A quasi-linear singular Cauchy problem (*)

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Summary. - *See the introduction.*

1. Introduction.

We consider the singular CAUCHY problem for the second order quasi-linear hyperbolic equation

$$u_x^{2m} u_{xx} - u_{yy} + f(x, y, u, u_x, u_y) = 0, \tag{1.1}$$

with the initial conditions

$$u(x, 0) = 0, \qquad u_y(x, 0) = \varphi(x), \qquad x \in I, \tag{1.2}$$

where m is any positive real number and $I = [\alpha, \beta]$ is a finite, closed interval. Using SCHAUDER's Fixed Point Theorem to solve a system of integral equations, we are able to show that under the appropriate conditions on f and φ, this problem has a unique solution in a neighborhood $(y > 0)$ of I.

A great deal of the work of this paper was motivated by the investigations of OGAWA [3]-[5]. OGAWA solved the comparable problems for the equations

$$r^2(x, y) u^{2m} u_{xx} - u_{yy} + f(x, y, u, u_x, u_y) = 0$$

and

$$u_x^{2m} u_{xx} - u_{yy} + f(x, y) = 0.$$

Let us denote by $\mathrm{Lip}(x_1, \ldots, x_n; K(y))$ the class of functions ξ which satisfy the LIPSCHITZ condition

$$|\xi(x_1, \ldots, x_n) - \xi(\bar{x}_1, \ldots, \bar{x}_n)| \leqq K(y)(|x_1 - \bar{x}_1| + \ldots + |x_n - \bar{x}_n|)$$

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on a given region. The norm $\|\cdot\|$ when applied to scalar functions shall mean the maximum modulus on a given region. The notation $\|\cdot\|_y$ shall mean

$$\| g(s, x, y) - h(s, x, y) \|_y = \max_{0 \leq s \leq y} | g(s, x, y) - h(s, x, y) |.$$

The symbols p and q are defined by

$$p = u_x, \qquad q = u_y.$$

To assure the hyperbolicity of (1.1) for $y > 0$, we suppose that there is a positive constant α such that $\varphi'(x) \geq \alpha$ on I. The CAUCHY problem is then singular in the sense that it is hyperbolic for $y > 0$, and along I, where we prescribe the initial conditions, the equation is parabolic. We also assume that φ has three derivatives on I and satisfies $|\varphi|, |\varphi'|, |\varphi''|, |\varphi'''| \leq \|\varphi\|$ and $\varphi''' \in \mathrm{Lip}(x; \|\varphi\|)$ for some constant $\|\varphi\|$. We choose constants A, A_0 and a_0 such that $A > \|\varphi\|$ and $A_0 > \varphi' \geq a > a_0 > 0$. Let D denote the open region bounded by the curves

$$x = \alpha + \frac{1}{m+1} A_0^m y^{m+1}, \qquad x = \beta - \frac{1}{m+1} A_0^m y^{m+1}, \qquad (y \geqq 0),$$

and the interval I. For each positive number δ, let D_δ be that portion of D for which $y < \delta$. Letting y_0 be the maximum ordinate of the points $\bar{D}$, we suppose that f is twice differentiable with respect to x, u, p and q on the region consisting of the points (x, y, u, p, q) which satisfy $(x, y) \in \bar{D}$, $|u| \leq Ay_0$, $|p| \leq Ay_0$ and $a_0 \leq q \leq A_0$. Assume moreover that f and its first and second partial derivatives with respect to x, u, p and q are bounded by $\|f\|$, and f is in $\mathrm{Lip}(y; \|f\|)$ and the foregoing second derivatives are in $\mathrm{Lip}(x, u, p, q; \|f\|)$ for some constant $\|f\|$. With these assumptions we shall prove the following existence theorem.

THEOREM 1. – Under the conditions

$$f_p(x, y, u, p, q) = o(y^{m-1}) \quad \text{as } y \to 0 \tag{1.3}$$

and

$$\frac{A_0}{a_0} < \frac{m+1}{m}, \tag{1.4}$$

there exists a $\delta > 0$, such that on D_δ the singular CAUCHY problem for (1.1), 1.2) has a solution $u(x, y)$ which is twice differentiable, with LIPSCHITZ continuous second derivatives on $\bar{D}_\delta$, and satisfies

$$\begin{aligned} &u_{xx}(x, y) \in \operatorname{Lip}(x;\ 0(y)), \\ (1.5) \qquad &u_{xy}(h_j, t) - u_{xy}(x, t) \in \operatorname{Lip}(x;\ 0(y^{m+1})), \\ &u_{xy}(h_j, t) - u_{xy}(x, t) \in \operatorname{Lip}(t;\ 0(y^m)), \qquad j = 1, 2. \end{aligned}$$

In (1.5), $h_j = h_j(t, x, y)$ are solutions of the ordinary differential equations

$$(1.6) \qquad \begin{aligned} &h_{jt} = (-1)^j t^{m^2/(m+1)} w^{m/(m+1)}(h_j, t), \qquad 0 \le t \le y, \\ &h_j(y, x, y) = x, \qquad (j = 1, 2), \end{aligned}$$

where the functions $w(x, y)$ are continuous and have continuous first partial derivatives with respect to x on $\bar{D}_\delta$ and satisfy

$$(1.7) \qquad \begin{aligned} &a_0^{m+1} y \le w \le A_0^{m+1} y, \\ &|w_x| \le My, \\ &w_x \in \operatorname{Lip}(x;\ Ny), \end{aligned}$$

M and N being constants.

