Classes of biorthonormal systems.

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Summary. - *The biorthogonal systems which are studied in this paper are composed of an Appel set of polynomials and of a sequence of derivatives. Conditions on one sequence are given which ensure the existence of the other. Expansion theorems in terms of derivatives and by means of summability methods are then proved.*

Introduction.

While linear integral equations with a symmetric kernel give rise to orthonormal systems of functions, a non-symmetric kernel leads to biorthonormal systems.

These last are composed of two sequences $f_n(t)$, $g_n(t)$ satisfying the relations

$$\int f_m(t) g_n(t) dt = \delta_{mn} ,$$

where the path of integration may be an interval of the real axis or a contour in the complex plane. In the case of complex-valued functions it is customary to write down the complex conjugate of $g_n(t)$, but this is not necessary for our purpose and we shall use the definition in the form as above; if the system f_n; g_n is biorthonormal in this sense, then the system f_n; $\bar{g}_n$ is biorthonormal in the customary sense.

The classes of biorthonormal systems we shall be concerned with are of a simple formal character; the first sequence will be an Appell set of polynomials and the second essentially the sequence of derivatives of a certain function associated with the Appell set. For the path of integration we shall consider the case of the whole real axis and the case of a circle with centre at the origin.

If $f_n \in L_2$ and if a_{jk} are the elements of the matrix of orthogonalisation, then necessary and sufficient conditions for the existence of the sequence g_n in L_2 are

$$\sum_{k=j}^{\infty} | a_{jk} |^2 < \infty ; \qquad j = 1, 2, \dots .$$

(*) This work has been sponsored by the European Office, Air Research and Development Command, U. S. Air Force, under Contract AF 61(052)-324.

For a proof see [7 ; 29, 30] [1]. Application of this theorem seems to be very difficult insofar as the computation of the elements a_{jk} is needed. In our more special case, we shall content ourselves with sufficient conditions on the generating function of the Appell set.

While extensive literature is available on the expansion of analytic functions in terms of Appell polynomials and generalisations of them [1], no expansion theorems in terms of derivatives could be found by the author in available publications. In these circumstances the problem was dealt with by the aid of summability methods, and as a consequence the completeness of sequences of derivatives in $L_2(-\infty, \infty)$ has been proved.

Since the author has been led to this subject by considering certain integral equations, and since the connection will appear again in the proof of Theorem 9, the first paragraph is devoted to a brief study of the integral equations. According to our definition of biorthonormality, the integral equation which is the transpose of that with the kernel $K(s, t)$ has the kernel $K(t, s)$, not $\overline{K}(t, s)$.

I. A class of integral equations.

In two papers [8; 9] the author studied integral equations of the type

$$f(s) - \lambda \int_{-\infty}^{\infty} K(bs - t) f(t)\,dt = 0. \tag{1}$$

where b is a parameter and the function $K(x)$ satisfies the conditions

(1) $|K(x)| < Ce^{-h|x|}$; $h > 0$; $-\infty < x < \infty$;

(2) $\int_{-\infty}^{\infty} K(x)\,dx \neq 0$.

We shall restrict the variables s, t and the parameter b to the real domain. The results needed in the present paragraph are as follows:

Let

$$\int_{-\infty}^{\infty} K(x)\,dx = \lambda_0^{-1};$$

then the integral equation (1) has the eigenvalues $\lambda_n = \lambda_0 b^{-n}$, $(n = 0, 1, 2, \ldots)$, and as corresponding eigenfunctions, an Appell set of polynomials $p_n(t)$. (The normalisation of $p_n(t)$ is that the coefficient of t^n is $1/n!$). The generating

[1] Brackets refer to the bibliography.

function of this Appell set is given by

(2) $$A(u) = \prod_{n=1}^{\infty} L(b^{-n}u); \qquad |b| > 1$$

(3) $$A(u) = \prod_{n=0}^{\infty} L^{-1}(b^{n}u): \qquad |b| < 1,$$

where

(4) $$L(u) = \lambda_0 \int_{-\infty}^{\infty} e^{-ut} K(t)dt; \qquad |\operatorname{Re} u| < h.$$

In the case $|b| > 1$, $A(u)$ is analytic in the strip $|\operatorname{Re} u| < bh$, and tends to zero in any strip $|\operatorname{Re} u| \leq b(h - \delta)$, $\delta > 0$; [9, Theorem 3; 12, Section 3]. If $|b| < 1$ and if $L(u) \neq 0$ in strip $|\operatorname{Re} u| < h$, then $A(u)$ is analytic in this strip. Furthermore, $A(u)$ satisfies the functional equation

(5) $$A(bu) = L(u)A(u)$$

for $|\operatorname{Re} u| < h$, if $|b| > 1$ and for $|\operatorname{Re} u| < b^{-1}h$, if $|b| < 1$.

Let us now consider equation (1) with $b > 1$; let us suppose that, in addition to conditions 1 and 2, the function $K(x)$ is differentiable and that its derivative satisfies a condition similar to condition 1.

We shall see that there exists then another set of eigenvalues for equation (1), and that the corresponding eigenfunctions are the derivatives of a certain function. We study first the asymptotic behaviour of $A(u)$ in the strip $|\operatorname{Re} u| < bh$.

