On the instability theory of differential polynomials [1].

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Summary. - *In this paper a class of n^{th} order non-linear differential equations is treated and solutions are sought which are asymptotically equivalent to logarithmic monomials.*

PART I - Preliminaries.

1. INTRODUCTION - In [5, 6], W. STRODT investigated the problem of finding those solutions of an *nth* order non-linear ordinary differential equation, which are of minimal rate of growth at a singular point at ∞, and furthermore are asymptotically equivalent ($\sim$) to logarithmic monomials (i.e. functions of the form $M(x) = Kx^{\alpha_0}(\log x)^{\alpha_1}(\log\log x)^{\alpha_2} \ldots (\log_p x)^{\alpha_p}$, for real α_j and non-zero complex K), as $x \to \infty$.

In this paper, we investigate the problem of finding *all* solutions of the equation which are asymptotically equivalent to logarithmic monomials. The class of equations treated in [5, 6] and in here, consists of equations $\Omega(y) = 0$, where Ω is a polynomial in an unknown function y and its derivatives, whose coefficients are functions defined and analytic in an unbounded region of the complex plane, and where, as $x \to \infty$, each coefficient has an asymptotic expansion in terms of logarithmic monomials and/or functions (called trivial) which are asymptotically smaller ($<$) than all powers of x. (For the rigorous concepts of «$<$» and «$\sim$», see [5, §§ 12-13]).

In [5, § 66], it was shown that Ω determines a finite set (denoted $pm(\Omega)$) of logarithmic monomials, M (called principal monomials) which are «approximate solutions» (i.e. $\Omega(M) < \Omega(0)$) and among all approximate solutions are of minimal rate of growth at ∞. These properties are shared by those exact solutions (called principal solutions) of $\Omega(y) = 0$ which are $\sim$ to principal monomials. An algorithm which produces $pm(\Omega)$ in a number of steps which

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can be bounded in advance was introduced in [5, § 66], and existence and uniqueness theorems for principal solutions were established in [5, § 127] and [6, § 122].

If $\Omega(y)=0$ possesses a solution $\backsim$ to a logarithmic monomial M (not necessarily a principal monomial), then at M, Ω must satisfy (see § 5(*c*)) a condition called ***instability***, which was introduced in [9, 10], and which means that for some function $f \backsim M$, $\Omega(f)$ is not $\backsim \Omega(M)$. Furthermore, an equivalent definition of instability (§ 3(*b*)) hints at the existence of solutions $\backsim$ to those monomials at which Ω is unstable. For these reasons, the concept of instability is chosen as our starting point, and we investigate ***all*** the logarithmic monomials (called ***critical***) at which Ω is unstable. (The problem of solutions is taken up in Parts VII-IX). The expected result that $pm(\Omega)$ constitutes the set of minimal critical monomials concludes Part II.

Methods for finding the critical monomials of Ω are developed. Two methods are required. One is for finding those critical monomials M, (called *parametric*) such that every constant multiple of M is also critical. The second method is for finding the non-parametric critical monomials (among which are included as a special case all the principal monomials). Both methods are of an algorithmic nature, and use the same basic principle as the algorithm for $pm(\Omega)$, namely repeated application of the change of variables $x=e^u$, $y=ve^{\alpha u}$, where α is a real number determined at each stage. When followed by multiplication by a suitable power of e^u, this change of variables transforms $\Omega(y)$ into a differential polynomial in v (denoted $[\alpha;\Omega]$), which again belongs to the class we are considering. Part III is devoted to the study of this, and the successive transforms $[\beta;[\alpha;\Omega]]$ etc. Their crucial property (§ 11) is that $M(x)=Kx^{\alpha_0}(\log x)^{\alpha_1}(\log_2 x)^{\alpha_2}\ldots$ is critical of Ω if and only if $N(u)=Ku^{\alpha_1}(\log u)^{\alpha_2}\ldots$ is critical of $[\alpha_0;\Omega]$. Hence if α_0 is known to satisfy a certain condition C, when M is critical of Ω, then α_1 satisfies C relative to $[\alpha_0;\Omega]$, and so on for $\alpha_2, \alpha_3, \ldots$. Both methods use this algorithmic property, and [5, § 61] (which is here strengthened and incorporated into § 13), is used to show that the process can be stopped at a predetermined point, and the conditions C are sufficient also.

Part IV is devoted to the method for parametric monomials. It is first shown (§ 15) that a necessary (but certainly not sufficient) condition for M to be parametric of Ω, is that it be parametric of at least one homogeneous part of Ω. For the moment, we focus our attention on finding the parametric monomials when Ω is homogeneous (§ 19). In this case, condition C takes a simple form, namely that α_0 be a root of an algebraic equation, which resembles the indicial equation at ∞ (see [4, § 161]) in the case of linear equations. When Ω is non-homogeneous (§ 21), our condition C is phrased in such a way that we are examining each homogeneous part of Ω for parametric monomials (using the method already developed in § 19), while simultaneously examining the behavior of the rest of Ω to determine if the parametric monomial produced by a homogeneous part will actually be parametric of the *whole* polynomial, Ω. The method in § 21 produces *each* parametric monomial in a number of steps which can be bounded in advance, but except in the case of linear or first order Ω, the number of steps required to produce the *set* of parametric monomials *may* be infinite (see § 17, Remark (2)).

Part V is devoted to the method for non-parametric critical monomials. Here condition C takes a form similar to that for the algorithm for $pm(\Omega)$, namely that α_0 should be the slope of a side of a NEWTON polygon. The resulting method (§ 26) produces the set of non-parametric critical monomials in a number of steps which can can be bounded in advance. (A simple example illustrating both methods is given in Part X).

Since we are ultimately interested in solutions of $\Omega(y) = 0$ which are $\sim$ to critical monomials M, and since the existence of such a solution is clearly equivalent to the existence of a solution < 1 of the equation $A(z) = 0$, which is obtained from $\Omega(y) = 0$ by the change of variables $y = M + Mz$, it is of importance to investigate such critical monomials of Λ as are < 1. This is done in Part VI (§§ 31, 33), and use is made of these results in Part VII.

Parts VII through IX are devoted to existence theorems for solutions $\sim M$ of $\Omega(y) = 0$. Here the coefficients of Ω are assumed to be defined and analytic in a sectorial region (more specifically, in an element of an $F(a, b)$, as defined in [5, § 94]), and the solutions obtained are of the same type.

In Part VII (§§ 36, 38, 39), the result obtained by STRODT in [7] (see § 35), is used to obtain solutions in certain first order cases, when the coefficients of Ω are of the type considered in [7].

In Part VIII, non-parametric critical monomials M, of an *nth* order Ω are considered. It is shown (§ 40) that when M and Ω satisfy the general conditions analogous to those for principal monomials in [5, § 85] (when $n=1$) or [6, § 116] (when $n > 1$), then under the change of variable $y = M + Mz$, $\Omega(y)$ is transformed into a differential polynomial to which [5, § 126] (when $n = 1$) or [6, § 115] (when $n > 1$), can be applied, thus obtaining solutions $\sim M$. These results are given in §§ 44-45.

Part IX concerns critical monomials of $\Omega(y) = \Phi(y) - g$, where Φ is an *nth* order linear differential polynomial whose coefficients, along with g, have asymptotic expansions in therms of real (but not necessarily integral) powers of x, and/or trivial functions. In [8], it was shown that for such an Ω (in the case where it possesses a principal monomial), the equation $\Omega(y) = 0$ has at least one principal solution. We utilize this, and other results in [8], to prove (§ 45) that corresponding to *any* critical monomial M, of Ω, the equation $\Omega(y) = 0$ has at least one solution $\sim M$. The connection between this and the FUCHS regularity theorem ([2, p. 143], and [3, p. 358], or 4, p. 365]), will be explored in a future paper.

2. UNIFORM HYPOTHESES

(*a*) M is a logarithmic monomial.

(*b*) $n \in \{0, 1, 2, \ldots\}$

(*c*) $W \in \{0, 1, 2, \ldots\}$

(*d*) $r \in \{-1, 0, 1, 2, \ldots\}$

(*e*) $S^{\#}$ is a complex neighborhood system of ∞ as defined in [5, § 3]. (That is, $S^{\#}$ is a filter base which converges to ∞ in the sense of [1, § 6], and which consists of unbounded regions, each disjoint from the non-positive real axis. The concept of asymptotic equivalence as $x \to \infty$, which we employ ([5, § 13]), is defined relative to such a filter base, and explicit mention of $S^{\#}$ will be omitted when no confusion is possible).

(*f*) Ω is an *nth* order differential polynomial in an unknown function y (that is, a polynomial in $y, dy/dx, \ldots, d^n y/dx^n$), whose coefficients are functions of x which belong to a logarithmic domain of rank r over $S^{\#}$ (briefly, an $LD_r(S^{\#})$), as defined in [5, § 49]. This condition ensures that each coefficient of Ω is either $\sim$ to a logarithmic monomial in $S^{\#}$ or is trivial in $S^{\#}$, and further ensures that under either change of variable, $y = M + z$ or $y = Mz$, $\Omega(y)$ is transformed into a differential polynomial whose coefficients again belong to a logarithmic domain (and therefore can be treated by our methods).

(*g*) At least one term in Ω is to have a non-trivial coefficient (briefly, we then say Ω is *non-trivial*). If we require that at least one term of *positive degree* in the indeterminates have a non-trivial coefficient, we will indicate this by the abbreviation *NTPD* (non-trivially of positive degree).

(*h*) W is the maximum of the weights of all terms in Ω, which have non-trivial coefficients.

PART II - **Critical Monomials.**

3. LEMMA - Assume § 2 and let Ω be *NTPD*. Then the following two conditions are equivalent:

(*a*) Ω is unstable at M.

(*b*) Either $\Omega(M)$ is trivial, or some $P \in pm(\Omega(M+z))$ is $< M$.

PROOF - Let $\Lambda(z) = \Omega(M+z)$. If (*b*) does not hold, there exists $P \in pm(\Omega(M+z))$ with $M \lesssim P$, (that is, $M < P$ or $M \sim kP$ for some non-zero constant k). Hence $g < M$ implies $g < P$, and therefore, by the properties of a principal monomial ([5, § 66]), $\Lambda(g) \sim \Lambda(0)$. Thus (*a*) does not hold.

Conversely, suppose (b) holds but (a) does not. Then $\Omega(M)$ must be trivial, for in the contrary case, Λ would have a principal monomial, $P < M$, and (a) would hold since $\Omega(M+P) < \Omega(M)$. Hence $\Lambda(V)$ is trivial for any $V < M$. If we chose a real number q so small that a principal monomial N, of $\Phi(z) = \Lambda(z) - x^q$ is $< M$, then $\Phi(N) \sim \Phi(0)$. This contradicts the definition of principal monomial, so (a) must hold.

4. DEFINITION - Assuming § 2 with Ω *NTPD* we say M is a *critical monomial* of Ω, if M and Ω satisfy either (and hence both) conditions of Lemma 3. The set of all critical monomials of Ω is denoted crit (Ω).

5. LEMMA - Assume § 2 with *NTPD*. Then under *any* of the following conditions, $M \in \operatorname{crit}(\Omega)$.

(a) There exist a constant c, and a function $g \sim M$ such that $\Omega(g) < \Omega(cM)$.

(b) There exists a function $h \sim M$ which is an *approximate* solution of Ω (i.e. $\Omega(h) < \Omega(0)$ if $\Omega(0) \neq 0$, and $\Omega(h) = 0$ if $\Omega(0) = 0$).

(c) There exists an *exact* solution of $\Omega(y) = 0$, which is $\sim M$.

(d) $M \in pm(\Omega)$.

PROOF - (a) Assume $M \notin \operatorname{crit}(\Omega)$. Then there exists $N \in pm(\Omega(M+z))$ with $M \lesssim N$. Since $g - M < M$, $g - M < N$. Thus $\Omega(g) \sim \Omega(M)$. Therefore, by hypothesis, $\Omega(M) < \Omega(cM)$. But the contradictory relation $\Omega(cM) \lesssim \Omega(M)$ follows from the fact that $(c-1)M \lesssim N$, and $N \in pm(\Omega(M+z))$, (see [5, § 67]), thus proving the result for (a).

(b) If $\Omega(0) \neq 0$, then (b) follows from (a), by taking $c = 0$. If $\Omega(0) = 0$, but $M \notin \operatorname{crit}(\Omega)$, then $\Omega(M)$ is non-trivial and therefore $\Omega(h) < \Omega(M)$. But then $M \in \operatorname{crit}(\Omega)$, by taking $c = 1$ in (a). This contradiction establishes the result for (b).

(c) and (d) follow from (b).

6. LEMMA - Assume § 2 with Ω *NTPD*, and let $\Omega(0)$ be non-trivial. Then,

(a) If $N \in pm(\Omega)$, while $M \in (\operatorname{crit}(\Omega) - pm(\Omega))$, then $N < M$.

