On minimum problems for families of functionals.

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Sommario. - Per una famiglia $\{I_\lambda\} \lambda \in \Lambda$ di funzionali reali definiti in uno spazio metrico X e dipendenti da un parametro $\lambda \in \Lambda$, dove Λ è uno spazio topologico, si pone il problema di come si comporta, al variare di λ , l'insieme dei punti di minimo di I_λ . Lo studio di tale problema è l'oggetto del presente lavoro e il risultato principale è dato dal Teorema 1.

In this paper we consider a family of functionals $(x, \lambda) \mapsto I(x, \lambda)$ defined on a metric space X , depending on a parameter λ in a topological space Λ . We investigate the dependence on λ (in a sense to be precisely defined later) of the set of minimum points.

The main result is Theorem 1 from which some consequences (Theorem 2 and Theorem 3) and applications are derived (Corollaries 1, ..., 8).

1. - We recall first some terminology.

Let $T: X \rightarrow Y$ be a continuous mapping of a metric space (X, ρ) into a metric space (Y, d) . Consider the equation

$$(1) \quad T(x) = y^*$$

where y^* is a given point of Y .

We recall that a point x_0 is called *quasi-solution* of (1) if $d(T(x_0), y^*) = \inf \{ d(T(x), y^*) \mid x \in X \}$, i.e. x_0 is a minimum point for the functional $I: X \rightarrow R$ defined by $I(x) = d(T(x), y^*)$ (see V. K. IVANOV [1]).

Let $F: X \rightarrow X$ be a continuous mapping of a metric space (X, d) into itself. The point x_0 is called an *almost fixed point* for F if it is a minimum point for the functional $I: X \rightarrow R$ defined by $I(x) = d(F(x), x)$.

Let $I: X \rightarrow R$ be a lower bounded, lower semicontinuous real functional, defined on a complete metric space (X, d) . The minimum problem for the functional I is said to be *well-posed (well-posed in the generalized sense)* if any minimizing sequence $\{x_n\}$ is convergent (compact).

Consequently the problem of finding the quasi-solutions (almost fixed points) of $T(x) = y^*$ ($F(x) = x$) is said to be well-posed, or well-posed in the

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generalized sense, according to whether the real functional $I: X \rightarrow R$ defined by $I(x) = d(T(x), y^*)$ ($I(x) = d(F(x), x)$) is well-posed or well-posed in the generalized sense (see A. N. TICHONOV [2]).

For our purposes it will be useful the following result proved in [3]:

The minimum problem for the functional I is well-posed (well-posed in the generalized sense) if and only if $\delta(\Omega_\rho) \rightarrow 0$ ($\alpha(\Omega_\rho) \rightarrow 0$) as $\rho \rightarrow 0$, where $\Omega_\rho = \{x \in X \mid I(x) \leq \inf I(X) + \rho\}$ ⁽¹⁾.

2. - Let X be a metric space (or, more generally, a topological space). By $\mathcal{S}(X)$ we denote the set of all nonempty subsets of X , with the *upper semi-finite topology* (u.s.f. topology), i.e. the topology which has as a basis all sets of the form $\{A \in \mathcal{S}(X) \mid A \subset U, U \text{ open in } X\}$ (see T. VAN der WALT [5]).

Let $I: X \times \Lambda \rightarrow R$ be a real functional defined on the cartesian product $X \times \Lambda$, where X is a complete metric space and Λ is a topological space. In this paper by I_λ we denote the partial mapping $I_\lambda: X \rightarrow R$ defined by $I_\lambda(x) = I(x, \lambda)$. We shall also use the following notations:

$$\Omega_\rho(\lambda) = \{x \in X \mid I_\lambda(x) \leq \inf I_\lambda(X) + \rho\};$$

$$\Omega(\lambda) = \{x \in X \mid I_\lambda(x) = \inf I_\lambda(X)\};$$

$$B(\Omega(\lambda), \varepsilon) = \{x \in X \mid d(x, \Omega(\lambda)) < \varepsilon\}.$$

