Invariants of the group of a third order differential element.

RODNEY ANGOTTI (Buffalo U.S.A.)

Summary. - *In this note we discuss the invariant properties of the group of homographies which leave a third order differential element E_3 invariant.*

1. - In [2] BOMPIANI discussed the projective invariants of configurations associated with a third order curvilinear differential element E_3 in the ordinary projective space P_3; in particular, he showed that the configuration C_1 of the element E_3, a line r, and a point $P \notin r$ has a projective invariant and that the configuration C_2 of the element E_3 and two lines has two invariants. Furthermore, it is shown in [1] that the configuration C_2 has another invariant in the event that the two lines are incident and, in addition, that the invariants of C_2 which BOMPIANI discusses are not geometric when the two lines are incident. The purpose of this note is to discuss the projective properties of the configuration consisting of the element E_3, a line r and a plane $\pi \not\supset r$; in fact, we show that this configuration has a projective invariant and give an analytical formulation of this. We remark here that the configuration C_3 is not the dual configuration of C_1. In fact the dual configuration of C_1 would be a line, a plane and, using BOMPIANI'S terminology, a particular pseudo-element, i.e., an equivalence class of differential elements. For a precise definition of pseudo-element, see [3].

The author wishes to acknowledge here a communique from Professor BOMPIANI which was instrumental in the preparation of this manuscript.

In the following we will indicate points of P_3 by the ordered tetrad (t, x, y, z) or, alternatively, by $x^i (i = 0, 1, 2, 3)$.

2. - Let C be a curve passing regularly through the point $0(1, 0, 0, 0)$ with tangent $y = z = 0$ and osculating plane $z = 0$. The equivalence class of curves (of at least class C^2 in a neighborhood of 0) which have a contact of order three with C at 0 is called a third order curvilinear differential element E_3 with center 0. This element can be represented analytically by the expansions $(t = 1)$

$$y = ax^2 + a_1x^3 + 0(x^4)$$

$$z = \qquad bx^3 + 0(x^4)$$

where the coefficients a, a_1, and b are determined by C and the symbol $0(x^4)$ indicates that the coefficients of the terms > 4 are arbitrary.

By changing the plane $y = 0$, we can always reduce the representation of the element E_3 to the form

$$\begin{aligned} y &= ax^2 + 0(x^4) \\ z &= bx^3 + 0(x^4) \end{aligned} \tag{2.1}$$

where $t = 1$, $a \cdot b \neq 0$.

The group of homographies $G(E_3)$ which leave the expansions (2.1) invariant (or leave the element E_3 invariant) is represented by

$$\begin{aligned} \lambda t' &= p_1 t + p_2 x + p_3 y + p_4 z \\ \lambda x' &= q_1 x + q_2 y + q_3 z \\ \lambda y' &= p_1^{-1} q_1^2 y + q_4 z \\ \lambda z' &= p_1^{-2} q_1^3 z \end{aligned}$$

where

$$q_4 = ab^{-1} p_1^{-2} q_1 (2ap_1 q_2 - p_2 q_1) .$$

Since each collineation of this group maps a generic line r, whose equations we can always write in the form

$$\begin{aligned} t - s_2 y - s_3 z &= 0 \\ x + h_2 y + h_3 z &= 0 , \end{aligned} \tag{2.2}$$

onto $t' = x' = 0$, we can express these homographies more conveniently by the equations

$$\begin{aligned} \lambda t' &= t - s_1 x - (s_2 + h_2 s_1) y - (s_3 + h_3 s_1) z \\ \lambda x' &= n(x + h_2 y + h_3 z) \\ \lambda y' &= n^2 (y + pz) \\ \lambda z' &= n^3 z \end{aligned}$$

where

$$p = ab^{-1}(s_1 + 2ah_3) . \tag{2.3}$$

For a fixed line r, equation (2.3) represents a projectivity between the pencil of planes through the tangent (depending upon p) and the pencil of planes through r (depending on s_1). The locus of line intersections of corresponding planes in this projectivity are generators of the quadric Q_r

$$az(t - s_2y - s_3z) + (by + 2a^2h_2z)(x + h_2y + h_3z) = 0. \tag{2.4}$$

By varying r, we obtain a linear system of quadrics of freedom three. This system is characterized geometrically in [2].

3. - Since $G(E_3)$ depends on six parameters, a generic line (which depends on four parameters) and a generic plane (which depends on three parameters) has an invariant under $G(E_3)$; in fact, we have the theorem:

THEOREM 1. - Let r be a generic line with equations (2.2) and π be a plane, not passing through r, with plane-coordinates $u_i (i = 0, 1, 2, 3)$, $u_0 \neq 0$, then the function

$$I(\pi, r) = \frac{[au_1\pi_1 + bu_0(\pi_2 - k\pi_1)]^2}{u_0\pi_1^3} \tag{3.1}$$

where

$$\pi_1 = u_2 - u_1h_2 + u_0s_2$$

$$\pi_2 = u_3 - u_1h_3 + u_0s_3$$

$$k = 2ab^{-1}h_2$$

is invariant under $G(E_3)$.