Let us call K_δ the set of all such functions h_j.

In section 8 we shall use Theorem 1 to prove our main theorem which removes condition (1.4) and establishes the uniqueness of our solution.

2. Properties of K_δ.

We now establish some estimates for the functions in K_δ which will be used repeatedly in the proof of Theorem 1. From a theorem in ordinary differental equations ([1], pp. 25-28), it follows that any pair of functions h_j $(j = 1, 2)$ in K_δ have continuous first partial derivatives respect to x and y given by

$$(2.1) \qquad h_{jx}(t, x, y) = \exp\left[(-1)^{j+1} \int_t^y s^{m^2/(m+1)} \frac{\partial w^{m/(m+1)}}{\partial h_j}(h_j(s, x, y), s)\, ds\right],$$

$$(2.2) \qquad h_{jy}(t, x, y) = (-1)^{j+1} y^{m^2/(m+1)} w^{m/(m+1)}(x, y) h_{jx}(t, x, y), \qquad (j = 1, 2).$$

LEMMA 1. - If $h_j \in K_\delta$ and $(x, y) \in \bar{D}_\delta$, then

(*i*) $(h_j(t, x, y), t) \in \bar{D}_\delta$,

(*ii*) $h_j(t, x, y) = x + 0(y^{m+1} - t^{m+1})$, (*iii*) $h_{jx}(t, x, y) = 1 + 0(y^{m+1} - t^{m+1})$,

(*iv*) $h_{jx}(t, x\ y) \in \mathrm{Lip}\,(x;\ 0(y^{m+1}))$,

with the constants depending on δ, a_0, A_0, M and N.

PROOF. - The first two statements are easily verified using the integrated form of (1.6),

$$h_j(t, x, y) = x + (-1)^{j+1}\int_t^y s^{m^2/(m+1)} w^{m/(m+1)}(h_j(s, x, y), s)ds, \tag{2.3}$$

and the properties (1.7) of w. The last two statements follow from (1.7), (2.1), (*i*) and (*ii*), and the theorem of the mean.

LEMMA 2. - If h_j, $\tilde{h}_j \in K_\delta$, then for δ sufficiently small

(*i*) $| h_j(t, x, y) - \tilde{h}_j(t, x, y) | \leq 0(y^m) \| w - \tilde{w} \|$,

(*ii*) $| h_{jx}(t, x, y) - \tilde{h}_{jx}(t, x, y) | \leq 0(y^m)(\| w - \tilde{w} \| + \| w_x - \tilde{w}_x \|)$,

where the norms on the right are the maximum moduli of the functions on $\bar{D}_\delta$.

PROOF. - Using the integrated form (2.3), conditions (1.7) and the theorem of the mean, we have

$$| h_j(t, x, y) - \tilde{h}_j(t, x, y) | \leq 0(y^{m+1}) \| h_j - \tilde{h}_j \|_y + \frac{y^m}{(m+1)a_0} \| w - \tilde{w} \|.$$

The inequality still holds if the left side is replaced by the expression max. $|h_j - \tilde{h}_j|$ which is on the right, so the first assertion is proved by choosing δ so small that $0(y^{m+1}) \leq \frac{1}{2}$. The second estimate is the result of applying the bound of $h_j - \tilde{h}_j$, the properties of (1.7), and the theorem of the mean to the expression for $h_{jx} - \tilde{h}_{jx}$ obtained through the formula (2.1).

3. The integral equation.

Let us introduce the functions u_0, u_1, u_2 and z by

$$u_0(x, y) = u(x, y), \qquad u_0(x, 0) = 0, \tag{3.1}$$

$$u_1(x, y) = u_y(x, y) + \frac{1}{m+1} u_x^{m+1}(x, y), \qquad u_1(x, 0) = \varphi(x), \tag{3.2}$$

$$u_2(x, y) = u_y(x, y) - \frac{1}{m+1} u_x^{m+1}(x, y), \qquad u_2(x, 0) = \varphi(x) \tag{3.3}$$

$$z(x, y) = \frac{m+1}{2} y^{-m}(u_1(x, y) - u_2(x, y)), \qquad z(x, 0) = 0. \tag{3.4}$$

The equation (1.1) can then be written as the first order system

$$\begin{aligned} u_{0y} &= \frac{1}{2}(u_1 + u_2), \\ u_{1y} - y^{m^2/(m+1)} z^{m/(m+1)} u_{1x} &= F(x, y), \\ u_{2y} + y^{m^2/(m+1)} z^{m/(m+1)} u_{2x} &= F(x, y), \end{aligned} \tag{3.5}$$

where

$$F(x, y) = f(x, y, u_0(x, y), \; y^{m/(m+1)} z^{1/(m+1)}(x, y), \; \frac{1}{2}(u_1(x, y) + u_2(x, y))).$$

