Asymptotic behavior of a perturbed linear differential equation with time lag on a product space.

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Summary. - *The objective of this paper is to study the asymptotic behavior of the solutions of the differential equation with finite time lag*

$$\begin{aligned}\dot{x}(t) &= L(t, x_t) + X(t, x_t, y_t) \\ \dot{y}(t) &= J(t, y_t) + Y(t, x_t, y_t)\end{aligned}$$

by knowing that the zero solution of $\dot{u}(t) = L(t, u_t)$ is uniformly stable, the zero solution of $\dot{v}(t) = J(t, v_t)$ is exponentially stable and X, Y satisfy certain given conditions.

1 - Introduction.

The results of this paper are closely related to the ones of J. HALE (2-b- II) concerning the behavior of the solutions of a class of differential equations with time lag on a product space, as $t \rightarrow \infty$.

The main tool used in this work is provided by a result of J. HALE (2-a-, Theorem II-3), in which the problem of determining the behavior of the solutions of a differential equation with time lag is reduced to the solution of an ordinary scalar differential equation. An application of this Hale's comparison principle to uniform stability problems was done by us in a previous paper [4].

Denote by C or C^n the space of continuous functions taking the interval $[-h, 0]$, $0 \leq h \leq \infty$, into R^n . For any $\varphi \in C$ we define $\|\varphi\| = \sup_{-h \leq \theta \leq 0} |\varphi(\theta)|$, where $|\cdot|$ is any vector norm in R^n . If x is any continuous function defined in the interval $[-h + t_0, t_0 + A]$, $t_0 \geq 0$, $0 < A \leq \infty$, x_t for $t_0 \leq t < t_0 + A$ is defined by $x_t(\theta) = x(t + \theta)$, $-h \leq \theta \leq 0$. For H , $0 < H \leq \infty$, $C_H = \{\varphi \in C \mid \|\varphi\| < H\}$.

Let $f(t, \varphi) \in R^n$ be a function defined on $[0, l) \times C_H$, $l \leq \infty$. Then

$$(1) \quad \dot{x}(t) = f(t, x_t)$$

is called a differential equation with finite time lag.

Let t_0 be any number, $0 \leq t_0 < l$, and $\varphi \in C_H$ be any given function. A continuous function $x(t)$ defined on $[t_0 - h, t_0 + A]$, $A > 0$, is said to be a solution of (1) with initial function φ at $t = t_0$ if:

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- i) for each t , $t_0 \leq t < t_0 + A$, we have $x_t \in C_H$;
- ii) $x_{t_0} = \varphi$;
- iii) $\dot{x}(t) = f(t, x_t)$ for $t_0 \leq t < t_0 + A$

Function $f(t, \varphi)$ is said to be locally Lipschitzian in $[0, l) \times C_H$, $l \leq \infty$, if for every H_1 , $0 < H_1 < H$, and every T , $0 < T < l$, there is $L = L(T, H_1)$, such that

$$|f(t, \varphi) - f(t, \psi)| \leq L \|\varphi - \psi\|$$

for $\varphi, \psi \in C_{H_1}$, $0 \leq t \leq T$.

In this work $f(t, \varphi)$ is supposed to be continuous and locally Lipschitzian in $[0, l) \times C_H$, $l \leq \infty$.

The unique solution of (1) with initial function φ at $t = t_0$ will be denoted by $x(t) = x(t, t_0, \varphi)$ and $x_t = x_t(t_0, \varphi)$; $[t_0 - h, t^+)$ will denote the maximal interval, open at the right, in which $x(t)$ is defined. If $l = \infty$ with $t^+ = \infty$, the solution is said to be defined in the future and if in addition $x(t)$ is bounded on $[t_0, \infty)$, the solution is said to be bounded in the future. If $x_t \in C_{H_1}$ for $t_0 \leq t < t^+$, $H_1 < H$, then $t^+ = l$.