PROOF: - Take a transformation $g \in G(E_3)$ which maps r onto $r' : [t = x = 0]$. We have

$$I(\pi', r') = [a\bar{u}_1\bar{u}_2 + b\bar{u}_0\bar{u}_3]^2/(\bar{u}_0\bar{u}_2^3)$$

where the $\bar{u}_i$ are the plane coordinates of $g\pi = \pi'$. A calculation shows that for any $f \in G(E_3)$

$$I(g\pi, gr) = I(\pi', r') = I(f\pi', fr').$$

By choosing $f = g^{-1}$, we have

$$I(g\pi, gr) = I(\pi, r).$$

Let $G(I(\pi, r))$ be the group of homographies which leave $I(\pi, r)$ invariant. Obviously $G(E_3) \subset G(I(\pi, r))$. In the following we show that $G(I(\pi, r)) \subset G(E_3)$ and, consequently, $G(I(\pi, r)) = G(E_3)$.

LEMMA. - A homography f which leaves the line r': $[t = x = 0]$ invariant and has the property that $I(\pi, r') = I(f\pi, r')$, identically with respect to π, belongs to $G(E_3)$.

PROOF. - If the homography f has the form $x^{i'} = a_{ij}x^j$, $(i,j = 0, 1, 2, 3)$, then $a_{ij} = 0$, for $i \neq j$, and

$$a_{33} = a_{11}^{-1}a_{22}^2$$

$$a_{44} = a_{11}^{-2}a_{22}^3$$

which imply that $f \in G(E_3)$.

Now take a homography $h \in G(I(\pi, r))$. The homography h has the property

$$I(h\pi, hr) = I(\pi, r)$$

for generic π and r. There obviously exist homographies $g_1, g_2 \in G(E_3)$ which map r and hr onto r': $[t = x = 0]$, respectively. Since g_1, g_2 leave (3.1) invariant, we have

$$I(g_1\pi, g_1r) = I(\pi, r) = I(h\pi, hr) = I(g_2h\pi, g_2hr).$$

The homography $f = g_2hg_1^{-1}$ maps g_1r onto $g_2hr = r'$ and leaves (3.1) invariant; therefore, by the Lemma it belongs to $G(E_3)$. The equality $h = g_2^{-1}fg_1$ implies that $h \in G(E_3)$. We have proved the following theorem:

THEOREM 2. - A homography of P_3 which leaves $I(\pi, r)$ invariant is a transformation of the group $G(E_3)$.

COROLLARY. - The invariant (3.1) of a plane and line completely characterizes the group $G(E_3)$.

We propose to show that the invariant $I(\pi, r)$ is transitive with respect to $G(E_3)$, i.e., there exists a transformation which maps a given plane-line pair π_1, r_1 onto any other plane-line pair π_2, r_2 for which $I(\pi_1, r_1) = I(\pi_2, r_2)$. Clearly there exist transformations, say f and g, of $G(E_3)$ which map r_1, r_2 onto r': $[t = x = 0]$, respectively. Since $f, g \in G(E_3)$, we have

$$I(\pi_1, r_1) = I(f\pi_1, fr_1) = I(g\pi_2, gr_2) = I(\pi_2, r_2).$$

There exists, however, a unique transformation $h \in G(E_3)$ which leaves r' invariant and is such that $hf\pi_1 = g\pi_2$. Therefore, the composition $g^{-1}hf$ maps

r_1 onto r_2, π_1 onto π_2 and has the property that

$$I(\pi_1, r_1) = I(g^{-1}hf\pi_1, g^{-1}hfr_1) = I(\pi_2, r_2).$$

COROLLARY. - The invariant $I(\pi, r)$ is the only (independent) invariant of a plane and line under transformations of $G(E_3)$.

4. - In order to give a geometrical construction of the invariant $I(\pi, r)$, consider the envelopes of the quadrics which contain the element E_3, the line r, and are tangent to the plane π. The residual intersection of this system is a cubic developable on which there are four well-determined planes; namely, the plane π, the osculating plane $z = 0$ and the two planes of the developable intersecting in the line r. It is only a matter of calculation to verify that the cross ratio of these four planes is a rational function of $I(\pi, r)$.

This invariant can be interpreted more directly as the cross ratio of four points on the unique skew cubic C^3 which contains the element E_3 and is tangent to the line r at the point $P = r \cap \pi$. In this case, the four points are the center 0 of the element E_3, the point P, and the two additional points, say Q, R, in which the plane π cuts C^3. This construction gives rise to several other equivalent interpretations of the invariant $I(\pi, r)$.

5. - If we fix the line r, the set of planes for which $I(\pi, r) = 0$ is given by

$$au_1(u_2 - u_1h_2 + u_0s_2) + bu_0[u_3 - ku_2 - (h_3 - h_2k)u_1 + (s_3 - ks_2)u_0] = 0$$

where $k = 2ab^{-1}h_2$. This locus is the envelope of the quadric Q_r obtained from the projectivity (2.3). More generally, the planes for which $I(\pi, r)$ is constant, r fixed, belong to the fourth class envelope in the pencil determined by the envelope of the quadric Q_r, counted twice, and the envelope composed of the star of planes through the center 0 of the element E_3 and the pencil of planes with axis r, counted three times.

REFERENCES

[1] R. ANGOTTI, *The Projective Invariants of Two Lines and a Third Order Differential Element*, «Rend. Semin. Padova», 35 (1965).

[2] E. BOMPIANI. *Configurations associées à un élément curviligne du troisieme ordre dans l'espace projectif*, «Journal des Math. pure et appl.», 41 (1962).

[3] E. BOMPIANI, *Nuovi Enti Geometrici: Pseudoelementi Differenziali*, «Sem. Mat. e Fisico» 33 (1963).