Using the initial conditions (1.2) and the characteristics $g_j(t, x, y)$ of equation (1.1), given by the solutions of (1.6) with $w = z$, we now express this system as the integral equations

$$\begin{aligned} u_0(x, y) &= \frac{1}{2} \int_0^y [u_1(x, t) + u_2(x, t)] dt, \\ u_1(x, y) &= \varphi(g_1(0, x, y)) + \int_0^y F(g_1(s, x, y), s) ds, \\ u_2(x, y) &= \varphi(g_2(0, x, y)) + \int_0^y F(g_2(s, x, y), s) ds. \end{aligned} \tag{3.6}$$

4. A Banach space

Let S_δ be the set of vector functions $\underline{u} = (u_0, u_1, u_2)$ which are continuous and have continuous first partial derivatives with respect to x on $\bar{D}_\delta$. We associate with each $\underline{u}$ in S_δ the function z defined by (3.4) and the functions $v_i(h_j)$ given by

$$\begin{aligned} v_i(h_j; t, x, y) &= \frac{1}{2} y^{-m}[u_{ix}(h_j(t, x, y), t) - u_{ix}(x, t)], \\ v_i(h_j; 0, x, 0) &= 0, \qquad (i = 0, 1, 2; \; j = 1, 2), \end{aligned} \tag{4.1}$$

where h_j is in K_δ.

Let $\| u_i \| = \max | u_i(x, y) |$, $(x, y) \in \bar{D}_\delta$, and define $\| u_{ix} \|$, $\| z \|$, and $\| z_x \|$ analogously, and set

$$\| v_i(h_j) \| = \max. | v_i(h_j;\ t,\ x,\ y) |, \qquad (x,\ y) \in \bar{D}_\delta, \qquad 0 \leq t \leq y.$$

The set S_δ is a BANACH space under the norm

$$\| \underline{u} \| = \max. (\ \| u_i \|,\ \| u_{ix} \|,\ \| z \|,\ \| z_x \|,\ \| v_i(h_j) \|),$$

the maximum being taken over $i = 0,\ 1,\ 2;\ j = 1,\ 2$, and all $h_j \in K_\delta$.

Next we denote by X_δ the set of elements of S_δ which satisfy on $\bar{D}_\delta$

$$\begin{aligned} &| u_0 | \leq Ay, \qquad u_0 \in \mathrm{Lip}\,(y;\ A), \qquad | u_{0x} | \leq Ay, \\ &u_{0x} \in \mathrm{Lip}\,(x;\ By), \qquad u_{0x} \in \mathrm{Lip}\,(y;\ A); \end{aligned} \tag{4.2}$$

$$\begin{aligned} &| u_i | \leq A, \qquad u_i \in \mathrm{Lip}\,(y;\ B), \qquad | u_{ix} | \leq A, \\ &u_{ix} \in \mathrm{Lip}\,(x,\ y;\ B)\,; \qquad (i = 1,\ 2); \end{aligned} \tag{4.3}$$

$$\begin{aligned} &a_0^{m+1} y \leq z \leq A_0^{m+1} y, \qquad z \in \mathrm{Lip}\,(y;\ B), \qquad | z_x | \leq My, \\ &z_x \in \mathrm{Lip}\,(x;\ Ny), \qquad z_x \in \mathrm{Lip}\,(y;\ C); \end{aligned} \tag{4.4}$$

$$| v_0(h_j) | \leq Q_0 y^2, \qquad v_0(h_j) \in \mathrm{Lip}\,(x,\ R_0 y^2), \qquad (j = 1,\ 2); \tag{4.5a}$$

$$| v_i(h_j) | \leq Q_i y, \qquad v_i(h_j) \in \mathrm{Lip}\,(x;\ R_i y), \qquad (i = 1,\ 2;\ j = 1,\ 2); \tag{4.5b}$$

$$v_0(h_j) \in \mathrm{Lip}\,(y,\ t;\ C_0 y), \qquad v_i(h_j) \in \mathrm{Lip}\,(y,\ t;\ C_i), \qquad (i = 1,\ 2;\ j = 1,\ 2), \tag{4.5c}$$

the capital letters being constants.

Since the function

$$\underline{u} = \left(\varphi y,\ \ \varphi' + \frac{(\varphi')^{m+1} y^{m+1}}{m+1},\ \ \varphi' - \frac{(\varphi')^{m+1} y^{m+1}}{m+1} \right)$$

satisfies conditions (4.2)-(4.5) for some set of constants, it follows that there are non-empty spaces X_δ. We also note that due to condition (4.4), the characteristics $g_j(t,\ x,\ y)$, $j = 1,\ 2$, corresponding to an element $\underline{u}$ of X_δ, defined as the solution of (1.6) with $w = z$, are in K_δ.

5. An equivalent solution.

We shall first show that Theorem 1 is proved if the system (3.6) has a solution in some space X_δ.

THEOREM 2. - The singular CAUCHY problem for equation (1.1), (1.2) has a solution u on D_δ, δ sufficiently small, twice differentiable with LIPSCHITZ continuous second derivatives and satisfying conditions (1.5) on $\bar{D}_\delta$ if and only if the integral equations (3.6) have a solution $\underline{u}$ in some space X_δ.