LEMMA 1. - *If $K(x)$ is differentiable for every real x and satisfies conditions* 1 *and* 2, *and if*

$$|K'(x)| < C_1 e^{-h|x|},$$

then the generating function $A(u)$ of the polynomial eigenfunctions of the integral equation (1), *with $b > 1$, is analytic in the strip $|\operatorname{Re} u| < bh$ and satisfies the relation*

$$A(u) = o(u^{-r}); \qquad |u| \to \infty: \qquad |\operatorname{Re} u| \leq b(h - \delta); \qquad \delta > 0,$$

for any positive r.

PROOF. - In the proof that $A(u) \to 0$ in the strip, we showed that $L(u) \to 0$ in this strip [19; Theorem 1]. In the same way, since $K'(x)$ satisfies a similar inequality for $K(x)$, we can show that the bilateral LAPLACE transform $N(u)$ of $K'(x)$ tends to zero in the strip, at infinity. Now, by a well-known theorem on the LAPLACE transform, we have

$$N(u) = L(u)u; \qquad |\operatorname{Re} u| < qh,$$

[3; p. 104; Satz 8], and since $N(u) = o(1)$, for $|\operatorname{Re} u| \leq h - \delta$; $\delta > 0$, we obtain

$$L(u) = u^{-1}N(u) = o(u^{-1}); \quad |u| \to \infty; \quad |\operatorname{Re} u| \leq h - \delta; \quad \delta > 0.$$

For every $n = 1, 2, \ldots$, we can now write

$$(6) \qquad |L(b^{-n}u)| < C_\delta b^n |u|^{-1}; \qquad |\operatorname{Re} u| \leq b(h - \delta) \leq b^n(h - \delta);$$

If r is any positive integer, we obtain from (2) and for $|\operatorname{Re} u| < bh$,

$$(7) \qquad |A(u)| = |A(b^{-r}u)| \prod_{n=1}^{r} |L(b^{-n}u)|.$$

By the boundedness of $A(u)$ in any strip $|\operatorname{Re} u| \leq b(h - \delta)$, we have

$$|A(ub^{-r})| < M,$$

where M depends on δ but not on r, since $|\operatorname{Re}(ub^{-r})| \leq |\operatorname{Re} u|$. From (6), (7), (8) follows

$$|A(u)| < MC_\delta^r b^{r(r+1)/2} |u|^{-r}; \qquad |\operatorname{Re} u| < b(h - \delta),$$

this inequality leading immediately to the lemma.

LEMMA 2. - *Under the same conditions as in Lemma 1, the generating function $A(u)$ is the bilateral Laplace transform of an infinitely differentiable function $q(t)$, which is given by*

$$(9) \qquad q(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{ut}A(u)du$$

PROOF. - Sufficient conditions for $f(u)$ to be representable as

$$(10) \qquad f(u) = \lim_{T \to \infty}\int_{-T}^{T} e^{-ut}F(t)dt,$$

with

$$F(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{ut} f(u)du\,; \qquad x_1 < x < x_2\,, \tag{11}$$

are that $f(u)$ should be analytic in the strip $x_1 < \operatorname{Re} u < x_2$ and vanish at infinity in this strip, and that $f(u)$ should be absolutely integrable along any straight line in the strip: [3; p. 261, Satz 1]. Now by Lemma 1 these conditions are satisfied for $f(u) = A(u)$; $x_1 = -hb$; $x_2 = hb$. The integral (11), with $x = 0$, (it is independent of x) converges here uniformly for every real t; furthermore the same is true for all the integrals

$$\int_{-i\infty}^{i\infty} u^{n-1} e^{ut} A(u)du, \qquad (n = 1,\ 2, \ldots),$$

so that the function $F(t) = q(t)$ is infinitely differentiable.

Taking $x = (-h + \delta)b$ in (11), we get

$$q(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{ut} A(u)du = \frac{1}{2\pi} e^{xt}\int_{-\infty}^{\infty} e^{iyt} A(x + iy)dy,$$

the last integral being bounded as a function of t, so that

$$q(t) = 0\,\{\exp(-b(k - \delta)t)\}\,; \qquad t \to +\infty\,; \qquad \delta > 0. \tag{12}$$

Putting $x = (h - \delta)b$ in (10), we obtain in a like manner

$$q(t) = 0\,\{\exp(b(h - \delta)t)\}\,; \qquad t \to -\infty\,; \qquad \delta > 0. \tag{13}$$

Hence (10) takes the form of an ordinary bilateral LAPLACE integral

$$A(u) = \int_{-\infty}^{\infty} e^{-ut} q(t)dt\,; \qquad |\operatorname{Re} u| < bh, \tag{14}$$

$q(t)$ being infinitely differentiable and vanishing exponentially a infinity as indicated in (13) and (12).

Next, we seek a new set of eigenfunctions of equation (1).