(b) $pm(\Omega)$ constitutes the set of minimal elements (relative to « $<$ ») of crit (Ω).

PROOF - It obviously suffices to prove (a). If N were not $< M$, then $M \lesssim N$. Since $M \notin pm(\Omega)$, $\Omega(M) \approx \Omega(0)$. Thus $\Omega(M)$ is non-trivial, and therefore $\Omega(M+z)$ has a principal monomial, G, with $G < M$. Hence $\Omega(M+G) < \Omega(M)$. But $M + G \sim M$, and therefore $M + G$ is not $\sim$ to any element of $pm(\Omega)$. Thus $\Omega(M+G) \approx \Omega(0)$, so $\Omega(0) < \Omega(M)$. This contradicts the relation $\Omega(M) \approx \Omega(0)$, previously established, thus proving (a).

PART III - **The transform** $[\alpha; \Omega]$.

7. NOTATION - Assume § 2.

(*a*) If $i^{\#} = (i_0, i_1, \ldots, i_n)$ is an $(n+1)$-tuple of natural numbers, then the coefficient of $y^{i_0}(y')^{i_1} \ldots (y^{(n)})^{i_n}$ in Ω is denoted $\Omega[i^{\#}]$, and as in [5, § 62], the degree $i_0 + i_1 + \ldots + i_n$ and the weight $i_1 + 2i_2 + \ldots + ni_n$ of $i^{\#}$, will be denoted by $d(i^{\#})$ and $w(i^{\#})$ respectively.

(*b*) If α is a real number, then by $\Omega(i^{\#}, \alpha]$, we will mean the quantity $\alpha d(i^{\#}) + \delta_0(\Omega[i^{\#}]) - w(i^{\#})$, where as in [5, § 23-24], $\delta_0(\Omega[i^{\#}])$ is $-\infty$ if $\Omega[i^{\#}]$ is trivial, while in the non-trivial case, it is the exponent of x in the logarithmic monomial to which $\Omega[i^{\#}]$ is asymptotically equivalent. (In general, $\delta_j(\ )$ is the exponent of $\log_j x$). $\Omega[*, \alpha]$ will denote the maximum, over all $i^{\#}$, of the numbers $\Omega[i^{\#}, \alpha]$.

(*c*) If $\Phi(v)$ is the polynomial in v, dv/du, ..., $d^n v/du^n$, obtained from $\Omega(y)$ by the change of variables $x = e^u$, $y = ve^{\alpha u}$, then the differential polynomial $\exp(-\Omega[*, \alpha]u)\Phi(v)$ is denoted $[\alpha; \Omega](v)$.

(*d*) If p is a natural number, we denote by $\Omega^{(p)}$, the sum of all terms in Ω which are of degree p in the indeterminates $y, y', \ldots, y^{(n)}$ (that is, $\Omega^{(p)}$ is the homogeneous part of total degree p of Ω). As usual, Ω will be called *homogeneous of degree p* if $\Omega = \Omega^{(p)}$, and simply, *homogeneous,* if it is homogeneous of some degree.

8. LEMMA - Assume § 2 and let α be a real number. Then,

(*a*) $[\alpha; \Omega]$ has coefficients in an LD_t (where $t = \max\{r-1, -1\}$), over the complex neighborhood system $\log S^{\#}$, defined in [5, § 8].

(*b*) $[\alpha, \Omega]$ is non-trivial in $\log S^{\#}$.

(*c*) If Ω is homogeneous of degree p, then so is $[\alpha; \Omega]$.

(*d*) $\mathrm{Max}\{w(i^{\#}): [\alpha; \Omega][i^{\#}]$ is non-trivial$\} \leq W$.

(*e*) If $p \geq 0$ and $\Omega^{(p)}[*, \alpha] < \Omega[*, \alpha]$, then all the coefficients of $[\alpha; \Omega]^{(p)}$ are trivial in $\log S^{\#}$.

(f) If $p \geq 0$ and $\Omega^{(p)}[*, \alpha] = \Omega[*, \alpha]$, then $\Omega^{(p)}$ is non-trivial in $S^{\#}$ and $[\alpha; \Omega]^{(p)} = [\alpha; \Omega^{(p)}]$.

PROOF - Under the change of variables $x = e^u$, $y = ve^{\alpha u}$, it is clear that $y^{(q)}$ becomes $F_q(v)e^{(\alpha-q)u}$, where $F_q(v)$ is a homogeneous linear polynomial in $v, v', \ldots, v^{(q)}$ with constant coefficients. Thus each coefficient of $[\alpha; \Omega]$ is a linear combination of functions of the form $g(i^{\#}, u) = \Omega[i^{\#}](e^u) \exp[(\alpha d(i^{\#}) - - w(i^{\#}) - \Omega[*, \alpha])u]$. If E^* is an $LD_r(S^{\#})$ which contains all the coefficients of Ω, then the coefficients of $[\alpha; \Omega]$ lie in the set $\log E^*$ (defined in [5, § 51]), which is an LD_t over $\log S^\#$. This follows because $\log E^*$ is the complex vector space generated by all functions which are either trivial in $\log S^\#$ or are of the form $h(e^u)\exp(-\delta_0(h)u)$, where h is a non-trivial element of E^*. If $\Omega[i^\#, \alpha] = \Omega[*, \alpha]$, then $g(i^\#, u)$ has this latter form, while $g(i^\#, u)$ is trivial if $\Omega[i^\#, \alpha] < \Omega[*, \alpha]$. This proves (a).

To prove (b), let $k^\#$ be the smallest $i^\#$ (relative to the lexicographic order) for which $\Omega[k^\#, \alpha] = \Omega[*, \alpha]$. Then $[\alpha; \Omega][k^\#] = g(k^\#, u) + f(u)$, where f is trivial, so $[\alpha; \Omega][k^\#]$ is non-trivial, proving (b).

Part (c) is clear, since each F_q is homogeneous and linear.

As seen in the proof of (a), each $[\alpha; \Omega][j^\#]$ is a liner combination of the functions $g(i^\#, u)$, and it is a routine computation to verify that $w(i^\#) \geq w(j^\#)$ and $d(i^\#) = d)j^\#)$ for each $g(i^\#, u)$ appearing non-trivially in this combination. Hence if $w(j^\#) > W$, then $[\alpha; \Omega][j^\#]$ is trivial, proving (d). If $\Omega^{(p)}[*, \alpha] < \Omega[*, \alpha]$, then $g(i^\#, u)$ is trivial if $d(i^\#) = p$, so all coefficients of terms of degree p in $[\alpha; \Omega]$ are also trivial, proving (e).

Finally, to prove (f), if $\Omega^{(p)}[*, \alpha] = \Omega[*, \alpha]$, then $\Omega^{(p)}[*, \alpha]$ is not $-\infty$ and so $\Omega^{(p)}$ is non-trivial. The relation $[\alpha; \Omega]^{(p)} = [\alpha; \Omega^{(p)}]$ follows easily, since $[\alpha; \Omega]^{(p)}$ and $[\alpha; \Omega^{(p)}]$ differ only by the multiplicative factor $\exp[(\Omega^{(p)}[*, \alpha] - \Omega[*, \alpha])u]$.

9. NOTATION - Assume § 2 and let $\alpha_0, \alpha_1, \ldots$ be a sequence of real numbers. By induction on Lemma 8, (a) and (b), the polynomial $[\alpha_i; [\alpha_{i-1}, \ldots, \alpha_0; \Omega]]$ is defined for all $i \geq 1$, and we denote it by $[\alpha_i, \alpha_{i-1}, \ldots, \alpha_0; \Omega]$. (For consistency, we let $[\alpha_{i-1}, \ldots, \alpha_0; \Omega]$ stand for Ω when $i = 0$).

If M is given, then $[M, i, \Omega]$ will stand for $[\delta_{i-1}(M), \ldots, \delta_0(M); \Omega]$.

10. LEMMA - Assume § 2. Let i and p be natural numbers, and let $\alpha_0, \alpha_1, \ldots, \alpha_i$ be real numbers. For each j, $0 \leq j \leq i+1$, let $\Omega_j = [\alpha_{j-1}, \ldots, \alpha_0; \Omega]$. Then the following conditions are equivalent.

(a) $(\Omega_{i+1})^{(p)}$ is non-trivial in $\log_{i+1} S^\#$.

(b) $(\Omega_j)^{(p)}[*, \alpha_j] = \Omega_j[*, \alpha_j]$ for each j, $0 \leq j \leq i$.

(c) $\Omega^{(p)}$ is non-trivial and $(\Omega_j)^{(p)} = [\alpha_{j-1}, \ldots, \alpha_0; \Omega^{(p)}]$ for each j, $0 \leq j \leq i+1$.

(d) $(\Omega_j)^{(p)}$ is non-trivial in $\log_j S^\#$ for each j, $0 \leq j \leq i+1$.

PROOF - (a) implies (b) by Lemma 8 (e).

(b) implies (c) by Lemma 8 (f).

(c) implies (d) by Lemma 8 (b).

(d) clearly implies (a).

11. LEMMA - Assume § 2 with Ω *NTPD*. Then,

(*a*) If $M \in \text{crit}(\Omega)$ with $\delta_0(M) = \alpha$, then $[\alpha; \Omega]$ is *NTPD* and $M_1(u) = e^{-\alpha u} M(e^u)$ is critical of $[\alpha; \Omega]$.

(*b*) If for some real number ν, $[\nu; \Omega]$ is *NTPD* and $N \in \text{crit}[\nu; \Omega]$, then $G(x) = x^\nu N(\log x)$ is critical of Ω.

PROOF - Both parts are proved using [5, § 19(*d*), (*e*)] which states that an asymptotic equivalence holds in $S^\#$ if and only if under the change of variable $x = e^u$, it holds in $\log S^\#$. To prove (*a*), we first show $[\alpha; \Omega]$ is unstable at M_1. Assume the contrary and let $h \backsim M$ in $S^\#$. Hence $h_1(u) = e^{-\alpha u} h(e^u) \backsim M_1(u)$ in $\log S^\#$. Therefore, $[\alpha; \Omega](h_1(u)) \backsim [\alpha; \Omega](M_1(u))$ in $\log S^\#$. This relation then holds in $S^\#$ (relative to x) when $u = \log x$. But using the definition of $[\alpha; \Omega]$, this implies $\Omega(h) \backsim \Omega(M)$ in $S^\#$, contradicting $M \in \text{crit}\, \Omega$. Thus $[\alpha; \Omega]$ is unstable at M_1. If $[\alpha; \Omega]$ were not *NTPD*, then by Lemma 8(*b*), only the term of degree zero in $[\alpha; \Omega]$ would be non-trivial, and this would imply the stability of $[\alpha; \Omega]$ at every logarithmic monomial and hence at M_1. This contradiction establishes that $[\alpha; \Omega]$ is *NTPD* and $M_1 \in \text{crit}\, [\alpha; \Omega]$, proving (*a*).

(*b*) is proved similarly by assuming G is not critical of Ω, and showing this would imply N is not critical of $[\alpha; \Omega]$.

12. LEMMA - Assume § 2 with Ω *NTPD*. Let $\alpha_0, \alpha_1, \ldots, \alpha_{s-1}$ be real numbers, where $s \geq r + 1$ and let $\Omega_s = [\alpha_{s-1}, \ldots, \alpha_0; \Omega]$. Then

(*a*) $\Omega_s = Q_s + R_s$ where Q_s is a non-zero differential polynomial with constant coefficients, while R_s has only trivial coefficients in $\log_s S^\#$. If Ω is homogeneous of degree p, so are Q_s and R_s.

(*b*) If k is a non-zero constant, then $Q_s(k) = 0$ if and only if $N(x) = k x^{\alpha_0} (\log x)^{\alpha_1} \ldots (\log_{s-1} x)^{\alpha_{s-1}}$ is critical of Ω.

PROOF - By Lemma 9(*a*) and [5, §§ 53-54], the coefficients of Ω_s lie in an LD_{-1} over $\log_s S^\#$, and hence each is of the form $c + T$ where c is a constant and T is trivial in $\log_s S^\#$. Part (*a*) now follows immediately.

To prove (*b*), suppose $Q_s(k) = 0$. Then $\Omega_s(k)$ is trivial in $\log_s S^\#$, and therefore $k \in \text{crit}(\Omega_s)$. By Lemma 11(*b*), $N \in \text{crit}\,(\Omega)$. Conversely, suppose $Q_s(k)$ is non-zero. Then $Q_s(k) \approx 1$. Now, $Q_s(k + z) = P(z) + Q_s(k)$, where each term of $P(z)$ has positive degree and a constant coefficient. If $G \prec 1$ in $\log_s S^\#$, then clearly $P(G) \prec 1$ in $\log_s S^\#$. Thus $Q_s(k + G) \backsim Q_s(k)$ for all $G \prec 1$. Therefore, Ω_s is stable at k, and so $N \notin \text{crit}\,(\Omega)$ by Lemma 11(*a*).