PROOF. - We have already seen that if u is a solution of the singular CAUCHY problem, then the vector $\underline{u}=(u_0, u_1, u_2)$ defined by (3.1)-(3.3) is a solution to the system (3.6). Moreover, straightforward calculations show that $\underline{u}$ is an element of some X_δ under conditions imposed on u.

Conversely, using the equations analogous to (2.1) and (2.2) relating the derivatives with respect to x and y of the characteristics corresponding to $\underline{u}$, we find that an element $\underline{u}=(u_0, u_1, u_2)$ of X_δ which is a solution of (3.6) is also a solution of the first order system (3.5). This in turn implies that

$$u(x, y)=u_0(x, y)$$

is a solution of the singular CAUCHY problem. The conditions (1.5) follow from the relations

$$(5.1)\qquad \begin{aligned} u_{xx} &= \frac{1}{m+1} y^{m/(m+1)} z^{-m/(m+1)} z_x , \\ u_{xy} &= \frac{1}{2}(u_{1x}+u_{2x}), \end{aligned}$$

and the properties of $\underline{u}$. The LIPSCHITZ continuity of the second derivatives of u are simple consequences of the same conditions for $\underline{u}$. This completes the proof of Theorem 2.

6. The continuous into mapping.

If $\underline{u}$ is in X_δ, we define

$$(6.1)\qquad T\underline{u}=\underline{U}=(U_0, U_1, U_2)$$

by the right-hand sides of the integral equations (3.6), g_1 and g_2 in the equations being the characteristics corresponding to $\underline{u}$. Moreover, let us denote by Z and V_i, respectively, the functions defined by the equations (3.4) and

(4.1) with u replaced by U. We note that T is well-defined, since by Lemma 1, $(g_i(t, x, y), t) \in \bar{D}_\delta$ $(i = 1, 2)$. To establish the existence of a solution of the integral equations, we shall prove that the mapping T has a fixed point. The proof is based on the fact that a continuous mapping of a convex, compact subset of a BANACH space into itself a fixed point.

THEOREM 3. - Under conditions (1.3) and (1.4), there exists a $\delta > 0$ and a space X_δ such that T is a continuous mapping of X_δ into itself.

PROOF. - We first prove that T maps some space X_δ into itself. If $u \in X_\delta$, then the functions $U = Tu$ and U_x are obviously continuous on $\bar{D}_\delta$, so $U \in S_\delta$.

It is easily verified that U_0 satisfies conditions (4.2) for δ sufficiently small by use of conditions (4.3) for u_1 and u_2, and the restrictions on a_0, A and B. Likewise, conditions (4.3) for U_1, U_2, U_{1x} and U_{2x} follow readily, for δ small, by a suitable choice of constants depending only on $\| \varphi \|$ and $\| f \|$, and the constants of Lemma 1. Let $F_x(x, y)$ denote the partial derivative of $F(x, y)$ with respect to x. Since $F_x \in \mathrm{Lip}(x; 0(1))$, $g_{iy} = 0(y^m)$, and $g_{ix} \in \mathrm{Lip}(y; 0(y^m))$, with the constants being independent of B, we see that $U_{ix} \in \mathrm{Lip}(x, y; B)$, $(i = 1, 2)$, for small δ, by selecting B adeguately large.

PROOF OF ESTIMATES FOR Z. - Writing

$$Z(x, y) = \frac{m+1}{2} y^{-m} \{ \varphi(g_1(0, x, y)) - \varphi(g_2(0, x, y)) + \int_0^y [F(g_1(s, x, y), s) - F(g_2(s, x, y), s)] ds \}$$

and using (2.3) to observe

$$\frac{2}{m+1} a_0^m y^{m+1} \leq g_1(t, x, y) - g_2(t, x, y) \leq \frac{2}{m+1} A_0^m y^{m+1}, \tag{6.2}$$

we easily obtain

$$a_0^{m+1} y \leq Z \leq A_0^{m+1} y.$$

The y-LIPSCHITZ continuity is readily demonstrated by using the above form of Z, Lemma 1, $g_{iy} = 0(y^m)$, and B sufficiently large.

Using (2.3), we note

$$| g_{1x}(t, x, y) - g_{2x}(t, x, y) | = 0(y^{m+1}). \tag{6.3}$$

Now differentiating the above form of Z and applying the obvious bounds, we have

$$|Z_x| \leq \frac{m\,\|\varphi'\|\,M}{a_0(m+1)}\,y + 0(y) + 0(y^2),$$

where $0(y)$ is independent of M. If we now choose M sufficiently large and y sufficiently small, and use condition (1.4), we get

$$|Z_x| \leq My$$

To prove the LIPSCHITZ continuity for Z_x, we write

$$Z_x(x,\ y) = \frac{m+1}{2}\,y^{-m}\,\{\varphi'(g_1(0,\ x,\ y))g_{1x}(0,\ x,\ y) - \varphi'(g_2(0,\ x,\ y))g_{2x}(0,\ x,\ y)$$

$$+\int_0^y [F_x(g_1(s,\ x,\ y),\ s)g_{1x}(s,\ x,\ y) - F_x(g_2(s,\ x,\ y),\ s)g_{2x}(s,\ x,\ y)]\}.$$

