi(\exp(\alpha_i)\cdot\exp(\alpha_2)) = \alpha_1 + \alpha_2$ whenever $\| \alpha_1 \|$, $\| \alpha_2 \| < b$ for some positive real number b. Put $V_2 = \exp(S_\beta(o))$ where $0 \leqq \beta < b, a$.

Suppose that U_1 is *not* a neighborhood of $e \in G$, and choose a sequence $\sigma_k \in U$ such that $\sigma_k \notin U_1$ with $\lim_k \sigma_k = e \in G$; suppose in addition $\sigma_k \in V_2$.

Now $\lim_k \Psi(\sigma_k) = 0 \in L(H)$ thus for k sufficiently large $\exp(\pi_1(\Psi(\sigma_k))) = \sigma'_k \in U_1$ and $\lim_k \sigma'_k = e \in G$. Put $\sigma''_k = (\sigma'_k)^{-1}\sigma_k$. For k sufficiently large, one has $\sigma''_k \notin T_1$ thus without loss of generality by a choice of a subsequence we may suppose since $\sigma'_k \in U_1$ that we have $\sigma''_k \neq e \in G$, $\sigma''_k \in U$ for k sufficiently large. By the choice of the chart (U, Ψ) at $e \in H$, we have

$$\sigma''_k = \exp(\pi_2(\Psi(\sigma_k))) \text{ and } \Psi(\sigma''_k) = \exp^{-1}(\sigma''_k).$$

From Proposition 1 it follows that the set of $\{ \exp^{-1}(\sigma_k) / \| \exp^{-1}(\sigma_k) \| \}$ has a cluster point $\alpha \in L(H)$.

Let α_k 1e a subsequence of $\left\{ \dfrac{\exp^{-1}(\sigma_k)}{\| \exp^{-1}(\sigma_k) \|} \right\}$ converging to α. By lemma 1, $\alpha \in H_1$; however, $\pi_1(\alpha) = \lim_k \pi_1(\alpha_k) = 0$ which is a contradiction since $\| \alpha \| = \lim \| \alpha_k \| = 1$. Q. E. D.

If $\sigma \in U_1$, put $f(\sigma) = \exp^{-1}(\sigma) \in H_1 f$ is a homeomorphism of U, onto a subset of H_1 containing the ball $\| \alpha \| < a$, $\alpha \in H_1$. Let a' be a positive real number such that $\| \exp^{-1}(\sigma) \|$, $\| \exp^{-1}(\sigma') \| < a'$ implies $\| \exp^{-1}(\sigma^{-1}\sigma^1) \| < a$ and designate by U'_1 the set of elements $\sigma \in U_1$ such that $\| \exp^{-1}(\sigma) \| < \alpha'$. As $\exp^{-1}(\sigma_1 \sigma_2^{-1})$ may expressed as an analytic function of $\exp^{-1}(\sigma_1)$ and $\exp^{-1}(\sigma_2)$ for $\| \exp^{-1}(\sigma_2) \|$, $\| \exp^{-1}(\sigma_2) \| < a'$. It follows that G is an analytic group (see [2]).

BIBLIOGRAPHY

[1] C. CHEVALLEY, *Theory of Lie Groups.*

[2] J. LESLIE, *Some Remarks on Function Spaces* (to appear).