13. LEMMA - (Weight reduction). Let $Q(y)$ be a non-zero *nth* order differential polynomial with constant coefficients. Let p and w be natural numbers such that each term of Q has degree p and weight w. Let α be a

real number. Then,

(*a*) $[\alpha; Q]$ has constant coefficients.

(*b*) Suppose $w > 0$. Then $[\alpha; Q]$ non-trivially involves a term of weight less than w unless $\alpha = 0$ and $Q(y) = c(y')^w y^{p-w}$ for some constant c.

PROOF - By direct calculation of $[\alpha; Q]$, it is clear that it has constant coefficients, and we can write $[\alpha; Q] = Q + Q_1$ where the non-zero terms of Q_1 (if any) have weight less than w. Let $w > 0$. If $\alpha \neq 0$ then (*b*) follows from [5, § 61]. Now assume $\alpha = 0$ and Q is not of the form $c(y')^w y^{p-w}$. Then for some constant b, we may write $Q(y) = b(y')^w y^{p-w} + G(y)$, where G is a non-zero polynomial in $y, y', \ldots, y^{(n)}$ with constant coefficients, each term of which has degree p, weight w and order ≥ 2. Then clearly, $[0; Q](v) = b(v')^w v^{p-w} + [0; G](v)$. Now assume (contrary to (*b*)) that $[0; Q]$ has no non-trivial therms of weight less than w. Therefore, $[0; Q](v) = Q(v)$ since $Q_1 \equiv 0$. Hence,

(*c*) $[0; G](v) = G(v)$.

If the derivatives of y in $G(y)$ are with respect to x, and if $P(u, v)$ is the polynomial in $v, dv/du, \ldots, d^n v/du^n$, obtained from $G(y)$ by the change of variables $y = v$, $x = e^u$, then by definition,

(*d*) $[0, G](v) = e^{wu} P(u, v)$.

The proof now proceeds in a way similar to that of [5, § 61]. Obviously, if $y = f(x)$ is a solution of $G(y) = 0$, then in view of (*c*) and (*d*), $y = f(\log x)$ is also a solution. Hence if B denotes the set of solutions of $G(y) = 0$, then $f(x) \in B$ implies $f(\log x) \in B$. Since $G(y)$ has constant coefficients, $f(x) \in B$ implies $f(a + x) \in B$ for each constant a. Finally $x \in B$ since every term of G has order ≥ 2.

Let $a_0, a_1, \ldots$ be complex numbers, and define functions $H_k(x, a_0, \ldots, a_k)$ recursively, as follows: $H_0(x, a_0) = a_0 + x$, $H_{k+1}(x, a_0, \ldots, a_{k+1}) = a_{k+1} + \log H_k(x, a_0, \ldots, a_k)$. It now follows from the preceeding that

(*e*) $y(x) = H_s(x, a_0, \ldots, a_s) \in B$

for any $s \geq 0$ and any complex numbers $a_0, \ldots, a_s$. (The proof is by induction on s).

We now prove that if $s \geq 0$ and $z = H_s$, then the Jacobian of z, $\partial z/\partial x, \ldots, \partial^s z/\partial x^s$ with respect to $a_0, \ldots, a_s$, is not identically zero as a function of $(x, a_0, \ldots, a_s)$. When shown, the proof will be completed since for fixed x, this implies the functional independence of z, $\partial z/\partial x, \ldots, \partial^s z/\partial x^s$, as functions of $a_0, \ldots, a_s$, which of course contradicts (*e*), for $s = n$.

Assume the Jacobian is identically zero. Then there exist functions $K_0, \ldots, K_s$ of $(a_0, \ldots, a_s)$ such that

(*f*) $K_0 \partial z/\partial a_0 + \ldots + K_s \partial z/\partial a_s \equiv 0$

in $(x, a_0, \dots, a_s$ with $|K_0| + \dots + |K_s| > 0$. If $a_0, \dots, a_s$ are fixed as positive numbers, and x ranges over large positive numbers, then clearly $H_0, \dots, H_s$ all $\to \infty$ as $x \to \infty$. Therefore, if $s \geq j > k$,

$$(g)\quad (\partial z/\partial a_k)(\partial z/\partial a_j)^{-1} = (H_k H_{k+1} \dots H_{j-1})^{-1} \to 0$$

as $x \to \infty$. But (g) clearly contradicts (f), and so the Jacobian is not identically zero.

PART IV - The parametric case.

14. DEFINITION - Assume § 2 with Ω *NTPD.*

(a) M is called a *parametric monomial* of Ω, if $kM \in \operatorname{crit}(\Omega)$, for every non-zero constant k. The set of all parametric monomials of Ω is denoted par (Ω).

(b) If $f \sim k x^{\alpha_0} (\log x)^{\alpha_1} \dots (\log_s x)^{\alpha_s}$ in $S^{\#}$, then the unit monomial $x^{\alpha_0}(\log x)^{\alpha_1} \dots (\log_s x)^{\alpha_s}$ is called the *gauge* of f and is denoted $]f[$. (This concept and notation were introduced [9, § 17]).

(c) If B is a finite non-empty set of unit monomials, then the *maximum* of B (denoted max B) is that element V of B, such that $N \in B$ implies either $N < V$ or $N = V$.

15. LEMMA - Assume § 2 with Ω *NTPD.* Let $M \in \operatorname{par}(\Omega)$. Then there exists $p > 0$ such that $\Omega^{(p)}$ is non-trivial and $M \in \operatorname{par}(\Omega^{(p)})$.

PROOF - Let I be the set of all $p > 0$ for which $\Omega^{(p)}$ is non-trivial, and assume the conclusion does not hold. Then if $p \in I$, there is a non-zero constant k for which $kM \notin \operatorname{crit}(\Omega^{(p)})$. But for any $h < M$ and any non-zero constant c, $\Omega^{(p)}(cM + h) = c^p k^{-p} \Omega^{(p)}(kM + g)$, where $g = c^{-1}kh$. Therefore it follows that $cM \notin \operatorname{crit}(\Omega^{(p)})$ for each constant c and each $p \in I$. In particular $\Omega^{(p)}(M)$ is non-trivial for $p \in I$. Let $N = \max\{]\Omega^{(p)}(M)[: p \in I \cup \{0\}\}$, and let J be the set of all $p \in I \cup \{0\}$ for which $]\Omega^{(p)}(M)[= N$. Then for $p \in J$, $\Omega^{(p)}(M) \sim b_p N$, where b_p is a non-zero constant. Let $f(a) = \Sigma\{b_p a^p : p \in J\}$, and let k_0 be a non-zero constant for which $f(k_0) \neq 0$. Then we assert that for any $h < M$, $\Omega(k_0 M + h) \sim f(k_0) N$. If proved, this implies $k_0 M \notin \operatorname{crit}(\Omega)$ which contradicts hypothesis, and thereby establishes the lemma. To prove the assertion, we note that if $p \in I$, then $\Omega^{(p)}(k_0 M + h) \sim \Omega^{(p)}(k_0 M)$, since $k_0 M \notin \operatorname{crit} \Omega^{(p)}$. Also, $\Omega^{(p)}(k_0 M$ is $\sim b_p k_0^p N$, if $p \in J$ and is $< N$ if $p \in I - J$. If $p = 0$, $\Omega^{(p)}(k_0 M + h)$ equals $\Omega^{(p)}(M)$, while for $p \notin I \cup \{0\}$, $\Omega^{(p)}(k_0 M + h)$ is trivial and therefore $< N$. The assertion now follows immediately, since $\Omega(k_0 M + h)$ is the sum (over p) of all $\Omega^{(p)}(k_0 M + h)$.

REMARK - The converse of this result is not true, for if $\Omega(y) = y' + 1$, then $1 \notin \operatorname{par}(\Omega)$ although $1 \in \operatorname{par}(\Omega^{(1)})$.

16. LEMMA - Assume § 2 with Ω *NTPD*. Let Ω be homogeneous.

Construct a polynomial $F(\alpha)$ as follows:

Let N be the maximum of the gauges of $x^{-w(i^{\#})}\Omega[i^{\#}]$ over all $i^{\#}$ for which $\Omega[i^{\#}]$ is non-trivial, and let I be the set of all $i^{\#}$ for which $]x^{-w(i^{\#})}\Omega[i^{\#}][=N$. For $i^{\#}\in I$, let $x^{-w(i^{\#})}\Omega[i^{\#}]\backsim c(i^{\#})N$, where $c(i^{\#})$ is a non-zero constant, and let $f(i^{\#},\alpha)=\alpha^{i_1}(\alpha(\alpha-1))^{i_2}\dots(\alpha(\alpha-1)\dots(\alpha-n+1))^{i_n}$ where $i^{\#}=(i_0,\dots,i_n)$. Define $F(\alpha)=\Sigma\{c(i^{\#})f(i^{\#},\alpha): i^{\#}\in I\}$.

Then, if $M\in\operatorname{crit}(\Omega)$, $F(\delta_0(M))=0$.

PROOF - Let p be the degree of Ω, and let $M=x^{\alpha}G$ where $\delta_0(G)=0$. Then, if $h\backsim M$, it follows by induction on q that $h^{(q)}=x^{\alpha-q}G(\alpha(\alpha-1)\dots\dots(\alpha-q+1)+E_q)$ where $E_q<1$. Hence $\Omega(h)=x^{\alpha p}G^pN(F(\alpha)+E)$, where $E<1$. If $F(\alpha)\neq 0$, then $\Omega(h)\backsim\Omega(M)$ for all $h\backsim M$, so $M\notin\operatorname{crit}(\Omega)$, proving the lemma.

17. DEFINITION - Under the hypothesis and notation of Lemma 16, the equation $F(\alpha)=0$ is called the *critical equation* of Ω.

REMARKS - (1) The converse of Lemma 16 is not true, for $\Omega(y)=x(\log_2 x)y'-y$ has no critical monomials, but zero is a root of its critical equation.

(2) It is possible for the critical equation to be satisfied by every complex number (e.g. $\Omega(y)=(y')^2-yy''-x^{-1}yy'$). However, if this is not the case (as for example, in linear or first order Ω), then the critical equation clearly has at most W roots.

18. LEMMA - Assume § 2 with Ω *NTPD*. Let Ω be homogeneous of degree p, and let $s\geq r+W+2$. For each i, $0\leq i<s$, let α_i be a real root of the critical equation of $[\alpha_{i-1},\dots,\alpha_0;\Omega]$. Then

(*a*) There exist $\beta\in\{1,2,\dots,p\}$ and a non-zero complex number c such that

$$[\alpha_{s-1},\dots,\alpha_0;\Omega](v)=cv^{p-\beta}(v')^{\beta}+R_s(v)$$

where the coefficients of R_s are all trivial in $\log_s S^{\#}$.

(*b*) Zero is a root and is the only root of the critical equation of $[\alpha_{s-1},\dots,\alpha_0;\Omega]$.

(*c*) $N(x)=kx^{\alpha_0}(\log x)^{\alpha_1}\dots(\log_{s-1}x)^{\alpha_{s-1}}\in\operatorname{par}(\Omega)$ for any non-zero k.

PROOF - Let $\Omega_i=[\alpha_{i-1},\dots,\alpha_0;\Omega]$ and $\beta_i=\Omega_i[*,0]$ for $0\leq i\leq s$. Then by Lemma 12(*a*), if $i\geq r+1$, $\Omega_i=Q_i+R_i$ where Q_i has constant coefficients and is homogeneous of degree p, while R_i is trivial in $\log_i S^{\#}$. Since δ_0 of a non-zero constant is 0, $-\beta_i$ is the minimum weight of all non-zero terms in Q_i. It is a routine computation to verify that the coefficient of the term of weight 0 in Ω_s is $F(\alpha_{s-1})+t$ where $F(\alpha)=0$ is the critical equation of

Ω_{s-1}, and t is trivial in $\log_s S^{\#}$. Since $F(\alpha_{s-1})=0$, clearly $-\beta_s>0$. Hence every constant is a solution of $Q_s(v)=0$, and therefore (c) follows from Lemma 12(b). Let Q_i^* be the sum of all terms of weight $-\beta_i$ in Q_i. Then, since $[\alpha_i; Q_i^*]$ has constant coefficients (by Lemma 13(a)), and since it is easily seen that $Q_i - Q_i^*$ and R_i are both transformed into the trivial part of Ω_{i-1}, we have

$$(1) \qquad [\alpha_i; Q_i^*] = Q_{i+1} \quad \text{for} \quad r+1 \leq i \leq s-1.$$

Thus by lemma 8(d), the sequence of weights $(-\beta_{r+1}, -\beta_{r+2}, \dots, -\beta_s)$ is a monotone decreasing sequence of elements of the set $\{1, 2, \dots, W\}$. If this sequence were strictly decreasing, it would have at least $W+2$ distinct coordinates (since $s \geq r+W+2$), which is clearly impossible. Hence $-\beta_j = -\beta_{j+1}$ for some $j \in \{1+1, r+2, \dots, s-1\}$. Therefore by Lemma 13, $\alpha_j=0$ and Q_j^* is of the form $Q_j^*(z)=cz^{p+\beta_j}(z')^{-\beta_j}$. Let $\beta=-\beta_j$. Then $\beta \in \{1, 2, \dots, p\}$ and since $\alpha_j=0$, $Q_{j+1}(v)=cv^{p-\beta}(v')^\beta$ by (1). Then $\alpha_{j+1}=0$ since its a root of the critical equation of Ω_{j+1}. It is now clear that for $1 \leq t \leq s-j$, $Q_{j+t}(v)=cv^{p-\beta}(v')^\beta$, the proof being by induction on t, using (1). For $t=s-j$, we obtain desired representation in (a). Part (b) follows from Part (a), and the fact that $\beta>0$.

