A strong form of spectral resolution

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Summary. - *We consider pseudo-measures T on totally disconnected sets E as finitely additive set functions whit the usual variation norm $\| \|_v$. If $E \subseteq R/2\Pi Z$, $m(E)=0$, $T=f'$, $f \in L^1$, and $\|T\|_v < \infty$ then T is a measure. For arbitrary locally compact abelian groups we give conditions that T be a measure in terms of the pseudo-measure norm; this result is known for $R/2\Pi Z$.*

Let G be a locally compact abelian group with dual group Γ and let $E \subseteq \Gamma$ be a compact totally disconnected set. We employ the following standard notation: $L^1(G)$ is the space of integrable functions on G with the usual L^1-norm $\| \|_1$; $A(\Gamma)$ is the space of Fourier transforms φ of $x \in L^1(G)$, where $\|\varphi\|_A \equiv \|x\|_1$; $C(\Gamma)$ is the algebra of continuous functions vanishing at infinity with sup-norm $\| \|_\infty$; $A'(\Gamma)$ is the space of pseudo-measures (the dual of $A(\Gamma)$) with dual norm $\| \|_{A'}$; $\mathfrak{M}(\Gamma) \subseteq A'(\Gamma)$ is the space of bounded Radon measures (the dual of $C(\Gamma)$) with dual norm $\| \|_1$; $A'(E)$ consist of those $T \in A'(\Gamma)$ with support contained in E; $A'_S(E)$ consists of those $T \in A'(E)$ where for all $\varphi \in A(\Gamma)$, $\varphi = 0$ on E, $< T, \varphi > = 0$; $\mathfrak{M}(E)$ consists of those $\mu \in \mathfrak{M}(\Gamma)$ with support contained in E.

E is a *spectral* set if $A'(E) = A'_S(E)$, a ***Helson*** set if $\mathfrak{M}(E) = A'_S(E)$, a set of *spectral resolution* if every closed subset of E is spectral, a set of *strong-spectral resolution* if $\mathfrak{M}(E) = A'(E)$, and a *Kronecker* set if every continuous complex valued function φ on E, where $|\varphi| = 1$, can be uniformly approximated by continuous characters.

In this paper we give conditions that E be a set of strong spectral resolution in terms of a suitable measure of the variation of a pseudomeasure. In § 1 we define pseudo-measure variation $\| \|_v$ and construct a cannonical map $A'(E) \to \mathfrak{M}(E)$ for those $T \in A'(E)$ with $\|T\|_v < \infty$. Some properties of $\| \|_v$ are presented in § 2 as well as general criteria for strong spectral resolution. We consider the special case $\Gamma = R$ (mod 2π) in § 3 and by two different methods get analogous results for arbitrary Γ in § 4.

Finally we treat the case of profinite groups in § 5 and illustrate the identity between pseudo-measure variation and Malliavin's recent notion of a strong c-lifting [3]. The result in this last section is preliminary and fur-

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ther related results will subsequently appear. Also we adapt a technique and lemma due to VAROPOULOS [5] for our Theorem 1.1 and Theorem 3.1.

By way of perspective we have the following situation in $\Gamma = R \pmod{2\pi}$. Helson showed that every Helson set is a set of uniqueness in the wide sense (clearly, these latter sets have Haar measure zero, and, thus, they are totally disconnected). It is not known if every Helson set is a set of strong spectral resolution, or, even if such sets are sets of uniqueness; obviously, strong resolution sets are of spectral resolution and Malliavin shoved that spectral resolution sets are sets of uniqueness. Further, Varopoulos proved that Kronecker sets are of strong spectral resolution and that there are sets of spectral resolution which are not Helson. References and discussion to these results are given in [1].

1. - Pseudo-measure variation.

A *finite decomposition of* E is $\{F_j : j = 1, \ldots, n\}$ where the $F_j \subseteq \Gamma$ are closed and mutually disjoint and such that $\cup F_j = E$. We let

$$\mathfrak{F} = \{F \subseteq E : F \text{ is closed and is an element of some finite decomposition of } E\}:$$

and for $T \in A'(E)$, $F \in \mathfrak{F}$, we define

$$T(F) \equiv \langle T, \psi_F \rangle,$$

where $\psi_F \in A(\Gamma)$, $\psi_F \equiv 1$ on some neighborhood of F, and $\psi_F \equiv 0$ on some neighborhood of $E - F$. Note that $E - F$ is closed. A *finite decomposition of* $T \in A'(E)$ is $\{T_j : j = 1, \ldots, n\} \subseteq A'(E)$ such that $T = \Sigma T_j$ and for which there is a finite decomposition of E, $\{F_j\}$, satisfying supp $T_j \subseteq F_j$.