Suppose that $f(t, 0) = 0$, with $l = \infty$. The following definitions of stability will be given with respect to the solution $x = 0$ of (1).

i) *Uniform Stability.* Given $\varepsilon > 0$ and $t_0 \geq 0$ there is a $\delta = \delta(\varepsilon)$ such that $t \geq t_0$ and $\varphi \in C_\delta$ imply $x_t(t_0, \varphi) \in C_\varepsilon$ for $t \geq t_0$.

ii) There is a $\rho > 0$ and for every $\varepsilon > 0$ there is a $T(\varepsilon) \geq 0$ such that $\varphi \in C_\rho$, $t \geq t_0$, imply $x_t(t_0, \varphi) \in C_\varepsilon$.

iii) *Uniform-Asymptotic Stability.* Both (i) and (ii) hold.

iv) *Exponential-Asymptotic Stability.* There are positive constants ρ , α , β such that if $\varphi \in C_\rho$, then

$$\|x_t(t_0, \varphi)\| \leq B e^{-\alpha(t-t_0)} \|\varphi\| \text{ for } t \geq t_0 \geq 0.$$

If $f(t, \varphi)$ is linear in φ , the (iii) $\Leftrightarrow$ (iv) [1, pg. 346].

II - Basic Lemmas

Consider the equation

$$(2) \quad \dot{u} = -\alpha u + b(t)u,$$

where $b(t)$ is continuous and nonnegative for $t \geq 0$ and $\alpha > 0$.

For $b(t)$ continuous and nonnegative consider the conditions:

$$(3) \quad \lim_{(t, v) \rightarrow (\infty, \infty)} \sup \frac{1}{v} \int_t^{t+v} b(s) ds < \alpha$$

(4) There are $\tau > 0$, $T > 0$ and $0 < \beta < \alpha$ so that

$$\frac{1}{\tau} \int_t^{t+\tau} b(s) ds \leq \beta \text{ for } t \geq T$$

$$(5) \quad \lim_{t \rightarrow \infty} \sup \int_t^{t+1} b(s) ds < \alpha$$

It is not hard to see that (3) and (4) are equivalent and that (5) implies (3).

The following lemma is essentially one communicated to us by A. STRAUSS and J. YORKE.

LEMMA 1. - *The solution $u=0$ of (2) is uniformly asymptotically stable if, and only if, $b(t)$ satisfies condition (3).*

PROOF. - Suppose that condition (3) holds.

$$u(t + t_0, t_0, u_0) = u_0 e^{\int_{t_0}^{t_0+t} b(s) ds} = u_0 e^{-t \int_{t_0}^{t_0+t} b(s) ds}$$

Let $\varepsilon > 0$ be chosen such that

$$\lim_{(t, v) \rightarrow (\infty, \infty)} \sup \frac{1}{v} \int_t^{t+v} b(s) ds = \alpha < \alpha - \varepsilon$$

and then choose $T_0 > 0$, $T > 0$ so that $t^{-1} \int_{t_0}^{t_0+t} b(s) ds \leq \alpha - \varepsilon$ for $t \geq T$, $t_0 \geq T_0$.

Let $k > 0$ be chosen so that $\int_{t_0}^{t_0+t} b(s) ds \leq k$ for $t_0 \geq T_0$ and $0 \leq t \leq T$. Then if $0 \leq t \leq T$, $t_0 \geq T_0$, we have $|u(t + t_0, t_0, u_0)| \leq |u_0| e^k$, while if $t \geq T$ we have $|u(t + t_0, t_0, u_0)| \leq |u_0| e^{-\varepsilon t}$.

Thus for all $t \geq 0$, $t_0 \geq T_0$, we have

$$|u(t + t_0, t_0, u_0)| \leq |u_0| e^{\varepsilon T} e^k e^{-\varepsilon t}$$

Hence $u=0$ is uniformly asymptotically stable for (2) for $t_0 \geq T_0$. The usual continuous dependence argument shows that $u=0$ is uniformly asymptotically stable for (2).