19. LEMMA - (Homogeneous case): Assume § 2 with Ω *NTPD*. Let Ω be homogeneous and let $s \geq r+W+2$. Then

(a) $M \in \operatorname{crit}(\Omega)$ if and only if $M(x)=kx^{\alpha_0}(\log x)^{\alpha_1} \dots (\log_{s-1} x)^{\alpha_{s-1}}$, where k is a non-zero constant and where α_i is a real root of the critical equation of $[\alpha_{i-1}, \dots, \alpha_0; \Omega]$, for each i, $0 \leq i \leq s-1$.

(b) $\operatorname{crit}(\Omega)=\operatorname{par}(\Omega)$.

PROOF - Part (a): The condition is sufficient by Lemma 18(c). To prove the necessity, let $M \in \operatorname{crit}(\Omega)$. Then by induction on Lemmas 16 and 11(b), $\delta_i(M)$ is a root of the critical equation of $[M, i, \Omega]$ for each $i \geq 0$. But then $\delta_i(M)=0$ for $i \geq s$ by Lemma 18(b), proving the necessity.

Part (b): This follows from Part (a).

REMARK - For an arbitrary Ω, Lemma 18(a) provides a method for finding $\operatorname{par}(\Omega^{(p)})$ for each p. The key step in adapting this method to the non-homogeneous case now follows.

20. LEMMA - Assume § 2 with Ω *NTPD*. Let $s \geq r+W+3$. Suppose there exists $p>0$ for which $M \in \operatorname{par}(\Omega^{(p)})$ and such that $[M, s, \Omega]^{(p)}$ is non-trivial on $\log_s S^{\#}$. Then:

(a) $M \in \operatorname{par}(\Omega)$

(*b*) There exists an integer $\beta > 0$ and a polynomial $C(y)$ in y alone, with constant coefficients, such that for any $t \geq s$, $[M, t, \Omega](y) = (y')^{\beta} C(y) + R_t(y)$, where all the coefficients of R_t are trivial in $\log_t S^{\#}$.

Proof - For $i \geq 0$, let $\Omega_i = [M, i, \Omega]$ and let $\Lambda_{iq} = [M, i, \Omega^{(q)}]$ when $\Omega^{(q)}$ is non-trivial in $S^{\#}$. Letting A be the set of all $q \geq 0$ for which $(\Omega_s)^{(q)}$ is non-trivial, it follows from Lemma 10(*a*)-(*c*) that for $q \in A$,

$$(1) \qquad (\Omega_j)^{(q)} = \Lambda_{jq} \quad \text{for} \quad 0 \leq j \leq s$$

and letting $\alpha_j = \delta_j(M)$,

$$(2) \qquad \alpha_j q + (\Omega_j)^{(q)}[*, 0] = \Omega_j[*, \alpha_j] \quad \text{for} \quad 0 \leq j \leq s-1.$$

By assumption, there exists $p \in A$ such that $p > 0$ and $M \in \operatorname{par} \Omega^{(p)}$. Hence by Lemmas 18(*a*) and 19(*a*), $\alpha_j = 0$ for $j \geq s-1$ and $\beta = -\Lambda_{s-1,p}[*, 0]$ is > 0, Let $q \in A$. Then since $\alpha_{s-1} = 0$, $\beta = -\Lambda_{s-1,q}[*, 0]$ by (1) and (2). Therefore, by Lemma 12(*a*), all non-trivial terms in $\Lambda_{s-1,q}$ have weight $\geq \beta$ and hence positive weight. Thus $1 \in \operatorname{par}(\Lambda_{s-1,q})$, and therefore $M \in \operatorname{par}(\Omega^{(q)})$ by Lemma 11(*b*). Hence $\Lambda_{s-1,q}(y) = c_q y^{q-\beta}(y')^{\beta} + R_q(y)$, where c_q is a constant, and R_q is trivial. But then Λ_{sq} also has this form since $\alpha_{s-1} = 0$. It now follows from (1) and the definition of A, that $\Omega_s(y) = (y')^{\beta} C(y) + T(y)$, where $C(y) = \Sigma\{c_q y^{q-\beta} : q \in A\}$, and T is trivial. This is the desired representation in (*b*), for $t = s$. For $t \geq s$, the representation in (*b*) follows easily by induction, since $\alpha_{t-1} = 0$. Finally, since $\beta > 0$, $1 \in \operatorname{par}(\Omega_s)$, and hence $M \in \operatorname{par}(\Omega)$ by Lemma 11(*b*), proving (*a*), and concluding the proof.

21. Theorem I (General case) - Assume § 2 with Ω *NTPD*. Let $s \geq r + W + 3$. Then $M \in \operatorname{par}(\Omega)$ if and only if $M(x) = kx^{\alpha_0}(\log x)^{\alpha_1} \dots (\log_{s-1} x)^{\alpha_{s-1}}$, where

(*a*) k is a non-zero constant,

(*b*) there exists $p > 0$ for which $\Omega^{(p)}$ is non-trivial, and such that for each i, $0 \leq i \leq s-1$,

(1) α_i is a root of the critical equation of $[\alpha_{i-1}, \dots, \alpha_0; \Omega]^{(p)}$, and

(2) $[\alpha_i, \dots, \alpha_0; \Omega]^{(p)}$ is non-trivial on $\log_{i+1} S^{\#}$.

Proof - Suppose (*a*) and (*b*) are satisfied for some $p > 0$. Then (2) implies

(*c*) $[\alpha_{j-1}, \dots, \alpha_0; \Omega]^{(p)} = [\alpha_{j-1}, \dots, \alpha_0; \Omega^{(p)}]$,

for $0 \leq j \leq s$, by Lemma 10(*a*), (*c*). Therefore (1) implies $M \in \operatorname{par}(\Omega^{(p)})$ by Lemma 19(*a*). Hence $M \in \operatorname{par}(\Omega)$ by Lemma 20(*a*).

Conversely, suppose $M \in \operatorname{par}(\Omega)$. Let $M_0 = M$ and $M_{i+1}(x) = \exp(-\delta_i(M)x)M(e^x)$ for $i \geq 0$. Then by Lemma 11(a), $M_i \in \operatorname{par}(\Omega_i)$ for all $i \geq 0$, where $\Omega_i = [M, i, \Omega]$. Letting A_i be the set of all $q > 0$ for which $(\Omega_i)^{(q)}$ is non-trivial and $M_i \in \operatorname{par}(\Omega_i)^{(q)}$, it follows from Lemma 15 that each A_i is non-empty (and each is clearly finite). Since A_0 is non-empty, it follows from Lemma (19(a) that $\delta_i(M) = 0$ for $i \geq s$, and we may write $M(x) = kx^{\alpha_0}(\log x)^{\alpha_1} \dots (\log_{s-1} x)^{\alpha_{s-1}}$. We now show $A_{i+1} \subset A_i$ for all i. If $p \in A_{i+1}$, then by Lemma 10, $(\Omega_i)^{(p)}$ is non-trivial and (c) holds for $0 \leq j \leq i+1$. Since $M_{i+1} \in \operatorname{par}(\Omega_{i+1})^{(p)}$, we have $M_i \in \operatorname{par}(\Omega_i)^{(p)}$ by (c) and Lemma 11(b). Hence A_i contains A_{i+1}. Therefore, the intersection of all the sets A_i contains an element p, which obviously satisfies (2). Since $M \in \operatorname{par}(\Omega^{(p)})$, it follows from ($c$) and Lemma 19($a$) that (1) is also satisfied.

REMARK - For an arbitrary Ω, Theorem I provides a method for finding $\operatorname{par}(\Omega)$, by considering separately, each $p > 0$ for which $\Omega^{(p)}$ is non-trivial, and finding all s-tuples $(\alpha_0, \dots, \alpha_{s-1})$ of real numbers which satisfy (1) and (2) relative to p (taking $s = r + W + 3$). Then corresponding to any such $(\alpha_0, \dots, \alpha_{s-1})$, $M(x) = kx^{\alpha_0}(\log x)^{\alpha_1} \dots (\log_{s-1} x)^{\alpha_{s-1}}$ is in $\operatorname{par}(\Omega)$. Conversely, for any $M \in \operatorname{par}(\Omega)$, the s-tuple $(\delta_0(M), \dots, \delta_{s-1}(M))$ must appear relative to some p.

PART V - The non-parametric case.

22. LEMMA - Assume § 2 with Ω *NTPD*. Let $M \in (\operatorname{crit}(\Omega) - \operatorname{par}(\Omega))$, and let $\delta_0(M) = \alpha$. Then there exist at least two distinct natural numbers p and q for which $\Omega^{(p)}[*, \alpha] = \Omega[*, \alpha] = \Omega^{(q)}[*, \alpha]$.

PROOF - Assume the conclusion is false. Then the set of all p for which $\Omega^{(p)}[*, \alpha] = \Omega[*, \alpha]$ reduces to $\{m\}$ for some m. Hence if $q \neq m$, then $\Omega^{(q)}[*, \alpha] < \Omega[*, \alpha]$ and therefore, $[\alpha; \Omega]^{(q)}$ is trivial. It follows that $\operatorname{crit}[\alpha; \Omega] = \operatorname{crit}[\alpha; \Omega]^{(m)}$, and therefore, $\operatorname{par}[\alpha; \Omega] = \operatorname{par}[\alpha; \Omega]^{(m)}$. But then $\operatorname{crit}[\alpha; \Omega] = \operatorname{par}[\alpha; \Omega]$, in view of Lemma 19($b$) (as applied to $[\alpha; \Omega]^{(m)}$). Since $M \in \operatorname{crit}(\Omega)$, it then follows from Lemma 11(a) that $e^{-\alpha u}M(e^u) \in \operatorname{par}[\alpha; \Omega]$, and therefore $M \in \operatorname{par}(\Omega)$ by Lemma 11(b). This contradicts hypothesis, and establishes the lemma.

23. DEFINITION - Assume § 2 with Ω *NTPD*. Then a real number α is called an *admissible value* of Ω, if the relation $\Omega^{(p)}[*, \alpha] = \Omega[*, \alpha] = \Omega^{(q)}[*, \alpha]$ holds for at least two distinct p and q.

24. LEMMA - Assume § 2 with Ω *NTPD*. Let $s \geq r + 2W + 2$. For each i, $0 \leq i < s$, let α_i be an admissible value of $\Omega_i = [\alpha_{i-1}, \dots, \alpha_0; \Omega]$, and let $\Omega_s = [\alpha_{s-1}, \dots, \alpha_0; \Omega]$. Then,

(a) There exist a natural number β, and a non-homogeneous polynomial $C(y)$, in y alone, with constant coefficients, such that $\Omega_s(y) = (y')^{\beta}C(y) + R_s(y)$, where R_s is trivial in $\log_s S^{\#}$.

(b) Zero is an admissible value, and is the only admissible value, of Ω_s.