PROPOSITION 1.1. - Let $T \in A'(E)$. For all $F \in \mathfrak{F}$, $T(F)$ is well defined, and for every finite decomposition $\{F_j\}$ of E, there is a unique finite decomposition of T given by $T_j \equiv T\psi_{F_j}$.

PROOF. - $T(F)$ is well-defined for if $\tilde{\psi}_F \in A(\Gamma)$, $\tilde{\psi}_F \equiv 1$ on a neighborhood of $E - F$, we have $\langle T, \psi_F - \tilde{\psi}_F \rangle = 0$.

For the unique finite decomposition, letting $\{S_j\}$ be a finite decomposition of T and $\varphi \in A(\Gamma)$, we have

$$\langle S_j, \varphi \rangle = \langle S_j, \varphi\varphi_j \rangle = \sum_k \langle S_k, \varphi\varphi_j \rangle = \langle T, \varphi\varphi_j \rangle = \langle T\psi_{F_j}, \varphi \rangle,$$

where $\varphi_j \in A(\Gamma)$, $\varphi_j \equiv 1$ on a neighborhood of F_j, and $\varphi_j \equiv 0$ on a neighborhood of $E - F_j$. *qed.*

The ***pseudo-measure variation*** of $T \in A'(E)$ is

$$\|T\|_v = \sup\{\Sigma |T(F_j)| : \{F_j\} \text{ is a finite decomposition of } E\};$$

and we let $\mathcal{R} \equiv \{T \in A'(E) : \|T\|_v < \infty\}$. Clearly, for all $\mu \in \mathfrak{M}(E)$, $\|\mu\|_v \leq \|\mu\|_1$, since for a given finite decomposition, $\{F_j\}$,

$$\Sigma |\mu(F_j)| = \Sigma < \mu, \psi_{F_j} e^{is_j} > = |<\mu, \psi>| \leq \|\mu\|_1,$$

where $\psi \in A(\Gamma)$, $\psi = \Sigma \psi_{F_j} e^{is_j}$ on a neighborhood of supp μ, and $\|\psi\|_\infty \leq 1$. In particular, $\mathfrak{M}(E) \subseteq \mathcal{R} \subseteq A'(E)$.

THEOREM 1.1. - There is a well-defined map

$$c : \mathcal{R} \to \mathfrak{M}(E)$$
$$T \mapsto c(T) \equiv \mu_T$$

and a subspace $V(E) \subseteq A(\Gamma)$, sup-norm dense in $C(\Gamma)$, such that supp $\mu_T \subseteq$ supp T and $T = \mu_T$ on $V(E)$.

PROOF. - Define

$$V(E) \equiv \{\varphi \in A(\Gamma) : \text{there is open } U_\varphi \supseteq E \text{ such that card } \varphi(U_\varphi) < \infty\}.$$

Clearly, $V(E)$ is a subalgebra of $C(\Gamma)$ and we use the Stone-Weierstrass theorem to show $\overline{V(E)} = C(\Gamma)$. It is only necessary to check that $V(E)$ separates points and this will follow since E is totally disconnected. In fact, letting γ, $\lambda \in E$ there are compact open E_γ, E_λ, disjoint with $E = E_\gamma \cup E_\lambda$, $\gamma \in E_\gamma$, $\lambda \in E_\lambda$; thus taking disjoint open sets about E_γ and E_λ we construct ([4; p. 49]) $\varphi \in V(E)$ for which $\varphi(\lambda) \neq \varphi(\lambda)$.