We shall use the symbol $0(1)$ in this paragraph only if it is independent of N. Straightforward calculations show

$$\varphi'(g_1(0,\ x,\ y)) - \varphi'(g_2(0,\ x,\ y)) \in \mathrm{Lip}\,(x;\ 0(1)y^{m+1}),$$

$$|\varphi'(g_2(0,\ x,\ y)) - \varphi'(g_2(0,\ \bar{x},\ y))|\,|g_{1x}(0,\ x,\ y) - g_{2x}(0,\ x,\ y)| \leq 0(y^{2m+2})\,|x-\bar{x}|,$$

$$g_{ix}(t,\ x,\ y) \in \mathrm{Lip}\left(x;\ \left[\frac{mN}{a_0(m+1)^2} + 0(1) + 0(y^{m+1})\right]y^{m+1}\right), \qquad i = 1,\ 2.$$

Combining the above results and using the decomposition

$$\varphi'(g_1(0,\ x,\ y))g_{1x}(0,\ x,\ y) - \varphi'(g_2(0,\ x,\ y))g_{2x}(0,\ x,\ y)$$

$$= [\varphi'(g_1(0,\ x,\ y)) - \varphi'(g_2(0,\ x,\ y))]g_{1x}(0,\ x,\ y)$$

$$+ \varphi'(g_2(0,\ x,\ y))[g_{1x}(0,\ x,\ y) - g_{2x}(0,\ x,\ y)],$$

we see this quantity is in

$$\mathrm{Lip}\left(x;\ \left[\frac{2mN\,\|\varphi'\|}{a_0(m+1)} + 0(1) + 0(y^{m+1})\right]y^{m+1}\right).$$

In a similar manner we consider the integrand

$$\begin{aligned} &F_x(g_1(s, x, y), s)g_{1x}(s, x, y) - F_x(g_2(s, x, y), s)g_{2x}(s, x, y) \\ (6.4) \qquad &= [F_x(g_1(s, x, y), s) - F_x(g_2(s, x, y), s)]g_{1x}(s, x, y) \\ &+ F_x(g_2(s, x, y), s)[g_{1x}(s, x, y) - g_{2x}(s, x, y)]. \end{aligned}$$

For brevity, let $\sigma = x, u, p$ or q, $g_i = g_i(s, x, y)$, $g_{ix} = g_{ix}(s, x, y)$, $\bar{g}_i = g_i(s, \bar{x}, y)$, $\bar{g}_{ix} = g_{ix}(s, \bar{x}, y)$ and $f_\sigma(g_i, ...) = f_\sigma(g_i, s, u_0(g_i, s), s^{m/(m+1)}z^{1/(m+1)}(g_i, s), \frac{1}{2}(u_1(g_i, s) + u_2(g_i, s)))$.

Differentiation with respect to x reveals that $f_\sigma(g_1, s) - f_\sigma(g_2, s)$ is in $\mathrm{Lip}\,(x;\ 0(y^{m+1}))$. Since

$$u_{ix}(g_1, s) - u_{ix}(g_2, s) = 2y^m[v_i(g_1; s, x, y) - v_i(g_2; s, x, y)] \qquad i = 0, 1, 2;$$

this difference is in $\mathrm{Lip}\,(x;\ 0(y^{m+1}))$, with $0(y^{m+1})$ depending on R_i. Thus the terms in f_u and f_q in $F_x(g_1, s) - F_x(g_2, s)$ are in $\mathrm{Lip}\,(x;\ 0(y^{m+1}) + 0(y^{m+2}))$. As for the terms in f_p, by the condition (1.3) and the bounds and LIPSCHITZ conditions on u_{0x} and z_x, we find they are in $\mathrm{Lip}\,(x;\ o(y^m))$, where $o(y^m)$ depends on N. Combining the above results, we have

$$F_x(g_1, s) - F_x(g_2, s) \in \mathrm{Lip}\,(x;\ o(y^m)).$$

Using (6.2)-(6.4), we see that the remaining terms of the integrand are all in $\mathrm{Lip}\,(x;\ 0(y^{m+1}))$. Thus $Z_x \in \mathrm{Lip}\,(x;\ 0(y) + o(y))$, where $o(y)$ depends on N and $0(y)$ is independent of N. Hence

$$Z_x \in \mathrm{Lip}\,(x;\ Ny)$$

for a suitable choice of N and δ.