THEOREM 1. — *Under the same conditions as in Lemma 1, equation (1) has the eigenvalues* $\lambda_0 b^n$ $(n = 1, 2, \ldots)$, *the corresponding eigenfunctions being* $q^{(n-1)}(t)$, $(n = 1, 2, \ldots)$, *where* $q(t)$ *is the function defined in Lemma 2, by* (9)

PROOF. - By (14), we have

$$A(bu) = \int_{-\infty}^{\infty} e^{-bus} q(s) ds \, ; \qquad | \operatorname{Re} u | < h.$$

Substituting $bs = v$, we obtain

$$A(bu) = b^{-1} \int_{-\infty}^{\infty} e^{-uv} q(vb^{-1}) dv.$$

By the convolution theorem for the bilateral LAPLACE transform, $L(u)A(u)$ is the transform of

$$\lambda_0 \int_{-\infty}^{\infty} K(v - t) q(t) dt$$

[14; p. 258, Theorem 16 b]. By the uniqueness theorem [14; p. 244, Theorem 6 b], we obtain from (5),

$$b^{-1} q(vb^{-1}) = \lambda_0 \int_{-\infty}^{\infty} K(v - t) q(t) dt$$

Re-substitutung s, we obtain

$$q(s) = b\lambda_0 \int_{-\infty}^{\infty} K(bs - t) q(t) dt \tag{15}$$

so that we see that $q(t)$ is an eigenfunction of (1) corresponding to the eigenvalue $\lambda_0 b$,

Differentiating (15) and using the differential equation $\frac{\partial K}{\partial s} + b \frac{\partial K}{\partial t} = 0$, for the kernel, we obtain

$$q'(s) + b^2 \lambda_0 \int_{-\infty}^{\infty} \frac{\partial K(bs - t)}{\partial t} q(t) dt = 0.$$

Integration by parts yields

$$q'(s) - b^2 \lambda_0 \int_{-\infty}^{\infty} K(bs - t) q'(t) dt = 0,$$

so that $q'(t)$ is an eigenfunction corresponding to $\lambda_0 b^2$. By complete induction we see that $q^{(n)}(t)$ is an eigenfunction corresponding to the eigenvalue $\lambda_0 b^{n+1}$.

Let us now consider the integral equation which is the transpose of (1).

Theorem 2. - *Under conditions* 1 *and* 2 *and for* $b > 1$, *the transpose of* (1) *has the eigenvalues* $\lambda_0 b^n$ $(n = 1, 2, ...)$ *and as corresponding eigenfunctions an Appell set of polynomials whose generating function* $A^*(u)$ *is related to that of* (1), *i. e.* $A(u)$, *by the equation*

$$A^*(u)A(-u) = 1\,; \qquad |\operatorname{Re} u| < bh. \tag{16}$$

Proof. - The transpose equation

$$g(s) - \lambda \int_{-\infty}^{\infty} K(bt - s)g(t)dt = 0 \tag{1*}$$

can be written in the same form as (1):

$$g(s) - \lambda \int_{-\infty}^{\infty} K^*(b^{-1}s - t)g(t)dt = 0,$$

with the parameter $b^* = b^{-1}$ instead of b $(0 < b^* < 1)$ and with

$$K^*(x) = K(-bx).$$

Now

$$1/\lambda_0^* = \int_{-\infty}^{\infty} K^*(x)dx = \int_{-\infty}^{\infty} K(t)b^{-1}dt = 1/b\lambda_0\,,$$

hence $\lambda_0^* = b\lambda_0$ and equation (1*) has the eigenvalues $\lambda_n^* = \lambda_0^*(b^*)^{-n} = \lambda_0 b^{n+1}$, $(n = 0, 1, 2, ...)$ and the corresponding polynomial eigenfunctions $p_n^*(t)$ have, by (3) (with b^* instead of b), the generating function

$$A^*(u) = \prod_{n=0}^{\infty} \{ L^*[(b^*)^n u] \}^{-1} \tag{17}$$

where

$$L^*(u) = \lambda_0^* \int_{-\infty}^{\infty} K^*(t)e^{-ut}dt = \lambda_0 b \int_{-\infty}^{\infty} K(-bt)e^{-ut}dt \tag{18}$$

$$= \lambda_0 b \int_{-\infty}^{\infty} K(v)e^{uv/b}b^{-1}dv = L(-b^{-1}u).$$

Finally we have, from (17), (18), (2),

$$A^*(u) = \prod_{n=0}^{\infty} L^{-1}(-b^{-n-1}u) = \prod_{n=1}^{\infty} L^{-1}(-b^{-n}u) = A^{-1}(-u).$$

By condition 1, $K^*(x)$ satisfies

$$| K^*(x) | < Ce^{-bh|x|},$$

so that $L^*(u)$ is analytic in the strip $|\operatorname{Re} u| < bh$, and, by (17), the same is true for $A^*(u)$, as it can be obtained in the same way as for $A(u)$, [9, Theorem 3; 12, Section 3].

As an immediate consequence of Theorems 1 and 2, we see that, under the conditions of Lemma 1, the integral equation (1) and its transpose have a common set of eigenvalues, $\lambda_0 b^{n+1}$ $(n = 0, 1, 2, \dots)$. (Cfr. FREDHOLM's theorems!), and that the two sets of eigenfunctions $q^{(n)}(t)$; $p_n^*(t)$ form a biorthogonal system; i. e.,

$$\int_{-\infty}^{\infty} p_m^*(t) q^{(n)}(t) dt = 0; \qquad m \neq n, \qquad (m, n = 0, 1, 2, \dots).$$

This follows in the classical way by multiplying the equation for p_m^* by $q^{(n)}(s)$, that of $q^{(n)}$ by $p_m^*(s)$, subtracting and integrating with respect to s.