For $T \in A'(E)$ we define the linear functional

$$c(T) : V(E) \to C$$
$$\varphi \mapsto < T, \varphi >.$$

$c(T)$ is obviously well-defined, and for fixed $\varphi \in V(E)$, with corresponding U_φ, we see that

(1) $$U_x \equiv U_\varphi \cap \varphi^{-1}(x) \text{ is open for all } x \in \varphi(E)$$

and

(2) $$E \subseteq \cup\{U_x : x \in \varphi(E)\}.$$

(2) is obvious. For (1), take $x \in \varphi(E)$ and an open neighborhood N of x such

that $N \cap (\varphi(U_\varphi) - \{x\}) = \emptyset$; then $U_\varphi \cap \varphi^{-1}(N)$ is open and we are done since $U_\varphi \cap \varphi^{-1}(N) = U_\varphi \cap \varphi^{-1}(x)$.

By definition of φ there are only finitely many U_x and the U_x are mutually disjoint.

Clearly, $\{F_x : x \in \varphi(E)$ and $F_x \equiv E \cap U_x\}$ is a finite decomposition of E, noting that F_x is closed since $F_x = E \cap \varphi^{-1}(x)$.

Thus, we use Proposition 1.1 and form the canonical finite decomposition $\{T_x\}$ of T. For $\varphi \in V(E)$ and $T \in A'(E)$ we then have

$$c(T)(\varphi) = \sum_{x \in \varphi(E)} x < T_x, 1 >$$

since

$$(3) \qquad c(T)(\varphi) = \sum_{x \in \varphi(E)} < T\psi_{F_x}, \varphi > = \sum_{x \in \varphi(E)} x < T, \psi_{F_x} >$$

where the last equality follows because $\varphi(\gamma) \equiv x$ on U_x.

Therefore, for all $\varphi \in V(E)$ and for all $T \in \mathfrak{R}$,

$$(4) \qquad |c(T)(\varphi)| \leq \sup_{\gamma \in E} |\varphi(\gamma)| \sum_{x \in \varphi(E)} |T(F_x)| \leq M_T \sup_{\gamma \in E} |\varphi(\gamma)|,$$

where M_T is a constant depending on T and where the last inequality follows since $T \in \mathfrak{R}$.

Hence $c(T) \in \mathfrak{M}(E)$ noting that $\overline{V(E)} = C(\Gamma)$ implies that the restriction of $V(E)$ to E is sup norm dense in $C(E)$

Because of (3), the first inequality in (4) can have E replaced by supp T, and thus $c(T) \in \mathfrak{M}(\text{supp } T)$. *qed.*

2. - Criteria for spectral resolution.

We begin by recording some trivial properties of the notions introduced in § 1.

PROPOSITION 2.1. -

a. $\| \ \|_v : \mathfrak{R} \to R^+$ is a semi-norm.

b. $c : \mathfrak{R} \to \mathfrak{M}(E)$ is a linear surjection and for all $\mu \in \mathfrak{M}(E)$, $c(\mu) = \mu$.

It is equally obvious that for $T \in \mathfrak{R}$, $c(T) \equiv 0$ yields $\hat{T}(0) = 0$, and that if $F \subseteq E$ then $V(E) \subseteq V(F)$. Further

PROPOSITION 2.2. -

a. For all $T \in A'(E)$, $\|T\|_v = \|c(T)\|_1$.

b. For all $T, S \in \mathfrak{R}$ with supp $T \cap$ supp $S = \emptyset$, $\|S + T\|_v = \|S\|_v + \|T\|_v$.

c. For all $T \in \mathfrak{R}$ and for any finite decomposition $\{T_j\}$ of T, $\Sigma \| T_j \|_v \leq \| c(T) \|_1$.

PROOF. - *a.* To show $\| c(T) \|_1 \leq \| T \|_v$ we first note that $B = \overline{V(E) \cap B}$, where B is the closed unit ball in $C(\Gamma)$.

Thus, for any $\varepsilon > 0$ there is $\varphi \in B$, $\psi \in B \cap V(E)$ such that

$$\| c(T) \|_1 \leq | c(T)(\varphi) | + \frac{\varepsilon}{2} \leq | c(T)(\psi) | + \varepsilon = < T, \psi e^{is} > + \varepsilon =$$

$$\Sigma < T, c_j \psi_{F_j} > + \varepsilon \leq \Sigma \, | c_j | \, | < T, \psi_{F_j} > | + \varepsilon \leq$$

$$\Sigma \, | < T, \psi_{F_j} > | + \varepsilon \leq \| T \|_v .$$

For the opposite inequality, if $\{F_j\}$ is a finite decomposition of E we have

$$\Sigma \, | < T, \psi_{F_j} > | = \Sigma < T, \psi_{F_j} e^{is_j} >$$

where $\varphi \equiv \Sigma \, \psi_{F_j} e^{is_j}$ is taken in B.