PROOF - Let $\beta_i(q) = (\Omega_i)^{(q)}[*, 0]$ and $\nu_i = \Omega_i[*, \alpha_i]$, for each i and q. Let A be the set of all q for which $(\Omega_s)^{(q)}$ is non-trivial, and let $q \in A$. Then by Lemma 10(a) and (b), $(\Omega_i)^{(q)}$ is non-trivial and $\alpha_i q + \beta_i(q) = \nu_i$ for $0 \leq i < s$. Now for $i \geq r+1$, $\Omega_i = Q_i + R_i$ where Q_i has constant coefficients, and R_i has trivial coefficients. Hence $-\beta_i(q)$ is the minimum weight of non-trivial terms in $(Q_i)^{(q)}$. Letting P_{iq} be the sum of all terms in $(Q_i)^{(q)}$ which have weight $-\beta_i(q)$, we have (as in (1) of Lemma 18), $[\alpha_i; P_{iq}] = (Q_{i+1})^{(q)}$. Hence, by Lemma 8($d$), $-\beta_{i+1}(q) \leq -\beta_i(q)$, for $q \in A$. Now A clearly has at least two elements. In what follows, assume t and q are any distinct elements of A, and let $m_i = -(\beta_i(t) + \beta_i(q))$. Then the sequence, $(m_{r+1}, m_{r+2}, \ldots, m_s)$ is a monotone decreasing sequence of elements of the set $\{0, 1, \ldots, 2W\}$. This sequence cannot be strictly decreasing, for otherwise, it would have at least $2W+2$ distinct coordinates (since $s \geq r + 2W + 2$), which is impossible. Hence for some j, $m_j = m_{j+1}$. Then clearly, $\beta_{j+1}(q) = \beta_j(q)$ and $\beta_{j+1}(t) = \beta_j(t)$. It now follows from Lemma 13(b), that $\alpha_j = 0$, and that $P_{jq}(z) = c(z')^{\beta}z^{q-\beta}$ (where $\beta = -\nu_j$), with a similar representation for $P_{jt}(z)$. Hence both $(Q_{j+1})^{(q)}$ and $(Q_{j+1})^{(t)}$ are also of this form, and by induction, so are $(Q_{j+k})^{(q)}$ and $(Q_{j+k})^{(t)}$ for $1 \leq k \leq s-j$. Since t and q were arbitrary elements of A, it follows, taking $k = s-j$, that $Q_s(z) = (z')^{\beta}C(z)$, where $C(z)$ is a non-homogeneous polynomial in z alone with constant coefficients, proving (a).

(b) follows immediately from (a).

25. DEFINITION - Under the hypothesis and notation of Lemma 24, the sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{i-1})$ is called an *admissible sequence* of Ω, and $(y')^{\beta}C(y)$ is called the *s-equation* of $(\alpha_0, \alpha_1, \ldots, \alpha_{s-1})$.

REMARK - β may be strictly positive in the s-equation, as evidenced from the example of $(0, 0, \ldots, 0)$ in $\Omega(y) = yy' - y' + x^{-2}$. (Note here that $1 \in \operatorname{par}(\Omega)$).

26. THEOREM II - Assume § 2 with Ω *NTPD*. Let $s \geq r + 2W + 2$. Then $M \in (\operatorname{crit}(\Omega) - \operatorname{par}(\Omega))$ if and only if $M(x) = kx^{\alpha_0}(\log x)^{\alpha_1} \ldots (\log_{s-1} x)^{\alpha_{s-1}}$, where $(\alpha_0, \ldots, \alpha_{s-1})$ is an admissible sequence of Ω, whose s-equation $(y')^{\beta}C(y)$ satisfies the conditions, $\beta = 0$ and $C(k) = 0$.

PROOF - The conditions are sufficient by Lemma 12(b).

Conversely, suppose $M \in (\operatorname{crit}(\Omega) - \operatorname{par}(\Omega))$. Then by Lemmas 11 and 22, $\delta_i(M)$ is an admissible value of $[M, i, \Omega]$ for all $i \geq 0$. Hence by Lemma 24(b),

$\delta_i(M) = 0$ for $i \geq s$. Clearly $\beta = 0$ in the s-equation of $(\delta_0(M), \ldots, \delta_{s-1}(M))$, for otherwise $M \in \mathrm{par}(\Omega)$ by Lemma 12(*b*). Finally $C(k) = 0$ by Lemma 12(*b*), since $M \in \mathrm{crit}\,(\Omega)$.

REMARK - It is clear that Theorem II provides a method for finding the set, $(\mathrm{crit}\,(\Omega) - \mathrm{par}\,(\Omega))$, in a number of steps which can be bounded in advance.

PART VI - **The associated function.**

27. LEMMA - Assume § 2 with Ω *NTPD*. Let $M \in \mathrm{crit}\,(\Omega)$, with $k = M(]M[)^{-1}$. Then there exist a natural number β, and a polynomial $C(y)$ in y alone, with constant coefficients, such that

(*a*) $\beta + m > 0$, where m is zero if $C(k) \neq 0$ and otherwise is the multiplicity of the root k in $C(y)$.

(*b*) For $s \geq r + 2W + 3$, we have $\delta_s(M) = 0$ and $[M, s, \Omega](y) = (y')^{\beta} C(y) + R_s(y)$, where R_s is trivial in $\log_s S^{\#}$.

PROOF - This follows from Theorem I and Lemma 20(*b*), in the case when M is parametric, and from Theorem II, in the non-parametric case.

28. DEFINITION - Under the hypothesis and notation of Lemma 27,

(*a*) $(y')^{\beta} C(y)$ is called the *associated function* of M in Ω, and is denoted $AF(M, \Omega, y)$.

(*b*) β is called the *exponent* of M.

(*c*) m is called the *multiplicity* of M.

(*d*) M is called an *ordinary* monomial if $m > 0$, and is called *simple if* $m = 1$.

REMARKS - (1) If $M \in pm(\Omega)$, then the associated function defined in [5, § 68(*e*)], coincides with that defined in Definition 28(*a*), for in this case, $[M, i+1, \Omega]$ is the first image (see [5, § 63]) of $[M, i, \Omega]$.

(2) Obviously, $\beta > 0$ if and only if $M \in \mathrm{par}\,(\Omega)$.

29. LEMMA - Assume § 2 with Ω *NTPD*.

(*a*) Let $s \geq r + 2W + 3$. Then M is an ordinary monomial of Ω if and only if $M(x) = kx^{\alpha_0} (\log x)^{\alpha_1} \ldots (\log_{s-1} x)^{\alpha_{s-1}}$, where $(\alpha_0, \ldots, \alpha_{s-1})$ is an admissible sequence of Ω, whose s-equation $(y')^{\beta} C(y)$ satisfies the condition $C(k) = 0$.

(*b*) Let $D(\Omega)$ (respectively, $d(\Omega)$), denote the maximum (respectively, the minimum) of the set of all p for which $\Omega^{(p)}$ is non-trivial. Then there are precisely $D(\Omega)-d(\Omega)$ ordinary monomials of Ω, provided each is counted as many times as its multiplicity indicates.

PROOF - (*a*) is obvious.

To prove (*b*), we first prove the following assertion (*A*). If $B=\{a_0, a_1, \dots, a_t\}$ is the set of admissible values of Ω, where $a_0 < a_1 < \dots < a_t$, then $(D(\Omega) - d(\Omega)) = \Sigma\{(D([a_i; \Omega]) - d([a_i; \Omega])\colon 0 \leq i \leq t\}$. First we show $D([a_i; \Omega]) = d([a_{i+1}; \Omega])$ for $0 \leq i \leq t-1$. If this relation fails to hold for i, then letting $p = D([a_i; \Omega])$ and $q = d([a_{i+1}; \Omega])$, we have $p < q$. But then using Lemma 10(*a*), (*b*), it is easily verified that the maximum of all the numbers, $(q-m)^{-1}(\Omega^{(m)}[*, 0] - \Omega^{(q)}[*, 0])$ for $p \leq m < q$, is an admissible value of Ω, which is strictly between a_i and a_{i+1}, contradicting our representation for B. Similarly, we prove $D([a_t; \Omega]) = D(\Omega)$ and $d([a_0; \Omega]) = d(\Omega)$, so assertion (*A*) follows immediately.

Now let B_i be the set of admissible sequences $(\alpha_0, \alpha_1, \dots, \alpha_{i-1})$ of Ω. If $s = r + 2W + 3$, then by (*a*), it is clear that the number N of ordinary monomials of Ω is precisely the sum, over all $(\alpha_0, \dots, \alpha_{s-1}) \in B_s$, of the numbers $D([\alpha_{s-1}, \dots, \alpha_0; \Omega]) - d([\alpha_{s-1}, \dots, \alpha_0; \Omega])$. This sum can be written as an interated sum, the inner one of which is over all α_{s-1} which are admissible in $[\alpha_{s-2}, \dots, \alpha_0; \Omega]$, and the outer sum is over all $(\alpha_0, \dots, \alpha_{s-2}) \in B_{s-1}$. But then applying assertion (*A*) to the inner sums, shows that N is the sum over all $(\alpha_0, \dots, \alpha_{s-2}) \in B_{s-1}$ of the numbers $D([\alpha_{s-2}, \dots, \alpha_0; \Omega]) - d([\alpha_{s-2}, \dots, \alpha_0\ \Omega])$. Repeated applications of this argument clearly leads to $N = D(\Omega) - d(\Omega)$.

30. LEMMA - Assume § 2 with Ω *NTPD*. Let $M \in \mathrm{crit}(\Omega)$, and let N be a logarithmic monomial, with $a = N(]N[)^{-1}$. Then,

(a) If $\Lambda = N\Omega$, we have $M \in \mathrm{crit}(\Lambda)$ and $AF(M, \Lambda, y) = a(AF(M, \Omega, y))$.

(*b*) If Φ is the N-multiplication transform of Ω (i.e. $\Phi(z) = \Omega(Nz)$), then $MN^{-1} \in \mathrm{crit}(\Phi)$, and $AF(MN^{-1}, \Phi, y) = AF(M, \Omega, ay)$.

PROOF - Part (*a*) is obvious.

Part (*b*) follows from the following assertion. If $\alpha = \delta_0(N)$ and $G(u) = e^{-\alpha u}N(e^u)$, then for any real number ν, $[\nu; \Phi]$ is the G-multiplication transform of $[\alpha + \nu; \Omega]$. (Part (*b*) then follows by induction, taking $\nu = \delta_0(MN^{-1})$). To prove the assertion, we note that $[\nu; \Phi]$ and the G-multiplication transform of $[\alpha + \nu; \Omega]$ differ only by the multiplicative factor $\exp[(\Omega[*, \alpha+\nu] - \Phi[*, \nu])u]$. Since both differential polynomials are non-trivial, this factor must be 1, proving the assertion.

31. LEMMA - Assume § 2 with Ω *NTPD*. Let $M \in \operatorname{crit}(\Omega)$ with exponent β and multiplicity m. Let $\Lambda(z) = \Omega(M + Mz)$, and let Φ be the sum of all terms in Λ of degree $\leq \beta + m$. Then,

(*a*) The set of critical monomials $\prec 1$ of Λ is precisely the set of critical monomials $\prec 1$ of Φ, (and the associated function in each is the same).

(*b*) Any ordinary monomial of Λ which is $\prec 1$ is an ordinary monomial of Φ. Any ordinary monomial of Φ is $\prec 1$ and is an ordinary monomial of Λ.

(*c*) If $\Omega(M)$ is non-trivial, then Λ has exactly $\beta + m$ ordinary monomials $\prec 1$ (counting multiplicity).

PROOF - By Lemma 30(*b*), 1 is a critical monomial of the M-multiplication transform of Ω, and its associated function is of the form $(y')^{\beta} C(y)$, where 1 is an m-fold root of $C(y)$. For $i \geq 0$, let $\Lambda_i = [1, i, \Lambda]$. Then for sufficiently large i, $\Lambda_i(y) = (y')^{\beta} C(1 + y) + T_i(y)$, where T_i is trivial (the proof of this being similar to that of Lemma 30(*b*)). Since 1 is an m-fold root of $C(y)$, obviously for all $i \geq 0$,

$$(\Lambda_i)^{(\beta+m)} \text{ is } NTPD. \tag{1}$$

Let the coefficients of Λ lie in an $LD_t(S^{\#})$, and let $s = t + 2W + 3$.

We first prove the following assertion. If G is a logarithmic monomial of rank $\leq s - 1$, and $G \prec 1$, then for every $q > \beta + m$, $[G, s, \Lambda]^{(q)}$ is trivial. Assume the contrary for some $q > \beta + m$. Then letting j be the smallest i for which $\delta_i(G)$ is non-zero, it follows from Lemma 10(*a*), (*b*) that $(\Lambda_j)^{(q)}[*, 0] > (\Lambda_j)^{(\beta+m)}[*, 0]$, and hence that $(\Lambda_{j+1})^{(\beta+m)}$ is trivial. This contradicts (1), and proves the assertion. Therefore, in view of Lemma 8(*b*), for such a $G \prec 1$ there is a $p \leq \beta + m$ such that $[G, s, \Lambda]^{(p)}$ is non-trivial (and this holds for $G \approx 1$ by (1), taking $p = \beta + m$). It now follows by induction that the relation,

$$[G, i, \Phi] = \Sigma \{ [G, i, \Lambda]^{(k)} \colon 0 \leq k \leq \beta + m \}, \tag{2}$$

is valid for any $G \precsim 1$ of rank $\leq s - 1$, and any i, $0 \leq i \leq s$.