Suppose that $b(t)$ fails to satisfy (3). Assume, if possible, that $u=0$ were uniformly asymptotically stable for (2). Thus there would exist $\sigma > 0$ and $B > 0$ such that

$$(6) \quad e^{\sigma t} u(t + t_0, t_0, 1) \leq B, \quad t_0 \geq 0 \text{ and } t \geq 0.$$

Since (3) fails there exist sequences $\{t_m\}$, $\{v_m\}$, $t_m \rightarrow \infty$, $v_m \rightarrow \infty$ as $m \rightarrow \infty$ such that

$$\frac{1}{v_m} \int_{t_m}^{t_m+v_m} b(s) ds \geq \alpha - \frac{1}{2} \sigma \text{ for all } m.$$

It would then follow that

$$\begin{aligned} e^{\sigma v_m} u(t_m + v_m, t_m, 1) &= e^{[\sigma v_m - \alpha v_m + \int_{t_m}^{t_m+v_m} b(s) ds]} = \\ &= e^{v_m[\sigma - \alpha + \frac{1}{v_m} \int_{t_m}^{t_m+v_m} b(s) ds]} \geq e^{1/2 \sigma v_m} \rightarrow \infty \text{ as } m \rightarrow \infty. \end{aligned}$$

This would contradict (6), completing the proof.

Consider the system

$$(7) \quad \dot{y}(t) = L(t, y_t)$$

where $L(t, \varphi)$ is continuous on $[0, \infty) \times C$ and linear in φ . One can see that if $y=0$ is uniformly stable, then (7) is uniformly stable, that is, every solution $y(t, t_0, \varphi)$ is uniformly stable. Concerning uniform stability of linear systems, the following lemma holds.

LEMMA 2. - *Equation (7) is uniformly stable if, and only if, there is a positive constant K , such that $\|y(t_0, \varphi)\| \leq K \|\varphi\|$ for every $t \geq t_0 \geq 0$ and $\varphi \in C$.*

Consider the system

$$(8) \quad \dot{x}(t) = L(t, x_t) + X(t, x_t)$$

where $L(t, \varphi)$ is continuous and linear in φ and $X(t, \varphi)$ is continuous and locally Lipschitzian in $[0, l) \times C_H$, $0 < l \leq \infty$.

The following lemma is a convenient version of the above mentioned Hale's result [2-a-, Th. II-3].

LEMMA 3. - Suppose that there exist a function $\alpha(t)$, with $\dot{\alpha}(t)$ continuous for $t \geq 0$, and a function $K(t)$, continuous for $t \geq 0$, such that for every $t_0 \geq 0$ and every $t \geq t_0 \geq 0$, $\varphi \in C$, the solution $y(t, t_0, \varphi)$ of (7) satisfies

$$(9) \quad \|y(t, t_0, \varphi)\| \leq K(t_0) e^{-[\alpha(t) - \alpha(t_0)]} \|\varphi\|.$$

Suppose that there exists a scalar function $W(t, r)$, continuous for $t \geq t_0 \geq 0$, $r \geq 0$, nondecreasing in r , such that

$$(10) \quad |X(t, \varphi)| \leq W(t, \|\varphi\|), \quad t_0 \leq t < l, \quad \varphi \in C_H.$$

Let $u(t, t_0, \varphi)$ be the maximal solution of the ordinary scalar differential equation

$$(11) \quad \dot{u} = -\dot{\alpha}(t)u + K(t) W(t, u),$$

$$u(t_0) = V(t_0, \varphi) = \sup_{\tau \geq 0} \|y_{t_0+\tau}(t_0, \varphi)\| e^{[\alpha(t_0+\tau) - \alpha(t_0)]}, \quad \varphi \in C_H.$$

Let $u(t) = u(t, t_0, \varphi) < H$ for $t_0 \leq t < l$.