Since $\varphi \in V(E)$ we have

$$\Sigma \, | < T, \psi_{F_j} > | = c(T)(\varphi) \leq \| c(T) \|_1$$

so that taking the sup of the left hand side over all finite decomposition gives $\| T \|_v \leq \| c(T) \|_1$.

b is clear from *a* since total variation is additive when dealing with measures having disjoint supports. *c* is a straightforward manipulation again using this additivity property of total variation. *qed.*

Thus for E a Helson set (and so there is $M > 0$ such that $\| \mu \|_1 \leq M \| \mu \|_{A'}$ for all $\mu \in \mathfrak{M}(E)$) we have $\| T \|_v \leq M \| c(T) \|_{A'}$ for all $T \in \mathfrak{R}$.

THEOREM 2.1. - The following are equivalent:

a. E is a set of strong spectral resolution.

b. $(A'(E), \| \; \|_v)$ is a normed space.

c. For all $T \in A'(E)$, $\| T \|_{A'} \leq \| T \|_v < \infty$.

PROOF. - The implications *a* to *c* to *b* are obvious by the above propositions.

To show *b* implies *a* we observe that $(A'(E), \| \; \|_v)$ nomed tells us $c : A'(E) \rightarrow \mathfrak{M}(E)$ is one to one. *qed.*

Hence, E is a set of strong spectral resolution if $\| T \|_v = \| T \|_{A'}$ for all $T \in A'(E)$.

Related to the pseudo-measure variation we define the *strong variation* *of* $T \in A'(E)$ to be

$$\|T\|_{sv} \equiv \sup\{\Sigma|<T, \varphi_{F_j}>| : \{F_j\} \text{ is a finite decomposition of } E\},$$

where $\varphi_F \in A(\Gamma)$ is zero on a neighborhood of $E - F$ and $|\varphi_F| \equiv 1$ on a neighborhood of F. Thus, $\|T\|_{sv} \geq \|T\|_{A'}$ and $\|T\|_{sv} \geq \|T\|_v$ for all $T \in A'(E)$, and we let $\mathcal{B} \equiv \{T \in A'(E) : \|T\|_{sv} < \infty\}$. These two inequalities and the technique of Theorem 2.1 give

PROPOSITION 2.3. - $\mathcal{B} = \mathfrak{M}(E)$.

3. - Strong spectral resolution in R (mod 2π).

We recall the following Varopoulos lemma [5] whose proof is based on a standard approximate identity result ([4; Theorem 2. 6. 5.]).

LEMMA - For all $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that for all $m, n \in Z$, $\theta \in R$ satisfying

$$|e^{in\gamma} - e^{i(\theta + m\gamma)}| < \delta(\varepsilon) \text{ for } \gamma \in F,$$

we have

$$|\widehat{T}(n) - e^{i\theta}\,\widehat{T}(m)| < \varepsilon\|T\|_{A'} \text{ for } T \in A'(F).$$

THEOREM 3.1. - Let $T \in A'(E)$ and let $N_T > 0$ have the property that for every finite decomposition $\{T_j\}$ of T, $\Sigma\|T_j\|_{A'} < N_T$. Then $T \in \mathfrak{M}(E)$.

PROOF. - By hypothesis $T \in \mathcal{B}$ since $\Sigma|<T, \psi_{F_j}>| = \Sigma|\widehat{T}_j(0)| \leq \Sigma\|T_j\|_{A'}$.

We show that for fixed $n \in Z$ and $\varepsilon > 0$, $|\widehat{T}(n) - \widehat{\mu}_T(n)| < \varepsilon$. Using the notation of the Lemma and the fact that $\overline{V(E)} = C(\Gamma)$ implies the restriction on $V(E)$ to E is sup norm dense in $C(E)$, we choose $\varphi \in V(E)$ such that

$$\sup_{\gamma \in E} |e^{in\gamma} - \varphi(\gamma)| < \delta\left(\frac{\varepsilon}{2N_T}\right) \equiv \delta \tag{5}$$