Hence, if $G \prec 1$, then since $[G, s, \Lambda]^{(q)}$ is trivial for $q > \beta + m$, we have

$$[G, s, \Lambda] = [G, s, \Phi] + T \tag{3}$$

where T is trivial. Part (*a*) of the lemma follows immediately from (3). Furthermore, (3) also implies that the ordinary monomials $\prec 1$ of Λ are precisely the ordinary monomials $\prec 1$ of Φ. Thus to conclude the proof of Part (*b*), we must show that every ordinary monomial of Φ is $\prec 1$.

From (2), it follows that if $G \approx 1$ then $[G, s, \Phi]$ is of the form $b(y')^{\beta}y^{m}+R(y)$, (where R is trivial), and hence there can be no ordinary monomial ≈ 1. Now assume Φ has an ordinary monomial N, with $1 < N$. Then $[N, s, \Phi]$ must involve at least two terms of different degree, non-trivially. Since Φ has no terms of degree $> \beta + m$, there exists $q < \beta + m$ for which $[N, s, \Phi]^{(q)}$ is non-trivial. But then letting j be the smallest i for which $\delta_i(N)$ is non-zero, it follows from Lemma 10(a), (b) that $[1, j, \Phi]^{(q)}[*, 0] > [1, j, \Phi]^{(\beta+m)}[*, 0]$, and hence that $[1, s, \Phi]^{(\beta+m)}$ is trivial in $\log_s S^{\#}$. But then by (2), $(\Lambda_s)^{(\beta+m)}$ is trivial, contradicting (1). This contradiction establishes Part (b).

Part (c) follows from Part (b) and Lemma 29(b).

32. LEMMA - Assume § 2 with Ω *NTPD*. Let $1 \in \operatorname{crit}(\Omega)$ with $AF(1, \Omega, y) = (y')^{\beta}C(y)$. Let $q \geq r + 2W + 3$, and let $\theta = \theta_q$ be the operator $\theta_q y = (x \log x \dots \log_{q-1} x)y'$ as defined in [5, § 15]. Then there is a unit monomial N such that when $N\Omega$ is written as a polynomial in $y, \theta y, \dots, \theta^n y$, it has the form $\Sigma\, t(k^{\#}, x) y^{k_0}(\theta y)^{k_1} \dots (\theta^n y)^{k_n}$, where

(a) $t(k^{\#}, x) \lesssim 1$ for all $k^{\#}$

(b) $t(k^{\#}, x) < 1$ if $k^{\#} \neq (k_0, \beta, 0, \dots, 0)$

(c) $C(y) = \Sigma \{ t(k^{\#}, +\infty) y^{k_0} \colon k^{\#} = (k_0, \beta, 0, \dots, 0) \}$.

PROOF - The change of variables $y = v$, $x = e^u$, transforms $\theta_{p+1}{}^{j} y$ into $\theta_p{}^{j} v$, for all p and j. Hence if we write $\Omega(y)$ as a polynomial in $y, \theta y, \dots, \theta^n y$, then we obtain a representation for $[1, q, \Omega]$ directly from the definition of $[1, q, \Omega]$ as a transform. Comparing this representation with that given by the associated function, and using [5, § 19(e)], we easily obtain the desired representation for $N\Omega$, when $N(x)$ is taken to be $x^{-\nu_0}(\log x)^{-\nu_1} \dots (\log_{q-1} x)^{\nu_{q-1}}$, where $\nu_i = [1, i, \Omega][*, 0]$.

33. LEMMA - Assume § 2 with $n = 1$ (i.e. let Ω be of order 1). Let Ω be *NTPD*. Let $M \in \operatorname{par}(\Omega)$, with exponent β and multiplicity zero. Let $G < 1$ be a parametric monomial of $\Omega(M + Mz)$, with exponent β_1 and multiplicity m_1. Then $\beta_1 + m_1 < \beta$.

In particular, the exponent of any critical monomial < 1 of $\Omega(M + Mz)$ is less than β.

PROOF - If Γ is the M-multiplication transform of Ω, then by Lemma 30(b), $1 \in \operatorname{par}(\Gamma)$, with $AF(1, \Gamma, y)$ of the form $(y')^{\beta}C(y)$, where $C(1) \neq 0$. From Lemma 34, it follows that for sufficiently large q, there is a unit monomial $H(x)$, such that the coefficients of $\Lambda(z) = H\Gamma(1 + z)$ satisfy the following asymptotic relations:

(a) $\Lambda[(k_0, k_1)] < (x \log x \dots \log_{q-1} x)^{k_1 - \beta}$ if $k_1 \neq \beta$.

(*b*) $\Lambda[k_0, \beta)] \lesssim 1$.

(*c*) $\Lambda[(0, \beta)] \approx 1$.

Suppose $G \prec 1$ is a parametric monomial of $\Omega(M + Mz)$ with exponent β_1 and multiplicity m_1. Then by Lemma 30(*a*), $G \in \mathrm{par}(\Lambda)$ with $AF(G, \Lambda, y)$ of the form $(y')^{\beta_1} C_1(y)$, where $C_1(y)$ has a non-zero m_1-fold root. Letting b be the degree of $C_1(y)$, we have $\beta_1 + b \leq \beta$ by Lemma 31(*a*). Assume that the conclusion $\beta_1 + m_1 < \beta$ does not hold. Then since $m_1 \leq b$, we have $\beta_1 + b = \beta$. Then $[G, i, \Lambda]^{(\beta)}$ is *NTPD* for all $i \geq 0$, and is of the form $c(y')^{\beta_1} y^b + R_i$ (where R_i is trivial) for sufficiently large i. But by Lemma 10(*a*), (*c*), $[G, i, \Lambda^{(\beta)}] = [G, i, \Lambda]^{(\beta)}$ and since $\beta_1 > 0$ it follows from Lemma 12(*b*) that $G \in \mathrm{par}(\Lambda^{(\beta)})$. Hence $\delta_i(G)$ is a root of the critical equation of $[G, i, \Lambda^{(\beta)}]$ for all i, by Lemma 19(*a*). Since $G \prec 1$, there exists j such that $\delta_i(G) = 0$ if $i < j$ while $\delta_j(G) < 0$. But a straightforward computation (using [5, § 19(*d*)]) shows that the relations (*a*)-(*c*) imply that for $i \leq j$, the critical equation of $[G, i, \Lambda^{(\beta)}]$ is of the form $a\,\alpha^\beta = 0$ (where a is a non-zero constant). Thus $\delta_j(G) = 0$ contradicting $\delta_j(G) < 0$. This contradiction establishes the relation $\beta_1 + m_1 < \beta$.

The second conclusion follows from the first.

REMARKS - (1) The requirement that Ω be of order 1 is essential in Lemma 33, for if $\Omega(y) = xy'' + 2y' + x^{-3}$, then $1 \in \mathrm{par}(\Omega)$ with $\beta = 1$ and multiplicity zero, while $x^{-1} \in \mathrm{par}(\Omega(1 + z))$, with exponent equal to one.

(2) The conclusion that $\beta_1 + m_1 < \beta$ in Lemma 33 holds only for *parametric* G, for if $\Omega(y) = (y')^2 - 2x^{-2}y' + x^{-5}y + x^{-4}$, then $1 \in \mathrm{par}(\Omega)$ with $\beta = 2$ and multiplicity zero, while $\Omega(1 + z)$ has a principal monomial of multiplicity two.

PART VII - Solutions in certain first order cases.

34. DEFINITION - Assume § 2 with Ω *NTPD*. Let $M \in \mathrm{crit}(\Omega)$. We say Ω is *asymptotically non-singular* at M, if $\partial\Omega/\partial y^{(n)}$, evaluated at $y = M$, is non-trivial, and $\partial\Omega/\Omega y^{(n)}$ is stable at M. (This is the obvious extension of the definition given in [5, § 77] for principal monomials).

35. REMARK - The next lemma depends only on the result proved in [7] (see below), and not on any results we have obtained thus far. It illustrates one method of proving the existence of solutions $\sim M$ of $\Omega(y) = 0$, namely by finding principal solutions of $\Omega(M + z) = 0$, and this is the main device of this section.

A Schwartzian-symmetric logarithmic differential field of rank p (briefly an $SLDF_p$) over $T^{\#} = F(-a, a)$, is a differential field E^*, containing all logarithmic monomials of rank $\leq p$, and having the property that if f is a non-zero element of E^*, then f is $\sim$ to a logarithmic monomial of rank $\leq p$, and E^* also contains the function whose value at the conjugate of x is the conjugate of $f(x)$. (For example, the set of all rational combinations, with complex coefficients, of logarithmic monomials of rank $\leq p$, is an $SLDF_p$).

It is proved in [7], that if a first order Ω with coefficients in an $SLDF_p$, possesses a principal monomial N, at which it is asymptotically non-singular, then $\Omega(y)=0$ possesses a principal solution $\sim N$, is some $F(c, d)$.

36. LEMMA - Let Ω be a first order differential polynomial with coefficients in an $SLDF_p$ over $F(-a, a)$. Let Ω be *NTPD* and let $M \in \operatorname{crit}(\Omega)$. Then if Ω is asymptotically non-singular at M, the equation $\Omega(y)=0$ has at least one solution $\sim M$ in some $F(c, d)$.

PROOF - Assuming $\Omega(M) \neq 0$, it is clear that $\Omega(M+z)$ is asymptotically non-singular at each of its principal monomials. Then if z_0 is any principal solution of $\Omega(M+z)=0$, the function $y_0 = M + z_0$ is a solution $\sim M$ of $\Omega(y)=0$.

37. LEMMA - Assume § 2 with $n=1$, and let Ω be *NTPD*. Let $M \in \operatorname{par}(\Omega)$ with $AF(M, \Omega, y)$ of the form $(y')^{\beta}C(y)$ and multiplicity m. Then

(*a*) For sufficiently large s,

$$[M, s, \partial\Omega/\partial y'](y) = (y')^{\beta-1}\beta C(y) + R_s(y),$$

where R_s is trivial in $\log_t S^{\#}$.

(*b*) If $\beta + m > 1$, then $M \in \operatorname{crit}(\partial\Omega/\partial y')$ and $AF(M, \partial\Omega/\partial y', y) = (y')^{\beta-1}\beta C(y)$.

(*c*) Ω is asymptotically non-singular at M if and only if $\beta=1$ and $m=0$.

PROOF - Here, for any differential polynomial Γ, we will use the notation $\Gamma_i = [1, i, \Gamma]$.

If Φ is the M-multiplication transform of Ω, then by Lemma 30(*b*), $1 \in \operatorname{par}(\Phi)$ and $AF(1, \Phi, y) = (y')^{\beta}k_0^{\beta}C(k_0 y)$ where $k_0 = M(\,]M[\,)^{-1}$. As in the proof of Lemma 32, we compute Φ_s (for sufficiently large s), and find that there is a unit monomial $g(x)$, such that if $\Lambda = g\Phi$, then

$$\Lambda_s = \Phi_s, \tag{1}$$

$$\Lambda_i[j^{\#}](u) = \Lambda[j^{\#}](e_i(u))\,(L_i(e_i(u)))^{\beta - j_1} \tag{2}$$

for each $j^{\#}$, and each i, $0 \leq i \leq s$. (Here, $L_i(x)$ is the function $x \log x \ldots \log_{i-1} x$, while $e_i(u)$ is defined recursively by $e_0(u) = u$, $e_{i+1}(u) = \exp e_i(u)$).

By comparing the representation for the coefficients of Λ_s given by (2) (for $i = s$), with that given by the associated function, we abtain asymptotic estimates on the functions $\Lambda[j^{\#}](e_s(u))$, in $\log_s S^{\#}$. Using [5, § 19(*e*)], we obtain the following relations for $0 \leq i \leq s$, in $\log_i S^{\#}$:

$$\Lambda[j^{\#}](e_i(u)) < [(L_{s-i}(u))(L_i(e_i(u)))]^{j_1-\beta} \quad \text{if} \quad j_1 \neq \beta, \tag{3}$$

$$\Lambda[j^{\#}](e_i(u)) \lesssim 1 \quad \text{if} \quad j_1 = \beta \tag{4}$$

$$k_0^{\beta} C(k_0 v) = \Sigma \{ \Lambda[j^{\#}](e_i(+\infty)) v^{j_0} : j_1 = \beta \}. \tag{5}$$

This last relation implies that for some $p \geq 0$,

$$\Lambda[(p, \beta)] \approx 1. \tag{6}$$

Using (2)-(6), it follows by induction that for $0 \leq i \leq s$, $\partial\Lambda_i/\partial y' = (\partial\Lambda/\partial y')_i$. In view of (1), we then see that $(\partial\Lambda/\partial y')_s$ is of the form $(y')^{\beta-1}\beta k_0^{\beta} C(k_0 y) + T_s(y)$, where $T_s(y)$ is trivial. But since $(gM)^{-1}\partial\Lambda/\partial y'$ is simply the M-multiplication transform of $\partial\Omega/\partial y'$, Part (*a*) now follows as in the proof of Lemma 30(*b*).