Then the solution $x(t, t_0, \varphi)$ of (8) satisfies

$$(12) \quad \|x(t, t_0, \varphi)\| \leq u(t, t_0, \varphi), \quad t_0 \leq t < l.$$

Consider the following hypothesis with respect to a continuous and non-negative function $a(t)$, $t \geq 0$

$$(13) \quad \sup_{t_0 \geq 0} \int_0^\infty e^{-\gamma t} a(t + t_0) dt < \infty, \quad \gamma > 0$$

or equivalently

$$(14) \quad \sup_{t_0 \geq 0} e^{\gamma t_0} \int_{t_0}^\infty e^{-\gamma t} a(t) dt < \infty.$$

A sufficient condition for (13) is

$$(15) \quad a(t) \leq M + c(t)$$

where $M > 0$ and $c(t) \geq 0$ with $\int_0^\infty c(t) dt < \infty$.

Conditions (13) and (15) are not equivalent. In fact, the function defined by

$$a(t) = n \text{ if } t = n + \frac{1}{2^n}$$

$$a(t) = 0 \text{ if } t \notin [n, n + \frac{1}{n}]$$

$a(t)$ is linear elsewhere, for $n = 1, 2, \dots$ satisfies (13) and not (15).

III - Asymptotic Behavior

Consider the systems

$$(16) \quad \dot{u}(t) = L(t, u_t)$$

$$(17) \quad \dot{v}(t) = J(t, v_t)$$

$$(18) \quad \begin{aligned} \dot{x}(t) &= L(t, x_t) + X(t, x_t, y_t) \\ \dot{y}(t) &= J(t, y_t) + Y(t, x_t, y_t) \end{aligned}$$

where $L(t, \varphi)$ is continuous in $[0, \infty) \times C^n$ and linear in φ ; $J(t, \psi)$ is continuous in $[0, \infty) \times C^m$ and linear in ψ ; $X(t, \varphi, \psi)$ and $Y(t, \varphi, \psi)$ are continuous and locally Lipschitzian with respect to (φ, ψ) in $[0, \infty) \times C_H^n \times C_H^m$.

THEOREM. - *Suppose that the following conditions are satisfied:*

- i) *the zero solution of (16) is uniformly stable;*
- ii) *the zero solution of (17) is exponentially stable, that is, there are positive constants σ, β such that for all $\varphi \in C$ we have*

$$(19) \quad \|v_i(t_0, \psi)\| \leq B e^{-\sigma(t-t_0)} \|\psi\|, \quad t \geq t_0 \geq 0;$$

- iii) $\|X(t, \varphi, \psi)\| \leq a(t) \|\psi\|^\beta + g(t)$, $\varphi \in C_H^n$, $\psi \in C_H^m$, $t \geq 0$, where $\beta > 0$, $a(t)$ satisfies condition (13) with γ sufficiently small and $g(t) \geq 0$ with $\int_0^\infty g(t) dt < \infty$.

- iv) $\|Y(t, \varphi, \psi)\| \leq b(t) \|\psi\| + G \|\psi\|^{1+\rho}$, $\varphi \in C_H^n$, $\psi \in C_H^m$, $t \geq 0$, where $b(t)$ satisfies condition (3) with $\alpha < \frac{\sigma}{B}$ and ρ, G are positive constants.

Then the following conclusions hold:

There are positive constants τ, μ, T_0 such that $\varphi \in C_\tau^n$, $\psi \in C_\mu^m$, $t_0 \geq T_0$, im-

ply $y(t, t_0, \varphi, \psi) \rightarrow 0$ as $t \rightarrow \infty$ and, in addition, if $|L(t, \varphi)| \leq \lambda(t) \|\varphi\|$, $\varphi \in C^n$, $t \geq 0$, with $\int_0^\infty \lambda(t) dt < \infty$, then $\lim_{t \rightarrow \infty} x(t, t_0, \varphi, \psi)$ exists. Here $(x(t, t_0, \varphi, \psi), y(t, t_0, \varphi, \psi))$, denotes the solution of (18) with initial function (φ, ψ) at $t = t_0$.