and

$$|\widehat{\mu}_T(n) - \mu_T(\varphi)| < \frac{\varepsilon}{2}. \tag{6}$$

Note that without loss of generality we can choose $|\varphi| \equiv 1$ in (5) and (6). To see this first choose $\psi \in V(E)$ such that $\psi(\gamma) \equiv c_j e^{is_j}$ on F_j, $\{F_j\}$ a finite decomposition of E and $\sup_{\gamma \in E} |e^{in\gamma} - \psi(\gamma)| < \frac{\delta}{2}$. Then $1 - \frac{\delta}{2} < c_j < 1 + \frac{\delta}{2}$ and we define $\varphi \in V(E)$ such that $\varphi(\gamma) \equiv (c_j + d_j)e^{is_j}$ on F_j where $|d_j| < \frac{\delta}{2}$ and $c_j + d_j = 1$. This does it.

For this finite decomposition of E we form the canonical finite decomposition $\{T_j\}$ of T so that (5), the Lemma, and the hypothesis that $\Sigma\|T_j\|_{A'} \leq N_T$ imply

$$|\widehat{T}(n) - \sum_j e^{is_j}\widehat{T}_j(0)| < \frac{\varepsilon}{2}.$$

Since $\varphi \in V(E)$, $\mu_T(\varphi) = <T, \varphi> = \sum_j e^{is_j}\widehat{T}_j(0)$.

Thus,

$$|\widehat{\mu}_T(n) - \widehat{T}(n)| \leq |\widehat{\mu}_T(n) - \mu(\varphi)| + |\mu(\varphi) - \widehat{T}(n)| < \varepsilon.$$

qed.

COROLLARY 3.1.1. - E is a set of strong spectral resolution if and only if for all $T \in A'(E)$ there is $N_T > 0$ such that for every finite decomposition $\{T_j\}$ of T we have $\Sigma\|T_j\|_{A'} \leq N_T$.

PROOF. - One way is immediate from the theorem. The other way is clear by taking $N_T \equiv \|T\|_1$ for $T \in \mathfrak{M}(E)$. *qed.*

In proving that Kronecker sets are sets of strong spectral resolution, Varopoulos [5] showed that

$$\Sigma\|T_j\|_{A'} = \|T\|_{A'} \tag{7}$$

for every finite decomposition of E; he used the Lemma in his proof. Obviously (7) is sufficient to apply Theorem 3.1 to Kronecker sets and hence $\|T\|_{A'} = \|T\|_v$ for such sets (since $\|\mu\|_{A'} = \|\mu\|_1$ for all $\mu \in \mathfrak{M}(E)$, E Kronecker). We can also easily deduce this equality by using only (7) and the Lemma.

4. Strong spectral resolution in Γ.

The proof of Theorem 3.1 depended on properties of R (mod 2π) because of the Varopoulos lemma. We now give two different proofs of Theorem 3.1 which are valid for any Γ.

Lemma 4.1 - Let X be a locally compact T_2 space and let $E \subseteq X$ be closed. Assume $E \subseteq \bigcup_{j=i}^{n} U_j$, where each U_j is a compact neighborhood of some $\lambda_j \in E$ and $\partial U_j \cap E = \emptyset$. Then

a. The sets $E_1 \equiv E$, $E_2 \equiv E_1 - E_1 \cap U_1$, $E_3 \equiv E_2 - E_2 \cap U_2$, ..., $E_n \equiv E_{n-1} - E_{n-1} \cap U_{n-1}$ form a decreasing sequence of closed sets.

b. There exist compact neighborhoods V_j of $E_j \cap U_j$, $j = 1, \ldots, n$, such that the V_j are mutually disjoint, each $V_j \subseteq U_j$, and $E \subseteq \bigcup V_j$.

PROOF. - Since $\partial U_1 \cap E = \emptyset$ we have $U_1 \cap E = U_1^0 \cap E$ and hence $U_1 \cap E$ is open in E. Thus $E = (E_1 \cap U_1) \cup E_2$ and $(E_1 \cap U_1)$, E_2 are disjoint closed sets of E.

By local compactness, $E_1 \cap U_1 \subseteq \tilde{V}_1 \subseteq \mathcal{C}E_2$ for some compact neighborhood $\tilde{V}_1$ of $E_1 \cap U_1$, and we set $V_1 \equiv \tilde{V}_1 \cap U_1$.