For this section of the proof of (4.4), let $\bar{g}_i = g_i(s, x, \bar{y})$, $\bar{g}_{ix} = g_{ix}(s, x, \bar{y})$, and $y > \bar{y}$. Then

$$\begin{aligned} &| Z_x(x, y) - Z_x(x, \bar{y}) | \\ &\leq \frac{m+1}{2} y^{-m} \{ [\,| \varphi'(g_1(0, x, y))g_{1x}(0, x, y) - \varphi'(g_2(0, x, y))g_{2x}(0, x, y) \\ &- (\varphi'(g_1(0, x, \bar{y})g_{1x}(0, x, \bar{y}) - \varphi'(g_2(0, x, \bar{y}))g_2(0, x, \bar{y}))\,| \\ &+ \int_0^y | F_x(g_1, s)g_{1x} - F_x(g_2, s)g_{2x} - (F_x(\bar{g}_1, s)\bar{g}_{1x} - F_x(\bar{g}_2, s)\bar{g}_{2x})\,| \, ds \end{aligned}$$

$$+\int_{\bar{y}}^{y} | F_x(g_1, s)g_{1x} - F_x(g_2, s)g_{2x} | ds]$$

$$+ | y^{-m} - \bar{y}^{-m} | [| \varphi'(g_1(0, x, \bar{y}))g_{1x}(0, x, \bar{y}) - \varphi'(g_2(0, x, \bar{y}))g_{2x}(0, x, \bar{y}) |$$

$$+\int_{0}^{\bar{y}} | F_x(\bar{g}_1, s)\bar{g}_{1x} - F_x(\bar{g}_2, s)\bar{g}_{2x} | ds] |.$$

Applying an analogous argument as was used for the φ'-terms for the x-LIPSCHITZ condition for Z_x, $|g_{iy}| = 0(y^m)$, $g_{ix} \in \mathrm{Lip}(y;\ 0(y^m))$, (6.2), and (6.3), we get

$$\varphi'(g_1(0, x, y))g_{1x}(0, x, y) - \varphi'(g_2(0, x, y))g_{2x}(0, x, y)) \in \mathrm{Lip}\,(y;\ 0(y^m)),$$

$f_\sigma(g_1, ...) - f_\sigma(g_2, ..) \in \mathrm{Lip}\,(y;\ 0(y^m))$, and hence

$$F_x(g_1, s)g_{1x} - F_x(g_2, s)g_{2x} \in \mathrm{Lip}\,(y;\ 0(y^m)).$$

Since

$$| F_x(g_1, s)g_{1x} - F_x(g_2, s)g_{2x} | \leq 0(y^{m+1}),$$

the last part of the expression $|Z_x(x, y) - Z_x(x, \bar{y})|$ is bounded by $0(1)|y - \bar{y}|$, with $0(1)$ independent of C. Combining the above results, we have

$$Z_x \in \mathrm{Lip}\,(y;\ C),$$

for C sufficiently large.

PROOF. OF. (4.5). - The bounds of (4.5) follow immediately from the LIPSCHITZ continuity of U_{ix} and Lemma 1. Since solutions of (1.6) through a given point are unique (for fixed j), we have

$$h_j(s, x, y) = h_j(s, h_j(t, x, y), t), \qquad 0 \leq s \leq t \leq y,$$

and therefore we can express $V_0(h_j)$ in the following form.

$$V_0(h_j) = \frac{1}{2} y^{-m} \int_0^t \sum_{i=1}^{2} [y^m v_i(h_j;\ s, x, y) - t^m v_i(h_j;\ s, h_j(t, x, y), t)]ds. \tag{6.5}$$

This integral is in $\mathrm{Lip}\,(x;\ 0(y^2))$ by the LIPSCHITZ conditions for v_i, $i = 1, 2$. Thus

$$V_0(h_j) \in \mathrm{Lip}\,(x;\ R_0 y^2),$$

for R_0 sufficiently large. The t-LIPSCHITZ condition for $V_0(h_j)$ follows in an analogous manner. The y-LIPSCHITZ condition for $V_0(h_j)$ is a simple consequence of the conditions on U_{0x} and the fact $h_{jy} = 0(y^m)$.

The proof of the x-LIPSCHITZ continuity of $V_i(h_j)$, $i = 1, 2$, is similar to the proof of $Z_x \in \mathrm{Lip}\,(x;\ Ny)$ and depends on the decomposition

$$
\begin{aligned}
(6.6)\qquad & F_x(g_i[s,\ h_j(t,\ x,\ y),\ t],\ s) - F_x(g_i(s,\ x,\ t),\ s) \\
&= [F_x(g_i[s,\ h_j(t,\ x,\ y),\ t],\ s) - F_x(h_j(t,\ x,\ y),\ s)] \\
&- [F_x(h_j[s,\ h_j(t,\ x,\ y),\ t],\ s) - F_x(h_j(t,\ x,\ y),\ s)] \\
&+ [F_x(h_j(s,\ x,\ y),\ s) - F_x(x,\ s)] - [F_x(g_i(s,\ x,\ t),\ s) - F_x(x,\ s)]
\end{aligned}
$$

Now

$$
\begin{aligned}
V_i(h_j;\ t,\ x,\ y) = \frac{1}{2} y^{-m} \{ & \varphi'(g_i(0,\ h_j(t,\ x,\ y),\ t))g_{ix}(0,\ h_j(t,\ x,\ y),\ t) \\
& - \varphi'(g_i(0,\ x,\ t))g_{ix}(0,\ x,\ t) \\
& + \int_0^t [F_x(g_i[s,\ h_j(t,\ x,\ y),\ t],\ s)g_{ix}(s,\ h_j(t,\ x,\ y),\ t) \\
& - F_x(g_i(s,\ x,\ t),\ s)g_{ix}(s,\ x,\ t)]ds \}.
\end{aligned}
$$