Given $\varepsilon > 0$ there are $T(\varepsilon) \geq 0$ and $\delta(\varepsilon) > 0$ so that $\varphi \in C_\delta^n$, $\psi \in C_\delta^m$, $t \geq t_0 \geq T(\varepsilon)$, imply $x(t, t_0, \varphi, \psi) \in C_\varepsilon^n$ and $y(t, t_0, \varphi, \psi) \in C_\varepsilon^m$. (In particular this says, by taking into account (3, pg. 128), that if $X(t, 0, 0) = 0$, $t \geq 0$, then the zero solution of (18) is uniformly stable).

PROOF. - Let $\varphi_0 \in C_H^n$, $\psi_0 \in C_H^m$ and $(\xi(t), \eta(t)) = (x(t, t_0, \varphi_0, \psi_0), y(t, t_0, \varphi_0, \psi_0))$, $t_0 - h \leq t \leq t^+$, where $[t_0 - h, t^+]$ is the maximal interval, open at the right, in which $(\xi(t), \eta(t))$ is defined.

Consider the sistem

$$(20) \quad \dot{y}(t) = J(t, y_t) + Y(t, \xi_t, y_t), \quad t_0 \leq t < t^+$$

From hypothesis (iv) it follows that.

$$\|Y(t, \xi_t, \psi)\| \leq W(t, \|\psi\|), \quad \psi \in C_H^m, \quad t_0 \leq t < t^+,$$

where $W(t, r) = [b(t) + GH^p]r$, $t \geq t_0$, $r \geq 0$.

In order to apply Lemma 3, consider the scalar equation

$$(21) \quad \begin{aligned} z &= -\sigma z + B[b(t) + GH^p]z = \\ &= -[\sigma - BGH^p]z + Bb(t)z \\ z(t_0) &= z(t_0, t_0, \psi) = V(t_0, \psi) \leq B\|\psi\| \end{aligned}$$

Without loss of generality we can suppose H small enough to have $B\alpha < [\sigma - BGH^p]$. Then from Lemma 1 it follows that $z = 0$ is exponentially stable for $\dot{z} = -[\sigma - BGH^p]z + Bb(t)z$. Therefore there exist positive constants k, α_0 such that

$$z(t) = z(t, t_0, \psi) \leq k e^{-\alpha_0(t-t_0)} z(t_0) \leq B k e^{-\alpha_0(t-t_0)} \|\psi\|,$$

for $t \geq t_0 \geq 0$.

Taking $\tau < \frac{H}{Bk}$, it follows that $\psi \in C_\tau^m$ implies $z(t, t_0, \psi) = z(t) < H$, for $t_0 \leq t < \infty$.

Therefore, by applying Lemma 3 with respect to (20) and $\psi = \psi_0$ it follows that

$$\begin{aligned}\|\eta_t\| &= \|y_t(t_0, \varphi_0, \psi_0)\| \leq z(t, t_0, \psi_0) \leq Bk e^{-\alpha_0(t-t_0)} \|\psi_0\| < \\ &< Bk\tau < H, \text{ for } \psi_0 \in C_\tau^m, \varphi_0 \in C_H^n, 0 \leq t_0 \leq t < t^+.\end{aligned}$$

Consider the system

$$(22) \quad \dot{x}(t) = L(t, x_t) + X(t, x_t, \eta_t)$$

We have that

$$\|X(t, \varphi, \eta_t)\| \leq a(t) \|\eta_t\|^\beta + g(t) \leq a(t) [\tau Bk]^\beta e^{-\beta\alpha_0(t-t_0)} + g(t)$$

for $\psi_0 \in C_\tau^m, \varphi \in C_H^n, t_0 \leq t < t^+$.