If we consider the set $q_n(t) = (-1)^n q^{(n)}(t)$; $(n = 0, 1, 2, \dots)$ the system $(p_m^*; q_n)$ is biorthonormal; this means that

$$\int_{-\infty}^{\infty} p_m^*(t) q_n(t) dt = \delta_{mn}.$$

The equalities for $m = n$ are obtained by integration by parts and by using the relation $(p_{n+1}^*)' = p_n^*$, $(n = 0, 1, 2, \dots)$ and the equality

$$\int_{-\infty}^{\infty} q(t) dt = 1,$$

which follows from (14), for $u = 0$, since by (4) and (2), $L(0) = A(0) = 1$. The function $q(t)$ can now be expressed directly in terms of $A^*(u)$ by the equality

$$q(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{ut}}{A^*(-u)} du,$$

which is a consequence of (9) and (16).

II. Completion of an Appell set and of sequence of derivatives to a biorthonormal system.

The results of Section I lead to the following questions and also suggest an answer to them.

Given an Appell set $p_n(t)$ with the generating function $A(u)$, sufficient conditions on $A(u)$ are to be found which ensure the existence of an infinitely differentiable function $q(t)$ such that the sequence $q_n(t) = (-1)^n q^{(n)}(t)$, $n = 0$, $1, 2, \dots$, should complete the sequence $p_n(t)$ to a biorthogonal system. Conversely, if $q(t)$ is given, under what conditions would there exist a set of polynomials completing the set $q_n(t)$ to a biorthonormal system?

The set $p_n(t)$ and the function $A(u)$ play here the roles of $p_n^*(t)$ and $A^*(u)$ of Section I.

To answer the first question, let us recall the class S of infinitely differentiable functions $f(y)$ along the whole real axis which, together with all their derivatives, vanish more rapidly than any negative power of y at infinity. As is well known, the FOURIER transform maps S onto itself in a one-to-one manner.

THEOREM 3. - *Let the generating function $A(u)$ of the Appell set $p_n(t)$ satisfy the conditions*

(a) *$A(u)$ is regular in a neighbourhood of the origin*; $(A(0) = 1)$,

(b) *$A^{-1}(iy)$ belongs to S, as a function of the real variable y.*

Then the function

$$q(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{-ut}}{A(u)}\,du = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-iyt}}{A(iy)}\,dy, \tag{19}$$

which belongs to S, satisfies the biorthonormality relations

$$\int_{-\infty}^{\infty} p_m(t) q_n(t)\,dt = \delta_{mn}, \qquad (m,\ n = 0,\ 1,\ 2, \dots), \tag{20}$$

where

$$q_n(t) = (-1)^n q^{(n)}(t).$$

PROOF. - Since, for real y,

$$A^{-1}(iy) = \int_{-\infty}^{\infty} e^{iyt} q(t)\,dt,$$

by FOURIER's integral theorem, we have, for $y = 0$,

$$\int_{-\infty}^{\infty} q(t)dt = A^{-1}(0) = 1. \tag{21}$$

By n integrations by parts, we obtain

$$\int_{-\infty}^{\infty} p_m(t)q_n(t)dt = \int_{-\infty}^{\infty} p_m^{(n)}(t)q(t)dt.$$

Hence if $n > m$, $p_m^{(n)}(t) = 0$, so that (20) holds for this case; if $n = m$, then $p_m^{(n)}(t) = 1$, and (20) again holds for this case. Let us now assume $n < m$, then $p_m^{(n)}(t) = p_{m-n}(t)$ so that we have to prove that $q(t)$ is orthogonal to all the Appell polynomials except for the first $p_0(t) = 1$. Since the explicit expression of the polynomials is [9, p. 56]

$$p_r(t) = \sum_{k=0}^{r} a_{r-k} \frac{t^k}{k!}; \qquad (r = 1, 2, ...),$$

where the coefficients a_n are the powers expansion coefficients of $A(u)$, we have

$$\int_{-\infty}^{\infty} p_r(t)q(t)dt = \sum_{k=0}^{r} a_{r-k} \frac{1}{k!} \int_{-\infty}^{\infty} t^k q(t)dt = \sum_{k=0}^{r} M_k a_{r-k}, \tag{22}$$

where

$$M_k = \frac{1}{k!} \int_{-\infty}^{\infty} t^k q(t)dt = \frac{1}{k!} \Big[A^{-1}(u)\Big]_{u=0}^{(k)}; \qquad (k = 0, 1, 2, ...).$$

The last equalities state that the numbers M_k are the powers expansion coefficients of $A^{-1}(u)$ which is analytic about the origin, since $A(u)$ is and $A(0) = 1$.

Using the CAUCHY multiplication formula for the power series of $A(u)$ and $A^{-1}(u)$ in the identity $A(u)A^{-1}(u) = 1$, one sees that the last member of (22) is zero for $r = 1, 2, ...$. This proves the theorem.

COROLLARY. - *Let the generating function $A(u)$ of the Appell set $p_n(t)$ satisfy the conditions*

(a) $A^{-1}(u)$ *is analytic in a strip* $|\operatorname{Re} u| < R$,

(b) $A^{-1}(u) = o(u^{-m})$; $|u| \to \infty$, $|\operatorname{Re} u| \leq r < R$; $m = 1, 2, ...$.