The portion of this expression involving φ' is in $\mathrm{Lip}\,(x;\ 0(y^{m+1}))$ by an argument analogous to the one for the corresponding quantity in Z_x. Each of the terms in brackets in the decomposition (6.6) is in $\mathrm{Lip}\,(x;\ 0(y^{m+1}) + o(s^m))$ by an argument similar to the one used on the integrand quantity for Z_x. Thus

$$
|\ V_i(h_j;\ t,\ x,\ y) - V_i(h_j;\ t,\ \bar{x},\ y)\ | \leq [O(y) + o(y)]\ |\ x - \bar{x}\ |,
$$

where $O(y)$ is independent of N and R_i. If we choose y small enough and R_i sufficiently large, then

$$
V_i(h_j) \in \mathrm{Lip}\,(x;\ R_i y), \qquad i = 1,\ 2.
$$

Using $g_{iy} \in \mathrm{Lip}\,(x;\ O(y^m))$, methods analogous to those used to prove the LIPSCHITZ continuity of Z_x, and $V_i(h_j) \in \mathrm{Lip}\,(x;\ R_i y)$, we see

$$
V_i(h_j) \in \mathrm{Lip}\,(y,\ t;\ C_i), \qquad i = 1,\ 2,
$$

for C_i sufficiently large and y sufficiently small.

This completes the proof that T maps X_δ into itself.

PROOF. OF CONTINUITY. - We now need to demonstrate that T is continuous. Let $T\underline{u} = \underline{U} = (U_0, U_1, U_2)$ and $T\tilde{\underline{u}} = \tilde{\underline{U}} = (\tilde{U}_0, \tilde{U}_1, \tilde{U}_2)$. Recall

$$\|\underline{U}\| = \max.(\|U_i\|, \|U_{ix}\|, \|Z\|, \|Z_x\|, \|V_i(h_j)\|) \qquad i = 0, 1, 2; \; j = 1, 2.$$

It is easily seen that

$$|U_0(x, y) - \tilde{U}_0(x, y)| \leq y \|\underline{u} - \tilde{\underline{u}}\|,$$

and that the same inequality holds for $U_{0x} - \tilde{U}_{0x}$.

Let $g_i = g_i(s, x, y)$, $\tilde{g}_i = \tilde{g}_i(s, x, y)$ and

$$\tilde{F}(x, y) = f(x, y, \tilde{u}_0(x, y), y^{m/(m+1)}\tilde{g}^{1/(m+1)}(x, y), \frac{1}{2}(\tilde{u}_1(x, y) + \tilde{u}_2(x, y)))$$

Using Lemma 2, we get

$$|F(g_i, s) - \tilde{F}(\tilde{g}_i, s)| \leq 0(1) \|\underline{u} - \tilde{\underline{u}}\|.$$

From this immediately follows

$$|U_i - \tilde{U}_i| \leq [O(y^m) + O(y)] \|\underline{u} - \tilde{\underline{u}}\|, \qquad i = 1, 2.$$

In an analogous fashion, we see that the same inequality holds for $|U_{ix} - \tilde{U}_{ix}|$.

Using Lemma 2 and the relation

$$|F(g_1, s) - F(g_2, s) - [\tilde{F}(\tilde{g}_1, s) - \tilde{F}(\tilde{g}_2, s)]| \leq O(y^m) \|\underline{u} - \tilde{\underline{u}}\|,$$

it readily follows that

$$|Z - \tilde{Z}| \leq 0(1) \|\underline{u} - \tilde{\underline{u}}\|.$$

To prove the continuity for Z_x, we observe

$$\begin{aligned} |Z_x - \tilde{Z}_x| \leq \frac{m+1}{2} y^{-m} \{ & |[\varphi'(g_2(o, x, y)) - \varphi'(g_2(o, x, y))]g_{1x}(o, x, y) \\ & + \varphi'(g_2(o, x, y))[g_{1x}(o, x, y) - g_{2x}(o, x, y)] \\ & - [\varphi'(g_1(o, x, y)) - \varphi'(\tilde{g}_2(o, x, y))]\tilde{g}_{1x}(o, x, y) \\ & - \varphi'(\tilde{g}_2(o, x, y))[\tilde{g}_{1x}(o, x, y) - \tilde{g}_{2x}(o, x, y)]| \\ & + \int_0^y |[F_x(g_1, s) - F_x(g_2, s)]g_{1x} + F_x(g_2, s)[g_{1x} - g_{2x}] \\ & - [\tilde{F}_x(\tilde{g}_1, s) - \tilde{F}_x(\tilde{g}_2, s)]\tilde{g}_{1x} - \tilde{F}_x(\tilde{g}_2, s)[\tilde{g}_{1x} - \tilde{g}_{2x}]| \, ds \}. \end{aligned}$$

Applying Lemma 2 to the expressions outside the integral, shows that this quantity is less than or equal to $0(1) \| \underline{u} - \underline{\tilde{u}} \|$. To get the bounds for the integrand, we shall first prove

$$| [F_x(g_1, s) - F_x(g_2, s)] g_{1x} - [\tilde{F}_x(\tilde{g}_1, s) - \tilde{F}_x(\tilde{g}_2, s)] \tilde{g}_{1x} | \\ \leq [O(y^m) + o(s^{m-1})] \| \underline{u} - \underline{\tilde{u}} \|.$$