From hypothesis (iii), with $\gamma = \beta\alpha_0$, it follows that there is $M = M(\gamma)$ so that

$$\int_{t_0}^{\infty} a(s) e^{-\beta\alpha_0(s-t_0)} ds < M, \quad t_0 \geq 0.$$

Hence there exist positive constants τ and μ , sufficiently small and $T_0 > 0$ sufficiently large so that

$$\begin{aligned}& K[\mu + (\tau Bk)^\beta \int_{t_0}^{\infty} a(s) e^{-\beta\alpha_0(s-t_0)} ds + \int_{T_0}^{\infty} g(s) ds] < \\ & < K[\mu + (\tau Bk)^\beta M + \int_{T_0}^{\infty} g(s) ds] < H, \text{ where } K \text{ is given as in Lemma 2 with respect to system (16).}\end{aligned}$$

By applying Lemma 3 with respect to (22), and $\varphi = \varphi_0$, where we take $K(t) = K, \alpha(t) = 0$, it follows that

$$\begin{aligned}\|\xi_t\| &= \|x_t(t_0, \varphi_0, \psi_0)\| \leq \\ &\leq K \left[\|\varphi_0\| + \int_{t_0}^t a(s) (\tau Bk)^\beta e^{-\beta\alpha_0(s-t_0)} ds + \int_{t_0}^t g(s) ds \right] \leq \\ &\leq K \left[\|\varphi_0\| + (\tau Bk)^\beta \int_{t_0}^{\infty} a(s) e^{-\beta\alpha_0(s-t_0)} ds + \int_{t_0}^{\infty} g(s) ds \right] < \\ &< K \left[\mu + (\tau Bk)^\beta M + \int_{T_0}^{\infty} g(s) ds \right] < H\end{aligned}$$

for $T_0 \leq t_0 \leq t < t^+$, $\varphi_0 \in C_\mu^n$, $\psi_0 \in C_\tau^m$.

Then $\varphi_0 \in C_\mu^n$, $\psi_0 \in C_\tau^m$, $t \geq t_0 \geq T_0$, imply $t^+ = \infty$ and consequently

$$\|x_t(t_0, \varphi_0, \psi_0)\| \leq K[\mu + (\tau Bk)^\beta M + \int_{T_0}^{\infty} g(s)ds] < H$$

$$\|y_t(t_0, \varphi_0, \psi_0)\| \leq Bke^{-\alpha_0(t-t_0)}\tau < Bk\tau < H \text{ for } t_0 \leq t < \infty.$$

This implies that for $\varphi_0 \in C_\mu^n$, $\psi_0 \in C_\tau^m$, $t_0 \geq T_0$ we have that $y_t(t_0, \varphi_0, \psi_0) \rightarrow 0$ as $t \rightarrow \infty$ and, in addition, if $|L(t, \varphi)| \leq \lambda(t)\|\varphi\|$, $\varphi \in C^n$, $t \geq 0$, with $\int_0^\infty \lambda(t)dt < \infty$, then $\lim_{t \rightarrow \infty} x(t, t_0, \varphi_0, \psi_0)$ exists.

Given ε , $0 < \varepsilon < H$, let $\delta = \delta(\varepsilon) < \frac{\varepsilon}{Bk}$, $T(\varepsilon) \geq T_0$ be chosen so that

$$K[\delta + (\delta Bk)^\beta M + \int_{T(\varepsilon)}^{\infty} g(s)ds] < \varepsilon$$

Then, proceeding as above we see that $\varphi_0 \in C_\delta^n$, $\psi_0 \in C_\delta^m$, $t \geq t_0 \geq T(\varepsilon)$ imply

$$\|x_t(t_0, \varphi_0, \psi_0)\| < \varepsilon$$

$$\|y_t(t_0, \varphi_0, \psi_0)\| < \varepsilon e^{-\alpha_0(t-t_0)} < \varepsilon$$

The proof is complete.

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