Clearly, $V_1 \subseteq U_1$, $V_1 \cap E_2 = \emptyset$ since $V_1 \subseteq \tilde{V}_1 \subseteq \mathcal{C}E_2$, and V_1 is a compact neighborhood of $E_1 \cap U_1$ since $\partial U_1 \cap E_1 = \emptyset$.

Now for $U_2 \cap E_2$ we have $\partial U_2 \cap E_2 = \emptyset$ because $\partial U_2 \cap E_1 = 0$, and thus $U_2 \cap E_2 = U_2^0 \cap E_2$ so that $U_2 \cap E_2$ is open in E_2. Consequently, $E = (U_1 \cap E_1) \cap (U_2 \cap E_2) \cap E_3$, a disjoint union of closed sets in E.

Again, by local compactness, $E_2 \cap U_2 \subseteq \tilde{V}_2 \subseteq \mathcal{C}(V_1 \cap E_3)$, for some compact neighborhood $\tilde{V}_2$ of $E_2 \cap U_2$, and we define $V_2 \equiv \tilde{V}_2 \cap U_2$.

As with V_1, $V_2 \subseteq U_2$, V_2 is a compact neighborhood of $U_2 \cap E_2$, and $V_1 \cap E_3 = \emptyset$, $V_2 \cap E_3 = \emptyset$. Further, since $\tilde{V}_2 \subseteq \mathcal{C}V_1$, $V_1 \cap V_2 = \emptyset$.

Continuing this process we find V_j, $j = 1, \dots, n-1$, where the V_j are mutually disjoint compact neighboroods of the $E_j \cap U_j$, each $V_j \cap E_n = \emptyset$, and $E = (U_1 \cap E_1) \cup \dots (U_{n-1} \cap E_{n-1}) \cup E_n$, a disjoint union of closed sets in E.

Now, $E_n \subseteq U_n$. If not there is $\lambda \in E_n$, $\lambda \notin U_h$ whereas $\lambda \in U_j$ for some $j < n$; but $\lambda \in E_n$ implies $\lambda \in E_j \cap U_j$, so that $\lambda \notin E_{j+1}$ and, consequently, $\lambda \notin E_n$.

Also, $\partial U_n \cap E_1 = \emptyset$ implies $\partial U_n \cap E_n = \emptyset$, so that

$$E_n = U_n \cap E_n = U_n^0 \cap E_n .$$

Therefore, since $E_n \cap (\bigcup_1^{n-1} V_j) = \emptyset$ we have $E_n \subseteq U_n^0 \cup \mathcal{C}(\bigcup_1^{n-1} V_j)$, and local compactness gives

$$E_n \subseteq \tilde{V}_n \subseteq U_n^0 \cap \mathcal{C}(\bigcup_1^{n-1} V_j),$$

$\tilde{V}_n$ a compact neighborhood of E_n. Finally, set $V_n \equiv \tilde{V}_n \cap U_n$ so that $V_n \subseteq U_n$.

qed.

THEOREM 4.1 -

a) Let $T \in A'(E)$. If there is N_T such that, for every finite decomposition $\{T_j\}$ of T, $\Sigma \|T_j\|_{A'} \leq N_T$ then

$$\| T \|_{A'} \leq \| T \|_v < \infty .$$

b) E is of strong spectral resolution if and only if for all $T \in A'(E)$ there is $N_T > 0$ such that, for every finite decomposition $\{T_j\}$ of T, $\Sigma \| T_j \|_{A'} \leq N_T$.

PROOF. -

a) Given $\varepsilon > 0$ and let $\varphi \in A(\Gamma)$ have the properties that $|\varphi| \leq 1$ and $| \| T \|_{A'} - < T, \varphi > | < \varepsilon/2$.

For each $\gamma \in E$ and W_γ, a neighborhood of γ, there is an open γ-neighborhood $\tilde{U}_\gamma$ and $h_\gamma \in A(\Gamma)$ such that $\|h_\gamma\| < \frac{\varepsilon}{2N_T}$, $h_\gamma = 0$ on $\complement W_\gamma$, and $\varphi = \varphi(\gamma) + h_\gamma$ on $\tilde{U}_\gamma$ [4; Theorem 2.6.5].