Then the function $q(t)$ *defined by* (19) *is infinitely differentiable, and satisfies* (20) *and the inequalities*

$$| q^{(n)}(t) | < C_{r,n} e^{-r|t|}, \qquad (n = 0, 1, 2, ...)$$

for every $r < R$.

PROOF. - Relation (20) follows from Theorem 3. The exponential vanishing of $q(t)$ and its derivatives are easily obtained from the representations

$$q^{(n)}(t) = \frac{(-1)^n}{2\pi i} \int_{\pm r - i\infty}^{\pm r + i\infty} \frac{u^n e^{-ut}}{A(u)} du; \qquad r < R$$

the integrals converging uniformly for all real t, by condition b (take $m = n + 2$).

We add here a simple direct proof of relations (20). Since $p_n(t)$ are the power series expansion coefficients of $A(u)e^{ut}$ as a function of u, one has

$$(23) \qquad \int_{-\infty}^{\infty} p_m(t) q_n(t) dt = \frac{1}{2\pi i} \int_{-\infty}^{\infty} q_n(t) dt \oint_{|u|=r} \frac{A(u) e^{ut}}{u^{m+1}} du = \frac{1}{2\pi i} \oint_{|u|=r} \frac{A(u)}{u^{m+1}} du \int_{-\infty}^{\infty} e^{ut} q_n(t) dt,$$

the inversion being justified by the uniform convergence of the last integral along the circle.

The uniformity of the convergence follows from the inequalities

$$| e^{ut} | \leq e^{r|t|}; \qquad | q_n(t) | < C_{n,r'} e^{-r'|t|}; \qquad r < r' < R.$$

Now we have

$$(24) \qquad A^{-1}(u) = \int_{-\infty}^{\infty} e^{ut} q(t) dt; \qquad | \operatorname{Re} u | < R,$$

and integrating by parts, n times, the last integral of (24), one obtains, by (23)

$$\int_{-\infty}^{\infty} p_m(t) q_n(t) dt = \frac{1}{2\pi i} \oint_{|u|=r} \frac{A(u)}{u^{m+1}} \frac{u^n}{A(u)} du = \delta_{mn}.$$

REMARKS.

(1) If no restrictions are required for the completing sequence $q_n(t)$, this sequence is not uniquely determined and does not have to be a sequence of derivatives, since there exists functions which are orthogonal to all the

polynomials, namely all the functions all of whose moments vanish; by adding such a function to every $q_n(t)$, not necessarily the same for all $q_n(t)$, we obtain a new sequence which completes the Appell set to a biorthonormal system. But under the conditions of the corollary, we saw that every derivative of the function $q(t)$ defined by (19) vanishes exponentially at infinity; now, we can claim that this sequence $q_n(t)$ is the unique having this last asymptotic property and belonging to $L(-\infty, \infty)$. For if a second sequence $q_n^*(t)$, with these properties would exist, then every function $q_n - q_n^*$ would be orthogonal to all Appell polynomials, hence such a function would have all its moments zero, vanish exponentially at infinity, and belong to $L(-\infty, \infty)$; but then the function $q_n - q_n^*$ must vanish almost everywhere 11; p. 131].

(2) That condition (b) of Theorem 3 or of its corollary is not a necessary condition for the existence of a completing set $q_n(t)$ is shown by the following example. Let $A(u) = 1$, so that condition (a) of Theorem 3, or of its corollary, is satisfied, but condition (b) does not hold. The Appell set is here $p_n(t) = t^n/n!$. As $q(t)$ we may take any function all of whos emoments vanish except for the first which is one, and which is infinitely differentiable. The author has given an explicit construction of such functions and shown [10, 12] that there exist entire functions with the required sequence of moments 1, 0, 0, 0, ...

It is easily verified, by n integrations by parts, that all the moments of $q^{(n)}(t)$ are zero, except for that of order n, which is $(-1)^n n!$. Hence, the sequence $q_n = (-1)^n q^{(n)}$ completes the Appell set $t^n/n!$ to a biorthonormal system. This example suggests that the conditions of Theorem 3 may be relaxed. In fact, we shall prove the following generalisation:

THEOREM 4. - *Let the generating function $A(u)$ of the Appell set $p_n(t)$ satisfy the conditions*

(a) *$A(u)$ is regular in a neighbourhood of the origin; $A(o) = 1$,*

(b) *$A^{-1}(iy)$ is infinitely differentiable with respect to y; $(- < \infty y < \infty)$.*

(c) *$d^n A^{-1}(iy)/dy^n = 0\ (|y|^{a_n})$; $|y| \to \infty$, for some real number a_n, $(n = 0, 1, 2, \ldots)$.*

Let $I(t)$ belong to S and have the moments 1, 0, 0, 0, ..., and let

$$\widetilde{I}(y) = \int_{-\infty}^{\infty} e^{iyt} I(t)\,dt.$$

Then the function

$$(19^*) \qquad q(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyt} A^{-1}(iy)\widehat{I}(y)dy$$

which belongs to S, satisfies the biorthonormality relations (20).