Moreover

a) $I_\lambda(x)$ converges to $I_{\lambda_0}(x)$ uniformly in X as $\lambda \rightarrow \lambda_0$ if for any $\varepsilon > 0$ there exists a neighborhood N of λ_0 such that $|I_\lambda(x) - I_{\lambda_0}(x)| < \varepsilon$ for any $x \in X, \lambda \in N$;

b) $\Omega_\rho(\lambda) \xrightarrow{\text{u.s.f.}} \Omega(\lambda_0)$ as $(\lambda, \rho) \rightarrow (\lambda_0, 0)$ if for any open set U containing $\Omega(\lambda_0)$, there exists a neighborhood N of λ_0 and $\delta > 0$ such that $\Omega_\rho(\lambda) \subset U$ for any $\lambda \in N, 0 < \rho < \delta$.

THEOREM 1. - *Let X be a complete metric space and Λ a topological space. Let $I: X \times \Lambda \rightarrow R$ be a real functional such that I_λ is lower bounded for any $\lambda \in \Lambda$. Let λ_0 be a given point of Λ . Assume that*

- i) I_{λ_0} is lower semicontinuous;
- ii) $I_\lambda(x)$ converges to $I_{\lambda_0}(x)$ uniformly in X as $\lambda \rightarrow \lambda_0$;
- iii) the minimum problem for I_{λ_0} is well-posed in the generalized sense.

⁽¹⁾ Let A be a subset of a metric space (X, d) . By $\delta(A)$ and $\alpha(A)$ we denote the diameter and the KURATOWSKI number of A , respectively (see [4]).

Then:

- a) $\Omega(\lambda_0)$ is nonempty and compact;
- b) $\Omega_\rho(\lambda) \xrightarrow{\text{u.s.f.}} \Omega(\lambda_0)$ as $(\lambda, \rho) \rightarrow (\lambda_0, 0)$.

PROOF. - The assertion a) follows immediately from the definition of well-posed problem in the generalized sense.

In order to prove the assertion b) let U be an open set containing $\Omega(\lambda_0)$. Since $\Omega(\lambda_0)$ is compact there exists $\varepsilon > 0$ such that $B(\Omega(\lambda_0), \varepsilon) \subset U$. But from iii) it follows that $\alpha(\Omega_\rho(\lambda_0)) \rightarrow 0$ as $\rho \rightarrow 0$ (see [3]); hence there exists $\delta > 0$ such that $\Omega_{3\delta}(\lambda_0) \subset B(\Omega(\lambda_0), \varepsilon)$ (see [4]). By ii) there exists a neighborhood N of λ_0 such that $\lambda \in N \Rightarrow |I_\lambda(x) - I_{\lambda_0}(x)| < \delta$ independently of $x \in X$. It remains to prove that if $0 < \rho < \delta$ and $\lambda \in N$, then $\Omega_\rho(\lambda) \subset B(\Omega(\lambda_0), \varepsilon) \subset U$, i.e. if $x \in \Omega_\rho(\lambda)$ we have $d(x, \Omega(\lambda_0)) < \varepsilon$.

Obviously

$$I_{\lambda_0}(x) - \inf I_{\lambda_0}(X) \leq |I_{\lambda_0}(x) - I_\lambda(x)| + I_\lambda(x) - \inf I_\lambda(X) + |\inf I_\lambda(X) - \inf I_{\lambda_0}(X)|.$$

Since $x \in \Omega_\rho(\lambda)$, it follows that

$$I_\lambda(x) \leq \inf I_\lambda(X) + \rho, \quad \text{i.e. } I_\lambda(x) - \inf I_\lambda(X) < \delta.$$

Moreover, it is easy to see that for any $\lambda \in N$ we have

$$|\inf I_\lambda(X) - \inf I_{\lambda_0}(X)| < \delta, \quad \text{since } |I_\lambda(x) - I_{\lambda_0}(x)| < \delta$$

for any $x \in X$, $\lambda \in N$. Hence, if $x \in \Omega_\rho(\lambda)$, $I_{\lambda_0}(x) - \inf I_{\lambda_0}(X) < 3\delta$.

Then $d(x, \Omega(\lambda_0)) < \varepsilon$.

As a particular case of Theorem 1 we have the following result proved in [6].