We now choose $U_\gamma \subseteq \tilde{U}_\gamma$, a compact neighborhood of γ, such that

$$\partial U_\gamma \cap E = \emptyset; \tag{8}$$

in particular, $U_\gamma^0 \cap E = U_\gamma \cap E$ and thus U_γ is a neighborhood of $U_\gamma \cap E$.

To see this note that by the total disconnectedness of E and because $\complement \tilde{U}_\gamma$ is closed, there is a compact open neighborhood C_γ of γ in E such that $C_\gamma \cap \complement \tilde{U}_\gamma = \emptyset$.

Letting $K_\gamma \equiv E - C_\gamma$, there are disjoint compact sets U_γ and N_γ which are neighborhoods of C_γ and K_γ, respectively.

Clearly, $\partial U_\gamma \cap C_\gamma = \emptyset$ and $\partial U_\gamma \cap K_\gamma = \emptyset$, and we have (8).

By the compactness of E let $\{U_j : j = 1, \ldots, n\} \subseteq \{U_\gamma : \gamma \in E\}$ cover E (where U_j is a neighborhood of γ_j).

Because of Lemma 4.1 there are mutually disjoint compact neighborhoods $\{V_j : j = 1, \ldots, n\}$ such that $E \subseteq \bigcup V_j$ and $V_j \subseteq U_j$.

If $F_j \equiv V_j \cap E$ we set $\psi_{F_j} \in A(\Gamma)$ to be 1 on a neighborhood of F_j and 0 on a neighborhood of the closed set $E - F_j$.

Then,

$$<T, \varphi> = <T, \Sigma\, \varphi\psi_{F_j}> = \Sigma <T, \varphi(\gamma_j)\psi_{F_j}> + \Sigma <T\psi_{F_j}, h_{\gamma_j}>.$$

Consequently,

$$|<T, \varphi>| \leq \Sigma |<T, \psi_{F_j}>| + \frac{\varepsilon}{2N_T} \Sigma \|T\psi_{F_j}\|_{A'},$$

By hypothesis we therefore have

$$|<T, \varphi>| \leq \|T\|_v + \varepsilon/2,$$

so that $\|T\|_{A'} \leq \|T\|_v$.

Obviously, $\Sigma \|T_j\|_{A'} \leq N_T$ implies that, for any finite decomposition $\{T\psi_{F_j}\}$ of T,

$$\Sigma |<T, \psi_{F_j}>| \leq \Sigma \|T\psi_{F_j}\|_{A'} \leq N_T,$$

and, hence, $\|T\|_v < \infty$.

b) Clear from *a*. *qed.*

Thus, in R (mod 2π), (7) holds for all $T\in A'(E)$ if and only if $\|S\|_{A'}=\|S\|_v$ for each $S\in A'(E)$.

THEOREM 4.2 - Let $T\in A'(E)$. If there is N_T such that, for every finite decomposition $\{T_j\}$ of T, $\Sigma\|T_j\|_{A'}\leq N_T$, then $T\in\mathfrak{M}(E)$.

PROOF. - For $\psi\in A(\Gamma)$, and using the notation of Theorem 1.1, we define the linear functional

$$\mu_{T\psi}: V(E)\to C$$

$$\varphi\mapsto <T, \psi\varphi>.$$

For fixed $\varphi\in V(E)$ we set $F_x\equiv E\cap\varphi^{-1}(x)$, $x\in\varphi(E)$, and define $S_x\equiv T\psi\psi_{F_x}$ since $\{F_x: x\in\varphi(E)\}$ is a finite decomposition of E.

Hence, $\mu_{T\psi}(\varphi)=<T\psi, \varphi>=\sum_{x\in\varphi(E)}<S_x, \varphi>=\sum_{x\in\varphi(E)}x<T\psi, \psi_{F_x}>$; and, therefore,

$$|\mu_{T\psi}(\varphi)|\leq\|\varphi\|_\infty\sum_{x\in\varphi(E)}|<T\psi, \psi_{F_x}>|$$

$$\leq\|\psi\|_A\|\varphi\|_\infty\sum_{x\in\varphi(E)}\|T\psi_{F_x}\|_{A'}$$

$$\leq N_T\|\psi\|_A\|\varphi\|_\infty\equiv K_{T,\psi}\|\varphi\|_\infty.$$

Thus, $\overline{V(E)}=C(\Gamma)$ implies $\mu_{T\psi}\in\mathfrak{M}(\Gamma)$ and

$$\mu_{T\psi}=\psi \text{ on } V(E).$$

Consequently, because $\mu_T=T$ on $V(E)$ we have

$$\psi\mu_T=\mu_{T\psi} \text{ on } V(E).$$

Let I be the closed ideal in $A(\Gamma)$ generated by $V(E)$. Clearly the zero-set of I is empty so that, by the Tauberian theorem, $I=A(\Gamma)$. Thus any $\varphi\in A(\Gamma)$ is represented as $\lim\varphi_n\psi_n=\varphi$ in $A(\Gamma)$ where $\varphi_n\in V(E)$ and $\psi_n\in A(\Gamma)$.