PROOF. - Since $\widehat{I}(o) = 1$, we have, from (19*),

$$\int_{-\infty}^{\infty} q(t)dt = A^{-1}(o)\widehat{I}(o) = 1.$$

It remains thus to prove, as in the proof of Theorem 3, that $q(t)$ is orthogonal to all the polynomials $p_1, p_2, \ldots$. Equality (22) holds true here with

$$M_k = \frac{1}{k!}\int_{-\infty}^{\infty} t^k q(t)dt = \frac{i^{-k}}{k!}\left[A^{-1}(iy)\widehat{I}(y)\right]_{y=0}^{(k)};$$

the differentiations being performed with respect to y. Now, since $\widehat{I}^{(k)}(o) = 0$, $(k = 1, 2, \ldots)$ and $\widehat{I}(o) = 1$, we see, by LEIBNITZ rule, that

$$M_k = \frac{i^{-k}}{k!}\left[A^{-1}(iy)\right]_{y=0}^{(k)} = \frac{1}{k!}\left[A^{-1}(u)\right]_{u=0}^{(k)}$$

exactly as in the proof of Theorem 3. The proof is now arrived at in the same way.

EXAMPLE. - Every rational function $A(u)$ which has no zero along the imaginary axis and for which $A(o) = 1$ obviously satisfies the three conditions of Theorem 4.

The simplest special case is that given before, i. e. $A(u) = 1$.

We shall now deal with the same problem in the case where the path of integration, for the biorthonormality relations, is a circle about the origin. For this purpose, we recall the definition of the BOREL-LAPLACE transform of en entire function of exponential type, [3, Kapitel 10]. Let the entire function

$$F(u) = \sum_{n=0}^{\infty} c_n u^n$$

satisfy the inequality

$$(25) \qquad |F(u)| < Ce^{h|u|}; \qquad h > 0.$$

It is then said to be of exponential type h at most. The coefficients then satisfy the inequality

$$\overline{\lim}\,(n!\,|\,c_n\,|\,)^{1/n} \leq h\,; \tag{26}$$

hence the function

$$f(t) = \sum_{n=0}^{\infty} n!\,c_n t^{-n-1}, \tag{27}$$

which vanishes at infinity, is analytic for $|t| > h$, and is called the BOREL-LAPLACE transform of $F(u)$. The following integral representations hold:

$$f(t) = \int_0^{\infty} e^{-ut} F(u)du\,; \qquad \mathrm{Re}\, t > h\,; \tag{28}$$

$$F(u) = \frac{1}{2\pi i}\oint_{|t|=R} e^{ut} f(t)dt\,; \qquad R > h. \tag{29}$$

If (25) holds for every positive h, $F(u)$ is said to be of minimal type, and the relations (26), (28), (29) hold with 0 instead of h.

THEOREM 5. - *Given an Appell set $p_n(t)$ whose generating function $A(u)$ is such that its reciprocal is an entire function of exponential type not exceeding $h > o$, or of minimal type; the Borel-Laplace transform $q(t)$ of $(2\pi i A(u))^{-1}$ then satisfies the biorthonormality relations*

$$\oint_{|t|=R} p_m(t) q_n(t) dt = \delta_{mn}\,; \qquad (m,\ n = 0,\ 1,\ 2, \ldots), \tag{30}$$

where $R > h$ or $R > o$ respectively.

The proof is essentially the same as the special proof given for the corollary of Theorem 3. Instead of (24), whe have here

$$A^{-1}(u) = \oint_{|t|=R} e^{ut} q(t) dt \tag{31}$$

and the inversion of the two integrals over $|\,u\,| = r < R$, and $|\,t\,| = R$ needs no special justification.

As the simplest example, let us take $A(u) = 1$; $p_m(t) = t^m/m!$; By (27), we have $q(t) = (2\pi i t)^{-1}$ and $q_n(t) = n!\ (2\pi i t^{n+1})^{-1}$, so that relations (30) are directly verified, with $R > 0$.

Let us now consider the converse problem. A sequence of polynomials $P_n(t)$, $n = 0, 1, 2, \ldots$ will be called a standard set if the degree of P_n is exactly n.

THEOREM 6. - *Let $q(t)$ belong to S and let it satisfy the condition*

$$\int_{-\infty}^{\infty} q(t)dt = 1\,; \tag{32}$$

There exists then a unique standard set, namely an Appell set, which completes the sequence $q_n(t) = (-1)^n q^{(n)}(t)$ to a biorthonormal system.

PROOF. - We prove first that every standard set p_m, satisfying (20), is necessarily an Appell set. In fact, from (20), follows

$$\int_{-\infty}^{\infty} q(t)p_m^{(n)}(t)dt = \delta_{mn}\,. \tag{33}$$

If $m = n$, and if C_n is the coefficient of t^n in $p_n(t)$, then (32) and (33) gives $C_n = 1/n\,!$. If $m > n$, then (33) gives

$$\int_{-\infty}^{\infty} q(t)\,\{\, p_m^{(m-1)}(t) - p_{m-1}^{(m-2)}(t)\,\}\, dt = 0, \qquad (m = 2,\ 3, \ldots) \tag{34}$$

But the two polynomials in brackets are of the first degree and by the former result the coefficient of t in both is 1, so that the difference of these polynomials is a constant; by (32) and (34) this constant must be zero. Thus $p_m^{(m-1)} = p_{m-1}^{(m-2)}$; $(m = 2, 3, \ldots)$ and integrating this identity, if $m \geq 3$, we obtain $p_m^{(m-2)} = p_{m-1}^{(m-3)} + C$. Using again (32) and (33), we obtain $C = 0$, and repeating this process yields $p'_m = p_{m-1}$, $(m = 2, 3, \ldots)$. This identity also holds true for $m = 1$, since $p_0 = 1$ and $p_1 = t + a_1$. Hence the sequence p_m is an Appell set.