Parts (*b*) and (*c*) follow easily from Part (*a*) and Lemma 11(*b*).

REMARK - Lemma 37(*c*) completely solves the problem of determining in advance those parametric monomials at which a first order Ω is asymptotically non-singular. For non-parametric critical monomials, there seems to be no way of determining this without actually computing the stability properties of $\partial\Omega/\partial y'$ at these monomials (using Theorems I and II, for example).

38. LEMMA - Let Ω be a first order differential polynomial with coefficients in an *$SLDF_p$* over $F(-a, a)$. Let Ω be *MTPD*. Then if $M \in \operatorname{par}(\Omega)$ with exponent 1 and multiplicity 0, the equation $\Omega(y) = 0$ has at least one solution $\sim M$ in some $F(c, d)$.

PROOF - This follows from Lemmas 36 and 37(*c*).

39. LEMMA - Let Ω satisfy the hypothesis of Lemma 38. Let $M \in \operatorname{par}(\Omega)$ with exponent 2 and multiplicity 0. Then under *either* of the following two conditions, the equation $\Omega(y) = 0$ has at least one solution $\sim M$ in some $F(c, d)$.

(*a*) M is a solution of $\partial\Omega/\partial y' = 0$

(*b*) $\Omega(M + z)$ has at least one simple ordinary monomial $< M$.

PROOF - Let $\Lambda(z) = \Omega(M + Mz)$, and $\Phi = \partial\Lambda/\partial z'$. In each case, we prove the existence of a critical monomial, $N < 1$ of Λ such that $N \notin \operatorname{crit}(\Phi)$. Then by Lemma 36, there is a solution $\sim N$ of $\Lambda(z) = 0$, and hence $\Omega(y) = 0$ has a solution $\sim M$. We first note that by Lemmas 37(*b*) and 33, any critical monomial < 1 of Φ has exponent 0, and hence, being ordinary, must be an ordinary monomial of $\Phi^{(1)} + \Phi^{(0)}$ by Lemma 31(*b*). (In what follows, we assume $\Omega(M) \neq 0$).

If (a) holds, then $\Phi^{(0)}=0$. Hence $\Phi^{(1)}+\Phi^{(0)}$ has no ordinary monomials by Lemma 29(b). Thus any principal monomial of Λ cannot be critical of Φ, so the result follows in this case.

If (b) holds, then Λ has two distinct ordinary monomials $\prec 1$ by Lemma 31(c). At least one of them is not in crit(Φ), since $\Phi^{(1)}+\Phi^{(0)}$ has at most one ordinary monomial, so the result follows if (b) holds.

REMARK - (a) is satisfied for $M=1$, when $\Omega(y)=(y')^2+\Sigma a_{ij}y^i(y')^j$, where $a_{i1}=0$ and $\delta_0(a_{ij})<j-2$ for all i and j.

(b) is satisfied when $\Omega(M+Mz)$ has no linear terms.

PART VIII - On solutions in the general non-parametric case.

40. LEMMA - Assume § 2 with Ω *NTPD*. Let M be a simple non-parametric critical monomial of Ω. Let $(\partial\Omega/\partial y^{(n)})(M)$ be non-trivial. Let $\Lambda(z)=\Omega(M+Mz)$, and let $F(x)=(\partial\Lambda/\partial z)(0)$. Then there is a logarithmic monomial $G\backsim F$ such that

(a) $G^{-1}\Lambda(0)\prec 1$, and $G^{-1}\Lambda^{(1)}(z)$ is unimajoral, having one or more principal factorization sequences, $(V_1, \ldots, V_n)$. (6, §§ 13, 28]).

(b) If Ω is of first order, and is asymptotically non-singular at M, then $G^{-1}\Lambda(z)$ is normal (in the sense of [5, § 83]), having divergence monomial $-V_1$.

PROOF - If Φ is the M-multiplication transform of Ω, then 1 is a simple non-parametric critical monomial of Φ, and $AF(1, \Phi, z)$ is of the form $C(z)$, where 1 is a simple root of $C(z)$. By Lemma 32, for s sufficiently large, there is a unit monomial N, such that when $N\Phi$ is written as a polynomial in $z, \theta_s z, \ldots, \theta_s{}^n z$, then each coefficient is $\precsim 1$, and $C(z)=\Sigma\, t_k(+\infty)z^k$, where $t_k(x)$ is the coefficient of z^k in this representation for $N\Phi$. Since 1 is a simple root of $C(z)$, $\Sigma\, kt_k(+\infty)=\lambda$ is non-zero. A simple computation shows that $F\backsim G$, if G is taken to be λN^{-1}.

Since $C(1)=0$, $G^{-1}\Lambda(0)\prec 1$. Let $G^{-1}\Lambda^{(1)}(z)=\Sigma\, H_j\theta_s{}^j z$. Since each coefficient of $N\Phi$ is $\precsim 1$, each $H_j\precsim 1$. Since $F\backsim G$, $H_0\backsim 1$. It then follows from [6, § 20], that $G^{-1}\Lambda^{(1)}(z)$ is unimajoral. The coefficient of $z^{(n)}$ in $G^{-1}\Lambda^{(1)}(z)$ is easily seen to be $G^{-1}M(\partial\Omega/\partial y^{(n)})(M)$, which is non-trivial by hypothesis. The existence of at least one principal factorization sequence for $G^{-1}\Lambda^{(1)}(z)$, therefore follows from [6, § 27], proving Part (a).

To prove Part (b), write $G^{-1}\Lambda(z)=\Sigma\, a_{ij}z^i(z')^j$. If (V_1) is a principal factorization sequence, then by definition, V_1 is in the divergence class, $a_{01}\sim -V_1^{-1}$ and $a_{10}\backsim 1$. By Part (a), $a_{00}\prec 1$, and since each coefficient of $N\Phi$ is $\precsim 1$, we have $a_{i0}\precsim 1$ for each i. To conclude the proof that $G^{-1}\Lambda(z)$ is normal, we must show there is a q for which $a_{ij} \lesseqgtr a_{01}(L_q)^{j-1}$ when $j \geq 1$ and $i+j \geq 2$. The proof of this follows from considering the transform $\Gamma(z)$ of $\partial\Omega/\partial y'$ under the change of variable, $y = M + Mz$. If Ω is asymptotically non-singular at M, then any principal monomial of Γ is not < 1, by Lemma 3. With this knowledge, the application of the algorithm of the principal monomial to Γ, readily produces the desired asymptotic relations for a_{ij}, thereby concluding the proof.

41. DEFINITION – Under the hypothesis and notation of Lemma 40,

(1) $(V_1, \ldots, V_n)$ is called a *type* for Ω at M.

(2) $G^{-1}\Lambda$ is called the *residual operator* for Ω at M.

(3) If $(V_1, \ldots, V_n)$ is a weak factorization sequence (see [6, § 88]), for $G^{-1}\Lambda$, then $(V_1, \ldots, V_n)$ is called an *asymptotically steady type* for Ω at M.

(These definitions extend those given in [6, § 116], for principal monomials).

42. THEOREM III – Let $S^\# = F(a, b)$, where $-\pi \leq a < b \leq \pi$. Let Ω be a first order differential polynomial which has coefficients in an $LD,(S^\#)$, and which is *NTPD*. Let M be a simple non-parametric critical monomial of Ω, at which Ω is asymptotically non-singular. Let (c, k, t) be the index (see [5, § 44]) of the type for Ω at M. Let $f(\theta) = \cos(\delta_{0k} t\theta + \arg(-c))$, for $a < \theta < b$, (where δ_{ij} is the Kronecker delta), and let $f(\theta) \not\equiv 0$. Then,

(a) For every point u in the open interval (a, b), there exists a positive number ν, and a function y_0, such that $\Omega(y_0) = 0$ and $y_0 \backsim M$ in $F(u-\nu, u+\nu)$.

(b) For each interval (a_1, b_1) in which f is positive there is a one-parameter family of solutions $\backsim M$ in $F(a_1, b_1)$, of the equation $\Omega(y) = 0$. For each interval (a_2, b_2) in which f is negative, there is a unique solution $\backsim M$ in $F(a_2, b_2)$, of the equation $\Omega(y) = 0$.

PROOF – By Lemma 40(b), the residual operator for Ω at M is normal, and its divergence monomial has index $(-c, k, t)$. Hence the theorem follows immediately from [5, § 126], concerning solutions of normal differential polynomials.

43. THEOREM IV – Let a, a_0 and b be real numbers such that $-\pi \leq a < a_0 < b \leq \pi$. Let $S^\# = F(a, b)$. Let Ω be an *nth* order differential polynomial with coefficients in an $LD(S^\#)$, and be *NTPD*. Let M be a simple non-parametric critical monomial of Ω, and let $(V_1, \ldots, V_n)$ be an asymptotically steady type for Ω at M. Let $(V_1, \ldots, V_n)$ be unblocked (see [6, § 98]) in (a, a_0, b). Then $\Omega(y) = 0$ has at least one solution $\backsim M$ in $S^\#$.

PROOF - Under the given conditions it follows from [6, § 115], that if $\Phi(z)$ is the residual operator for Ω at M, then $\Phi(z) = 0$ has a solution $\prec 1$ in $S^{\#}$. The theorem now follows immediately.

PART IX - Solution in the linear case.

The main result of this part is,

45. THEOREM V - Let $S^{\#} = F(a_1, a_2)$ where $-\pi \leq a_1 < a_2 \leq \pi$. Let $(A_0, A_1, .., A_n, g)$ be a sequence of $(n+2)$ functions lying in an $LD_0(S^{\#})$ such that A_n is non-trivial. Let $\Omega(y) = \Sigma\{A_j y^{(j)}: 0 \leq j \leq n\}$, and let M be any critical monomial of $\Omega(y) - g$. Then the equation $\Omega(y) = g$ has at least one solution $\sim M$ in some $F(a_3, a_4)$, where $a_1 \leq a_3 < a_4 \leq a_2$.

We need the following lemma.

44. LEMMA - Let $(B_0, B_1, ..., B_n, \varphi)$ be a sequence of $(n+2)$ functions lying in an $LD_0(S^{\#})$ (where $S^{\#}$ is arbitrary). Let the maximum of the numbers $\delta_0(B_i)$ be 0. Let $\Lambda(y) = \Sigma\{B_i \theta^i y : 0 \leq i \leq n\}$, where θ is the operator $\theta y = xy'$, and let M be a parametric monomial of $\Lambda(y) - \varphi$. Then,

(a) $M(x) = cx^{\delta}(\log x)^b$, for some real δ, some $b \in \{0, 1, ..., n-1\}$ and some constant c.

(b) $h = \varphi - \Lambda(M)$ is $\prec M$ and if h is non-trivial, say $\delta_0(h) = \alpha$, then there exists a polynomial $P(x)$, in x alone, with constant coefficients, such that $y^* = x^{\alpha} P(\log x)$ is $\prec M$ and $\delta_0(\Lambda(y^*) - h) < \alpha$.

PROOF - For each i, $B_i = b_i + w_i$ where b_i is constant and $\delta_0(w_i) < 0$. Let $\Phi(y) = \Sigma b_i \theta^i y$, $\Gamma(y) = \Sigma w_i \theta^i y$ and $\Omega(y) = \Lambda(y) - \varphi$. A straightforward computation shows $\Lambda[*, 0] = 0$. Let $\delta = \delta_0(M)$. Then $\Lambda[*, \delta] = \delta$. By Theorem I (§ 21), $[\delta; \Omega]^{(1)}$ is $NTPD$ on $\log S^{\#}$. Hence $\Omega[*, \delta] = \delta$ and therefore, $\delta_0(\varphi) \leq \delta$. Letting $\Xi(v)$ be the transform of $\Phi(y)$ under $y = ve^{\delta u}$, $x = e^u$, we have $[\delta; \Omega](v) = e^{-\delta u}\Xi(v) + T(v) - G(u)$, where $T(v)$ is trivial and $G(u) = e^{-\delta u}\varphi(e^u)$. Furthermore, we can write $e^{-\delta u}\Xi(v) = \Sigma\{\nu_j v^{(j)}: 0 \leq j \leq n\}$, for constants ν_j. Letting t be the smallest j for which $\nu_j \neq 0$, then the critical equation of $[\delta; \Omega]^{(1)}$ is $\nu_t s_{tt}(\alpha) = 0$, (where for $j \geq i$, $s_{ji}(\alpha)$ is the elementary symmetric function of degree i in $\alpha, \alpha - 1, ..., \alpha - j + 1$). By Theorem I, $b = \delta_1(M)$ is a root of $s_{tt}(\alpha) = \alpha(\alpha - 1) ... (\alpha - t + 1) = 0$, so $b \in \{0, 1, ..., t-1\}$. Then clearly ku^b is a solution of $\Xi(v) = 0$ for each constant k, so

$$\Phi(kx^{\delta}(\log x)^b) = 0. \tag{1}$$

We now prove,

$$\delta_0(\varphi) < \delta \text{ and } G(u) \text{ is trivial in } \log S^{\#}. \tag{2}$$

By Theorem I, $[b, \delta; \Omega]^{(1)}$ is *NTPD* and so $[\delta; \Omega][*, b] = b - t$. Hence $\delta_0(G) < 0$, since $b < t$, and (2) follows easily.