For such φ_n, ψ_n we have

$$<T, \psi_n\varphi_n>=<T\psi_n, \varphi_n>=<\mu_{T\psi_n}, \varphi_n>=<\psi_n\mu_T, \varphi_n>=<\mu_T, \varphi_n\psi_n>.$$

Therefore $<T, \varphi>=<\mu_T, \varphi>$ for all $\varphi\in A(\Gamma)$. *qed.*

5. - Pseudo-measure variation in profinite groups.

Let $\{\Gamma_\alpha : \alpha \in A\}$ be a family of finite abelian groups each with the discrete te topology, $(A, \geq)$ a directed set, and

$$C_\beta^\alpha : \Gamma_\alpha \to \Gamma_\beta, \ \alpha \geq \beta,$$

a family of continuous homomorphism for whic C_α^α is the identity and $C_\gamma^\beta \cdot C_\beta^\alpha = C_\gamma^\alpha$. $\Pi\Gamma_\alpha$ is a compact abelian group, and the *projective limit* Γ *of* $\{\Gamma_\alpha\}$ is that subgroup of $\Pi\Gamma_\alpha$ consisting of all $\{\gamma_\alpha\} \in \Pi\Gamma_\alpha$ for which $C^\alpha(\gamma_\alpha) = = \gamma_\beta$ if $\alpha \geq \beta$. It is easy to check that Γ is closed in $\Pi\Gamma_\alpha$, and thus it is compact. A *profinite abelian group* is a projective limit of finite abelian groups. For our purposes we also note that profinite groups are totally disconnected; this is a trivial consequence of the fact that compact groups are totally disconnected if and only if the identity is the intersection of all its compact open neighborhoods [2; p. 118].

THEOREM 5.1. - Let Γ be profinite. E is a set of strong spectral resolution if and only if for all $T \in A'(E)$, $\|T\|_v < \infty$.

PROOF. - Let $T \in A'(E)$. Strong spectral resolution implie $\|T\|_v < \infty$ by Proposition 2.2.

Conversely, since Γ is totally disconnected and because $V(E)$, as a subalgebra of $A(\Gamma)$, is self-apjoint, separates points, and not all its elements vanish at any fixed $\gamma \in \Gamma$, we apply the Katznelson-Rudin theorem [4; pp. 239-241] directly and have $\overline{V(E)} = A(\Gamma)$.

Thus, since the hypothesis allows us to apply Theorem 1.1, we have $T = \mu_T$ because $T = \mu_T$ on $V(E)$.

qed.

Therefore, in light of Malliavin's characterization of strong spectral resolution in terms of strong c-liftings (e.g. [3] for details) we have necessary and sufficient conditions that pseudo-measure variation be finite in terms of inequalities dealing with measures. Also, by the Katznelson-Rudin theorem, the statement of Theorem 5.1 holds for any totally disconnected Γ.

BIBLIOGRAPHY

[1] J. BENEDETTO, ***Pseudo-Measures and harmonic synthesis***, Lecture Notes No. 5, Dept. of Math., U. of Maryland, 1968.

[2] J. W. S. CASSELS and A. FRÖHLICH (editors), ***Algebraic number theory***, Thompson Book Co., Washington, D.C., 1967.

[3] P. MALLIAVIN, ***Sur les ensembles ne portant pas de pseudo-measures***, C. R. Acad. Sc. Paris, 267 (1968) 813-815.

[4] W. RUDIN, ***Fourier analysis on groups***, J. Wiley and Sons, New York, 1962.

[5] N. VAROPOULOS, ***Sur les ensembles parfaits et les séries trigonométriques***, C. R. Acad. Sc. Paris, 260 (1965) 3831-3834.