Next we prove that this Appell set is uniquely determined if such a set satisfying (20) exists. From (20), for $n = 0$, we have

$$\int_{-\infty}^{\infty} p_r(t)q(t)dt = 0\,; \qquad (r = 1,\ 2, \ldots).$$

By (22), these relations are equivalent to the system

$$\sum_{k=0}^{r} M_k a_{r-k} = 0\,; \qquad (r = 1,\ 2, \ldots) \tag{35}$$

with $a_0 = M_0 = 1$, which determine uuiquely the coefficients a_k and so the Appell set itself.

Finally, since from (32) and (35) there follows (20), the existence of the required standard set is established.

Corollary. - *If, in addition to the conditions of Theorem* 6, *the further condition*

$$q(t) = 0\ (e^{-R|t|})\,; \qquad |\,t\,| \to \infty\,; \qquad R > 0,$$

is satisfied then the Appell set which completes the sequence $q_n(t)$ *to a biorthonormal system has the generating function* $A(u)$ *defined by* (24), *this function being analytic in the strip* $|\,Re\, u\,| < R$.

Proof. - From the analyticity of $A^{-1}(u)$ about the origin, $(\,|\,\mathrm{Re}\, u\,| < R)$ and from the equality $A(o) = 1$, it follows that $A(u)$ is also analytic about $u = 0$. Let $p_n^*(t)$ be the Appell set generated by $A(u)$. Since $q(t)$ belongs to S, so does $A^{-1}(iy)$ as a function of y.

Hence by Theorem 3, we see that the set $p_n^*(t)$ completes the set $(-1)^n q^{(n)}(t)$ to a biorthonormal system. Since, by Theorem 6, the set of polynomials is unique, one has $p_n^* = p_n$.

For the case of the biorthonormality conditions (30), the following theorem can be proved in the same manner as Theorem 6 and its corollary.

Theorem 7. - *Let the function* $q(t)$ *satisfy the following conditions*

(1) *It is regular for* $|\,t\,| > h \geq o$.

(2) *It vanishes at infinity.*

(3) $\oint\limits_{|t|=R} q(t)dt = 1\,;\ R > h,$

There exists then a unique standard set $p_n(t)$, *namely an Appell set, which satisfies relations* (30) *and whose generating function* $A(u)$ *is given by* (31). *The function* $A^{-1}(u)$ *is entire; of exponential type not greater than* h *if* $h > o$, *and of minimal type if* $h = o$.

III. Fourier expansions in terms of a sequence of derivatives, for functions regular at infinity.

Let $q(t)$ satisfy the conditions of Theorem 7 and let $f(t)$ satisfy the two first conditions. Consider the Fourier series of $f(t)$ in terms of $q_n(t) = (-1)^n q^{(n)}(t)$,

$$\sum_{n=0}^{\infty} d_n q_n(t)\,; \tag{36}$$

$$d_n = \oint_{|s|=R} f(s) p_n(s) ds\,; \qquad R < h, \tag{37}$$

where the sequence p_n is the Appell set associated with $q(t)$ by Theorem 7.

Before we deal with the problem of the representability of $f(t)$ by means of (36), it is instructive to work out a simple example, which will show that convergence of these FOURIER series is not to be expected, in general, and that a strong summability method may be needed to sum it up.

Let $q(t) = (2\pi i)^{-1}(t^{-1} + t^{-2})$; the three conditions of Theorem 7 are satisfied with $h = o$. By (31) we have, for any $R > 0$,

$$A^{-1}(u) = \oint_{|t|=R} e^{ut}(2\pi i)^{-1}(t^{-1} + t^{-2}) dt = 1 + u.$$

The power expansion coefficients of $A(u) = (1 + u)^{-1}$ are $a_n = p_n(o) = (-1)^n$. Let us compute the FOURIER coefficients of $f(t) = t^{-1}$; by (37), they are

$$d_n = \oint t^{-1} p_n(t) dt = \oint t^{-1} p_n(o) dt = 2\pi i a_n = 2\pi i (-1)^n.$$

The series (36) takes the form

$$\sum_{n=0}^{\infty} (-1)^n n!\,[t^{-n-1} + (n+1) t^{-n-2}] = \sum_{n=0}^{\infty} (-1)^n n! \left(t^{-n-1} - \frac{d}{dt} t^{-n-1} \right). \tag{38}$$

Such a series is not convergent and not even (B) summable; for BOREL's method, see [2; 311, 401]. But it is summable, by a generalisation of the (B) method, to t^{-1} for $\operatorname{Re} t \geq o$; $t \neq o$, as it will be shown now.

The (B^2)-method is defined by the relation

$$(B^2) \sum_{n=0}^{\infty} c_n = \iint_Q e^{-x-y} \left\{ \sum_{n=0}^{\infty} \frac{c_n}{(n!)^2} x^n y^n \right\} dx dy, \tag{39}$$

where Q is the first quarter $x \geq o$; $y \geq 0$; this method is known to be regular [2; 405].