A simple calcutation now shows that in $[b, \delta; \Omega](v)$, the coefficient of v is trivial, while that of v' is of the form $a + g$, where $a = \nu_t s_{t,t-1}(b)$ and g is trivial. Now b is clearly a simple root of $s_{tt}(\alpha) = 0$, and since $s_{t,t-1}(\alpha) = = ds_{tt}(\alpha)/d\alpha$, we have $a \neq 0$. Hence the critical equation of $[b, \delta; \Omega]^{(1)}$ is $a\alpha = 0$, and therefore $\delta_2(M) = 0$, by Theorem I. It follows that $[\delta_2(M), b, \delta; \Omega](v) = = av' + R(v)$, where R is trivial. Repeated applications of Theorem I, now imply $\delta_i(M) = 0$ for $i \geq 2$, proving Part (a).

By (1), $\Phi(M) = 0$, and therefore $\Lambda(M) = \Sigma\{w_i \theta^i M : 0 \leq i \leq n\}$. Now $\theta^i M$ is a linear combination (with constant coefficients) of functions of the form $s_{ij}(b) x^{\delta} (\log x)^{b-j}$ for $0 \leq j \leq i$. Since $\delta_0(w_i) < 0$, $\delta_0(\Lambda(M)) < \delta$. Hence by (2), $\delta_0(h) < \delta$, where $h = \varphi - \Lambda(M)$, so $h < M$. Suppose now h is non-trivial, with $\delta_0(h) = \alpha$. Since $s_{jj}(b) = 0$ $j > b$, it follows that $h(x)$ is representable as a sum of functions of the form $f_j(x)(\log x)^j$ for $0 \leq j \leq n-1$, where each non-trivial f_j is of the form $c_j x^{d_j} + K_j$, where $\delta_0(K_j) < d_j$. Since some f_j is non-trivial, let d be the maximum of of the d_j, and let $Q(x) = \Sigma\{c_j x^j : d_j = d\}$. Then $h(x) = x^d Q(\log x) + K(x)$, where $\delta_0(K) < d$. Hence $\alpha = d$. It is proved in [8], that the differential equation $\Phi(y) = x^{\alpha} Q(\log x)$, possesses a solution of the form $y^* = x^{\alpha} P(\log x)$, where $P(x)$ is a polynomial. Then $y^* < M$ since $\alpha < \delta$. Finally, $\Lambda(y^*) - h = \Gamma(y^*) - K$, from which it easily follows that $\delta_0(\Lambda(y^*) - h) < \alpha$, thereby concluding the proof of Part (b).

PROOF OF THEOREM V - If $M \in pm(\Omega(y) - g)$, then the result is proved in [8].

If M is not a principal monomial, then $M \in \mathrm{par}(\Omega(y) - g)$, for by Lemma 29$(b)$, $\Omega(y) - g$ has no ordinary monomials if g is trivial, and has precisely one, namely its principal monomial, if g is non-trivial.

Let θ be the operator $\theta y = xy'$, and let $\Omega(y) = \Sigma\{B_j \theta^j y : 0 \leq j \leq n\}$. Then B_n is non-trivial. Let t be the j for which $\delta_0(B_j)$ is maximum, and let $\nu = \delta_0(B_t)$. Then letting $\Lambda(y) = x^{-\nu}\Omega(y)$ and $\varphi(x) = x^{-\nu} g(x)$, it is clear that $M \in \mathrm{par}(\Lambda(y) - \varphi)$ by Lemma 30(a), and that $\Lambda(y) - \varphi$ satisfies the hypothesis of Lemma 44.

If $h = \varphi - \Lambda(M)$ is trivial, then it is proved in [8], that $\Lambda(y) = h$ has a trivial solution y_0, in some $F(a_3, a_4)$, and so $y = M + y_0$ is a solution $\sim M$ of $\Omega(y) = g$.

If h is non-trivial with $\delta_0(h) = \alpha$, then by Lemma 44, $h < M$ and there is a polynomial $P(x)$, for which $y^* = x^{\alpha} P(\log x)$ is $< M$ and $\delta_0(\Lambda(y^*) - h) < \alpha$. Under the substitution $y = y^* + z$, the equation,

$$\Lambda(y) = h \tag{1}$$

becomes

$$\Lambda(z) = f \tag{2}$$

where $f = h - \Lambda(y^*)$. Thus $\delta_0(f) < a$. Now it is proved in [8] that there exists a finite set G of real numbers such that for any real α not in G, there is a non-zero constant k_α for which the linear differential polynomial $\Gamma_\alpha(w) = x^{-\alpha}\Lambda(k_\alpha x^\alpha w)$ is unimajoral and has a non-exceptional principal factorization sequence. Choose a real α not in G such that $\delta_0(f) < \alpha < a$. Under the substitution $z = k_\alpha x^\alpha w$, (2) is transformed into $x^\alpha \Gamma_\alpha(w) = f$, or equivalently

$$\Gamma_\alpha(w) = x^{-\alpha} f. \tag{3}$$

Letting $(V_1, \ldots, V_n)$ be a non-exceptional principal factorization sequence for Γ_α, clearly there exist a_3, a_4 such that $a_1 \leq a_3 < a_4 \leq a_2$ and $(V_1, \ldots, V_n)$ is unblocked in (a_3, a, a_4) for all $a) \in a_3, a_4)$. But by choice of α, $x^{-\alpha} f < 1$ and so $(V_1, \ldots, V_n)$ is a strong factorization sequence (see [6, § 88(*b*)]), for $\Gamma_\alpha(w) - x^{-\alpha} f$. Thus by [6, § 99] there is a function $w_0 < 1$ in $F(a_3, a_4)$ such that $\Gamma_\alpha(w_0) = x^{-\alpha} f$. Then it is clear from (1)-(3) that the function $y_1 = M + y^* + k_\alpha x^\alpha w_0$ is a solution of $\Omega(y) = g$, and satisfies $y_1 \sim M$ in $F(a_3, a_4)$, since $y^* < M$, $w_0 < 1$ and $\alpha < a$.

PART X - A simple example.

In this part, T_i will stand for a differential polynomial which is trivial in $\log_i S^\#$, and the sequence $(0, 0, \ldots)$ will be denoted $(0_1, 0_2, \ldots)$.

Let $\Omega(y) = x^{-9/2} y^3 + yy''' - x^{-2}$. We first apply Theorem I to find $\operatorname{par}(\Omega)$. The term of degree 3 will not contribute any parametric monomials, since the critical equation of $\Omega^{(3)}$ has no roots. The critical equation of $\Omega^{(2)}$ has the three roots, 0, 1 and 2. To test the root 0, we find $[0; \Omega](v) = e^{(-5/2)u} v^3 + (vv''' - 3vv'' + 2vv') e^{-u} - 1$. Since $[0; \Omega]^{(2)}$ is trivial, the process stops here for the root 0 (i.e. 0 is not the first coordinate of an s-tuple which satisfies § 21(*b*), relative to $p = 2$). Testing the root 1, we find $[1; \Omega](v) = e^{(-1/2)u} v^3 + vv''' - vv' - e^{-u}$. Hence $[1; \Omega]^{(2)}$ is non-trivial and we can continue. The critical equation of $[1; \Omega]^{(2)}$ has 0 as its only root, and $[0, 1; \Omega](v) = -vv' + T_2(v)$. Hence $[0, 1; \Omega]^{(2)}$ is non-trivial and 0 is the only root of its critical equation. It is now clear, by continuing this process, that $(1, 0, 0, \ldots, 0)$ is an s-tuple which satisfies § 21(*b*) relative to degree 2, and therefore $kx \in \operatorname{par}(\Omega)$ for each k. Clearly, $AF(kx, \Omega, y) = -yy'$. Since 0 was the only root of the critical equation of $[0_i, 0_{i-1}, \ldots, 0_1, 1; \Omega]^{(2)}$ for $i \geq 1$, kx are the only parametric monomials corresponding to the root 1. Finally, testing the root 2, we find $[2; \Omega]^{(2)}$ is trivial so the process stops. Since we have tested all the non-trivial homogeneous parts of Ω which are of positive degree, we conclude that $\mathrm{par}(\Omega)=\{kx\colon \text{all } k\}$. (We note that no logarithms appeared in the parametric monomials, and of course this is due to the fact that 0 was the only root of the critical equation of $[0_i, \ldots, 0_1, 1; \Omega]^{(2)}$, for $i \geq 1$. If however, $\Omega^{(2)}(y)$ had been $yy''' + x^{-1}yy''$, for example, then its critical equation would have the two roots 0 and 1, as before 0 would not contribute anything, but since $[1; \Omega]^{(2)}$ would now be $vv'' + vv'''$ it is clear that kx and $kx \log x$ would be in $\mathrm{par}(\Omega)$).

We now apply Theorem II to find $\mathrm{crit}(\Omega) - \mathrm{par}(\Omega)$. Since $\Omega^{(3)}[*, \alpha] = 3\alpha - 9/2$, $\Omega^{(2)}[*, \alpha] = 2\alpha - 3$, $\Omega^{(0)}[*, \alpha] = -2$, while all other $\Omega^{(q)}[*, \alpha]$ are $-\infty$, there are two admissible values, namely $\alpha = 3/2$ (from $\Omega^{(3)}$ and $\Omega^{(2)}$) and $\alpha = 1/2$ (from $\Omega^{(2)}$ and $\Omega^{(0)}$). For $\alpha = 3/2$, we find $[3/2; \Omega](v) = v^3 - (3/8)v^2 - (1/4)vv' + (3/2)vv'' + vv''' - e^{-2u}$. This has only 0 as an admissible value, and $[0, 3/2; \Omega](v) = v^3 - (3/8)v^2 + T_2(v)$. Again, this has only 0 as an admissible value, and it is now clear that $(3/8)x^{3/2} \in (\mathrm{crit}(\Omega) - \mathrm{par}(\Omega))$, with $AF((3/8)x^{3/2}, \Omega, v) = v^3 - (3/8)v^2$. This is the only contribution from $\alpha = 3/2$. Treating $\alpha = 1/2$ similarly, we find that $\pm(8/3)^{1/2}x^{1/2} \in (\mathrm{crit}(\Omega) - \mathrm{par}(\Omega))$, with associated function $(3/8)y^2 - 1$. (Of course, $\pm(8/3)^{1/2}x^{1/2}$ are the principal monomials of Ω). Hence there are three non-parametric critical monomials of Ω, and of course, each is simple.

Since $\partial\Omega/\partial y''' = x$, Ω possesses a type at each of the non-parametric critical monomials by Lemma 42(*a*). Computing the residual operators in each case, and using [6, § 44] to find the types, it is easily verified that Theorem IV can be applied to assert the existence of a solution $\sim M$ in $F(-\pi, \pi)$, for each $M \in (\mathrm{crit}(\Omega) - \mathrm{par}(\Omega))$. However in this particular example, more information about these solutions can be obtained by a more detailed investigation of the residual operators. In each case, it is found, using [6, § 105] that each of the residual operators is, in fact, uniformly quasi-linear. Hence [6, § 99] may be applied in each case, to assert that the equation $\Omega(y) = 0$ has (*a*) a unique solution $\sim (8/3)^{1/2}x^{1/2}$ in $F(-\pi, \pi)$. (*b*) a unique solution $\sim -(8/3)^{1/2}x^{1/2}$ in $F(-\pi, \pi)$, and (*c*) a one-parameter family of solutions $\sim (3/8)x^{3/2}$ in $F(-\pi, \pi)$.

For the parametric monomials, we consider $\Lambda_k(z) = \Omega(kx + z)$. It is found that Λ_k has a unique (simple) principal monomial, $N_k = (-8/3)k^2x^{1/2}$, at which it has a type. Following the same procedure as above, we find that the equation $\Lambda_k(z) = 0$ has a one parameter family of solutions $\sim N_k$ in $F(-\pi, \pi)$. Thus for each non-zero k, the equation $\Omega(y) = 0$ possesses a one-parameter family of solutions $\sim kx$ in $F(-\pi, \pi)$.

Hence in this example, for each $M \in \mathrm{crit}(\Omega)$, the equation $\Omega(y) = 0$ possesses at least one solution $\sim M$ in $F(-\pi, \pi)$.

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