COROLLARY 1. - Let $\{T_n\}$ be a sequence of mappings of a complete metric space (X, d) into itself, which converges uniformly to a continuous mapping T . Put $\Omega_\rho = \{x \in X \mid d(T(x), x) \leq \rho\}$. If $\alpha(\Omega_\rho) \rightarrow 0$ as $\rho \rightarrow 0$, then any sequence $\{x_n\}$, such that $d(T_n(x_n), x_n) \rightarrow 0$, is compact and all its limit points are fixed for T .

PROOF. - Let $\Lambda = \left\{0, 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\right\}$ with the relative topology as a subset of $\mathbb{R}$. Put

$$I_\lambda(x) = \begin{cases} d(T(x), x) & \text{if } \lambda = 0 \\ d(T_n(x), x) & \text{if } \lambda = 1/n. \end{cases}$$

Clearly, since $\alpha(\Omega_\rho) \rightarrow 0$ as $\rho \rightarrow 0$, the minimum problem for the functional I_0 is well-posed in the generalized sense and, since

$$|d(x, T(x)) - d(x, T_n(x))| \leq d(T(x), T_n(x)),$$

$I_\lambda(x)$ converges to $I_0(x)$ uniformly in X as $\lambda \rightarrow 0$. Obviously $x_n \in \Omega_{\rho_n}(1/n)$, where $\rho_n = d(T_n(x_n), x_n) \rightarrow 0$. Then, by Theorem 1, $\Omega(0)$ is nonempty and compact and $d(x_n, \Omega(0)) \rightarrow 0$. Therefore $\{x_n\}$ is compact and all its limit points belong to $\Omega(0)$.

3. - The following theorem is a consequence of Theorem 1 and it is less general, but more useful for some applications.

THEOREM 2. - *Let $I: X \times \Lambda \rightarrow R$ be a real functional, where X is a complete metric space and Λ is a topological space. Let I_λ be a lower bounded, lower semicontinuous for all $\lambda \in \Lambda$. If for any $\lambda_0 \in \Lambda$, $I_\lambda(x)$ converges to $I_{\lambda_0}(x)$ uniformly in X as $\lambda \rightarrow 0$ and the minimum problem for I_{λ_0} is well-posed in the generalized sense, then the set-valued mapping $\varphi: \Lambda \rightarrow \mathcal{S}(X)$, defined by $\varphi(\lambda) = \Omega(\lambda)$, is upper semicontinuous, i.e. is continuous in the u.s.f. topology of $\mathcal{S}(X)$.*

PROOF. - Since the minimum problem for the functional I_λ is well-posed in the generalized sense for any $\lambda \in \Lambda$ then the sets $\Omega(\lambda)$, $\lambda \in \Lambda$, are nonempty and compact. On the other hand $\Omega(\lambda) \subset \Omega_\rho(\lambda)$ for any λ and $\rho > 0$. Hence, by Theorem 1, the set-valued mapping φ is upper semicontinuous.

A continuous mapping $T: D \rightarrow X$, defined on a subset D of a metric space X , is called *densifying* if $\alpha(T(A)) < \alpha(A)$ for any bounded set $A \subset D$, $\alpha(A) > 0$ (see [7]).

It is easy to see that contractive mappings and completely continuous mappings are densifying, as well as sum of these two types of mappings in Banach spaces.

COROLLARY 2. - *Let D be a closed and bounded set of a Banach space X . Let $T: D \rightarrow X$ be a densifying mapping and consider the equation*

$$(1) \quad x + T(x) = y,$$

where y is a given point of X . Let $E = \{x + T(x) \mid x \in D\}$. Then the set-valued mapping $\Gamma: E \rightarrow \mathcal{S}(D)$, defined by $\Gamma(y) = \{x \in X \mid x + T(x) = y\}$ is upper semicontinuous, i.e. is continuous in the u.s.f. topology of $\mathcal{S}(D)$.

PROOF. - Put $F_y(x) = y - T(x)$. Then the equation (1) can be written as follows

$$(2) \quad x = F_y(x),$$

where F_y is a densifying mapping for all $y \in E$. Hence, for any $y \in E$, the minimum problem for the lower bounded, continuous functional $I_y: D \rightarrow R$,

defined by $I_y(x) = \|x - F_y(x)\|$ is well-posed in the generalized sense (see [8]).