Writing

$$| [f_\sigma(g_1, ...) - f_\sigma(g_2, ...)] - [\tilde{f}_\sigma(\tilde{g}_1, ...) - \tilde{f}_\sigma(\tilde{g}_2, ...)] |$$

as a difference of integrals, we find it is of order $O(y^m) \| \underline{u} - \underline{\tilde{u}} \|$, $\sigma = x$, u, p, or q. Using this result, the conditions on v_0 and the decomposition

$$f_u(g_1, ...) u_{0x}(g_1, s) - f_u(g_2, ...) u_{0x}(g_2, s) \\ = [f_u(g_1, ...) - f_u(g_2, ...)] u_{0x}(g_1, s) \\ + 2y^m f_u(g_2, ...) [v_0(g_1) - v_0(g_2)],$$

we have that the difference between this expression in f_u and the corresponding one in $\tilde{u}$ is of order $0(y^m) \| \underline{u} - \underline{\tilde{u}} \|$. In a similar manner the difference of terms with f_q and $\tilde{f}_q$ are of the same order. Finally, the difference of terms with f_p and $\tilde{f}_p$ are bounded by $[O(y^m) + o(s^{m-1})] \| \underline{u} - \underline{\tilde{u}} \|$ by virtue of condition (1.3). Using the fact that the other terms of the integrand are bounded by $O(y^m) \| \underline{u} - \underline{\tilde{u}} \|$, (6.7), and the result for the terms outside the integral, we get

$$| Z_x - \tilde{Z}_x | \leq 0(1) \| \underline{u} - \underline{\tilde{u}} \|.$$

From (6.5), the properties of the $v_i(h_j)$ and Lemma 2, it follows that

$$| V_0(h_j) - \tilde{V}_0(\tilde{h}_j) | \leq O(y) \| \underline{u} - \underline{\tilde{u}} \|.$$

Now

$$| V_i(h_j) - \tilde{V}_i(\tilde{h}_j) | \leq 0(1) \| \underline{u} - \underline{\tilde{u}} \|, \qquad i = 1, 2; \; j = 1, 2,$$

follow by means of the decomposition (6.6) and application of the techniques used to prove the continuity for Z_x.

This completes the proof that T is continuous mapping of X_δ into itself.

7. Application of Schaudar's Theorem.

By conditions (4.2)-(4.5) imposed on the functions in X_δ and the definition of the norm on X_δ, we observe that X_δ is convex and that the functions in X_δ are uniformly bounded and LIPSCHITZ continuous with respect to x, y, and t. Since the functions in X_δ are uniformly LIPSCHITZ continuous, X_δ is a uniformly equicontinuous family of functions. Therefore, by ARZALA's Theorem, X_δ is a convex, compact subset of the BANACH space S_δ.

The existencs of a a solution of the integral equations (3.6) is established by means of the following form of SCHAUDER's Fixed Point Theorem [6]:

A continuous mapping of a convex, compact subset of a BANACH space into itself has a fixed point.

Therefore, under conditions (1.3) and (1.4), our integral equations (3.6) have a fixed point and so, by Theorem 2, our original CAUCHY problem for (1.1), (1.2) has a solution.

This completes the proof of Theorem 1.

8. The theorem.

The theorem we desire to obtain is Theorem 1 without condition (1.4) in the hypotheses and with uniqueness of solution. It is as follows.

THEOREM 4. - Under the condition

$$f_p(x,\ y,\ u,\ p,\ q) = o(y^{m-1}) \quad \text{as } y \longrightarrow 0,$$

there exists a $\delta > 0$, such that on D_δ the singular CAUCHY problem for (1.1), (1.2) has a unique solution $u(x, y)$ which is twice differentiable with LIPSCHITZ continuous second derivatives on $\bar{D}_\delta$, and satisfies the conditions of (1.5).

PROOF. - The uniqueness of the solution of Theorem 1 follows from a paper by the author [2]. The restriction

$$\frac{A_0}{a_0} < \frac{m+1}{m}$$

can be removed by breaking $I = [\alpha, \beta]$ into a finite number of overlapping closed subintervals $I_i = [\alpha_i, \beta_i]$ $(i = 1, 2, \ldots, n;\ \alpha_1 = \alpha$ and $\beta_n = \beta)$, on each of which this condition is satisfied. Applying Theorem 1 and the above mentioned uniqueness of solution to each subinterval, there exists a unique solution u_i of the problem with domain D_{δ_i}, where D_{δ_i} is the open region bounded by I_i, $y = \delta_i$ and the curves

$$x = \alpha_i + \frac{1}{m+1} A_0^m y^{m+1}, \qquad x = \beta_i - \frac{1}{m+1} A_0^m y^{m+1}.$$

The function $u(x, y)$ given by

$$u(x, y) = u_i(x, y), \qquad (x, y) \in \bar{D}_\delta \cap \bar{D}_{\delta_i},$$

where

$$\delta = \min_i \delta_i$$

is then the desired solution on D_δ.

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