Clearly $I_y(x)$ converges to $I_{y_0}(x)$ uniformly in D as $y \rightarrow y_0$, for any $y_0 \in E$. The assertion follows by Theorem 2.

COROLLARY 3. - *Let $f: X \rightarrow Y$ be a continuous mapping of a compact metric space (X, ρ) onto a metric space (Y, d) . Then the set-valued mapping $\tilde{f}: y \rightarrow \mathcal{S}(X)$ defined by $\tilde{f}(y) = \{x \in X \mid f(x) = y\}$ is upper semicontinuous.*

PROOF. - Put $I_y(x) = d(f(x), y)$ and let y_0 be a given point. We have

$$|d(f(x), y) - d(f(x), y_0)| \leq d(y, y_0).$$

Hence $I_y(x)$ converges to $I_{y_0}(x)$ uniformly in X as $y \rightarrow y_0$. Moreover, by the compactness of X , the minimum problem for the functional I_y is well-posed in the generalized sense for any $y \in Y$. The assertion follows by Theorem 2.

Corollary 3 can be extended as follows.

COROLLARY 4. - *Let $f: X \rightarrow Y$ be a continuous mapping of a complete metric space (X, ρ) onto a metric space (Y, d) . Put*

$$\Omega_\varepsilon(y) = \{x \in X \mid d(f(x), y) < \varepsilon\}$$

i.e. $\Omega_\varepsilon(y)$ is the reverse image of the set $B(y, \varepsilon)$. If, for any $y \in Y$, $\alpha(\Omega_\varepsilon(y)) \rightarrow 0$ as $\varepsilon \rightarrow 0$, then the set-valued mapping $\tilde{f}: Y \rightarrow \mathcal{S}(X)$ defined by $\tilde{f}(y) = \{x \in X \mid f(x) = y\}$ is upper semicontinuous.

PROOF. - Put $I_y(x) = d(f(x), y)$. Since $\alpha(\Omega_\varepsilon(y)) \rightarrow 0$ as $\varepsilon \rightarrow 0$, for any $y \in Y$ the minimum problem for the functional I_y is well-posed in the generalized sense (see [3]). Moreover, as in Corollary 2, $I_y(x)$ converges to $I_{y_0}(x)$ uniformly in X as $y \rightarrow y_0$, for any $y_0 \in Y$. The conclusion follows again by Theorem 2.

Let $(R^n, |\cdot|)$ be the n -dimensional, real euclidean space and let $C_n[a, b]$ be the Banach space of all continuous mappings $\varphi: [a, b] \rightarrow R^n$ with norm

$$\|\varphi\| = \max \{ |\varphi(t)| : t \in [a, b] \}.$$

Consider the differential equation

$$(3) \quad \dot{x} = f(t, x)$$

where $f: [a, b] \times R^n \rightarrow R^n$ is bounded and continuous.

Let $D \subset C_n[a, b]$ be the set of all solutions of the differential equation (3). Then, as a consequence of Corollary 4, we have

COROLLARY 5. - *The mapping $\sigma: R^n \rightarrow \mathcal{S}(D)$ defined by $\sigma(\lambda) = \{\varphi \in D \mid \varphi(a) = \lambda\}$ is continuous in the u.s.f. topology of $\mathcal{S}(D)$.*

COROLLARY 6. - *Let $T: Q \times \Lambda \rightarrow Q$ be a continuous mapping, where Q is a compact and convex set of a Banach space X and Λ is a topological space. If, for any $\lambda_0 \in \Lambda$, $T_\lambda(x)$ converges to $T_{\lambda_0}(x)$ uniformly in Q as $\lambda \rightarrow \lambda_0$, then the set-valued mapping $\varphi: \Lambda \rightarrow \mathcal{S}(Q)$ defined by $\varphi(\lambda) = \{x \in Q \mid T_\lambda(x) = x\}$ is upper semicontinuous.*

PROOF. - Put $I_\lambda(x) = \|T_\lambda(x) - x\|$ and let $\lambda_0 \in \Lambda$ be a given point. Clearly we have

$$|I_\lambda(x) - I_{\lambda_0}(x)| = |\|T_\lambda(x) - x\| - \|T_{\lambda_0}(x) - x\|| \leq \|T_\lambda(x) - T_{\lambda_0}(x)\|.$$

Hence $I_\lambda(x)$ converges to $I_{\lambda_0}(x)$ uniformly in X as $\lambda \rightarrow \lambda_0$. But, for any $\lambda \in \Lambda$, T_λ is a continuous mapping of Q into itself; by the SCHAUDER fixed point theorem, there exists $x_\lambda \in Q$ such that $T_\lambda(x_\lambda) = x_\lambda$. Then $\Omega(\lambda) \neq \emptyset$.

Therefore, by the compactness of Q , the minimum problem for the functional I_λ is well-posed in the generalized sense for any $\lambda \in \Lambda$. Hence φ is continuous in the u.s.f. topology of $\mathcal{S}(Q)$ by Theorem 2.

4. - As a consequence of Theorem 2 we have

THEOREM 3. - *Let $I: X \times \Lambda \rightarrow R$ be a real functional, where X is a complete metric space and Λ is a topological space. Let, for all $\lambda \in \Lambda$, I_λ be lower bounded, lower semicontinuous. If, for any $\lambda_0 \in \Lambda$, $I_\lambda(x)$ converges to $I_{\lambda_0}(x)$ uniformly in X as $\lambda \rightarrow \lambda_0$ and the minimum problem for I_{λ_0} is well-posed, then the mapping $\psi: \Lambda \rightarrow X$, where $\psi(\lambda)$ is the unique minimum point of I_λ , is continuous.*

PROOF. - The uniqueness of the minimum point for I_λ follows immediately from the definition of well-posed problem. Since the injection $i: X \rightarrow \mathcal{S}(X)$ defined by $i(x) = \{x\}$ is an imbedding the assertion follows by Theorem 2.

COROLLARY 7. - *Let $T: X \times \Lambda \rightarrow X$ be a continuous mapping, where X is a complete metric space and Λ is a topological space. If for any $\lambda_0 \in \Lambda$, T_{λ_0} is a contractive mapping and $T_\lambda(x)$ converges to $T_{\lambda_0}(x)$ uniformly in X as $\lambda \rightarrow \lambda_0$, then there exists a unique mapping $\psi: \Lambda \rightarrow X$, such that $\psi(\lambda) = T(\psi(\lambda), \lambda)$ and ψ is continuous.*

PROOF. - The uniqueness follows immediately from BANACH contraction principle. Put $I_\lambda(x) = d(x, T_\lambda(x))$. It is easy to see that $\delta(\Omega_0(\lambda)) \leq 2\rho/(1 - k_\lambda)$, where $k_\lambda < 1$ is the LIPSCHITZ constant of T_λ . Therefore the minimum problem for I_λ is well posed for every $\lambda \in \Lambda$ (see [3]) and so, by Theorem 3, ψ is continuous.

Let $T: X \rightarrow X$ be a continuous mapping of a complete metric space (X, d) into itself, with a unique fixed point $\xi \in X$. We recall that the uniqueness of ξ is said to be *strong* (P. ZECCA [9]) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, T(x)) < \delta \Rightarrow d(x, \xi) < \varepsilon$$

COROLLARY 8. (S. DRAGONI - [10]) *Let $T: X \times \Lambda \rightarrow X$ be a continuous mapping, where X is a complete metric space and Λ is a metric space. For any $\lambda_0 \in \Lambda$ suppose that $T_\lambda(x)$ converges to $T_{\lambda_0}(x)$ uniformly in X . If T_λ has a unique fixed point ξ_λ for any λ , and the uniqueness of ξ_λ is strong, then the mapping $\psi: \lambda \rightarrow \xi_\lambda$ is continuous.*

PROOF. - It is easy to see that the strong uniqueness implies that the minimum problem for the functional $I_\lambda: X \rightarrow R$ defined by $I_\lambda(x) = d(x, T_\lambda(x))$ is well posed for any $\lambda \in \Lambda$. Then, by Theorem 3, the mapping ψ is continuous.

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