Applying (39) to series (38) the convergent series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{xy}{t} \right)^n = e^{-xy/t}$$

appears, and we obtain

$$\sum_{n=0}^{\infty} \frac{c_n}{(n!)^2} x^n y^n = \frac{1}{t} e^{-xy/t} - \frac{d}{dt}\left(\frac{1}{t} e^{-xy/t}\right). \tag{40}$$

Restricting now t by the conditions $t \neq o$; $\operatorname{Re} t \geq o$, one has

$$t^{-1} \iint_Q e^{-x-y-xy/t} dx dy = t^{-1} \int_0^\infty e^{-y} dy \int_0^\infty e^{-x-xy/t} dx = \int_0^\infty (t+y)^{-1} e^{-y} dy = F(t). \tag{41}$$

Hence the double integral (39) is equal to $F(t) - F'(t)$. But

$$F'(t) = -\int_0^\infty (t+y)^{-2} e^{-y} dy = \left[(t+y)^{-1} e^{-y}\right]_0^\infty + \int_0^\infty (t+y)^{-1} e^{-y} dy = -t^{-1} + F(t).$$

We see that in fact, the series (38) is (B^2)-summable to

$$F(t) - F'(t) = t^{-1}, \text{ for } t \neq o; \qquad \operatorname{Re} t \geq o.$$

Let us now generalise.

Let $P(u)$ be a HURWITZ polynomial, i. e. a polynomial all of whose roots have a negative real part. (Such polynomials play a fundamental role in physical theories of stability; for criteria to determine whether a polynomial is of this kind or not, see [5; p. 395, ...]). With $P(u)$ we associate an angular region $\mathcal{A}(P)$ of the t-plane, as follows; let $u_1, \ldots, u_r$ denote the roots of $P(u)$; then $\mathcal{A}(P)$ is the common part of the half-planes $\operatorname{Re}(tu_k) \leq o$; $(k = 1, 2, \ldots, r)$. If $\alpha = \min \arg u_k$; $\beta = \max \arg u_k$, $(k = 1, 2, \ldots, r)$, $\frac{\pi}{2} < \arg u_k < \frac{3\pi}{2}$; it is easily verified that $\mathcal{A}(P)$ is the set of all points t such that

$$-\alpha + \frac{\pi}{2} \leq \arg t \leq -\beta + \frac{3\pi}{2}.$$

With these concepts, we can formulate and prove the following expansion theorem.

THEOREM 8. - *Let $P(u)$ be an Hurwitz polynomial with simple roots only, let $P(o) = 1$ and*

$$q(t) = \frac{1}{2\pi i} \int_0^\infty e^{-ut} P(u) du.$$

If $f(t)$ is regular for $|t| > h \geq o$ and vanishes at infinity, then its Fourier series (28) in terms of $q(t)$ is (B^2)-summable to $f(t)$ in the part of $\mathfrak{A}(P)$ outside any circle $|t| = R > h$.

PROOF. - 1. The theorem will be proved first for $f(t) = t^{-1}$, $(h = o)$. As in the example calculated above, the expansion coefficients for t^{-1} are $d_n = 2\pi i a_n$, where $a_n = p_n(o)$ are the power series coefficients of the generating function $A(u)$ of the Appell set corresponding to $q(t)$ by Theorem 7.

By (31), with any $R > o$,

$$A^{-1}(u) = \oint_{|t|=R} e^{ut} q(t) dt = P(u).$$

Since $P(u)$ has simple roots only, we have $P'(u_k) \neq 0$, and

$$A(u) = P^{-1}(u) = \sum_{k=1}^{r} \frac{1}{P'(u_k)(u - u_k)}. \tag{42}$$

If u_1 denotes the root nearest to the origin, then, for $|u| < |u_1|$,

$$A(u) = \sum_{k=1}^{r} \frac{-1}{u_k P'(u_k)} \sum_{n=0}^{\infty} \left(\frac{u}{u_k}\right)^n = -\sum_{n=0}^{\infty} u^n \sum_{k=1}^{r} \frac{1}{u^{n+1} P'(u_k)}.$$

Hence

$$a_n = -\sum_{k=1}^{r} \frac{1}{u_k^{n+1} P'(u_k)}. \tag{43}$$

The function $q(t)$ can be expressed as

$$2\pi i q(t) = P(-D) t^{-1}$$

where D denotes the differentiation operator with respect to t; in fact $D^k t^{-1} = (-1)^k t^{-k-1} k!$, so that

$$P(-D) t^{-1} = \sum_{k=0}^{r} b_k (-D)^k t^{-1} = t^{-1} \sum_{k=0}^{r} b_k k! \, t^{-k} = 2\pi i q(t).$$

Hence

$$q_n(t) = (-1)^n q^{(n)}(t) = \frac{1}{2\pi i} P(-D) n! \, t^{-n-1}; \quad (n = 0, 1, 2, \ldots). \tag{44}$$

The FOURIER series of t^{-1} is thus, by (43) and (44),

$$(45)\quad \sum_{n=0}^{\infty} d_n q_n(t) = \sum_{n=0}^{\infty} a_n n!\, P(-D) t^{-n-1} = \sum_{k=1}^{r} \frac{1}{P'(u_k)} \sum_{n=0}^{\infty} n!\, P(-D)(tu_k)^{-n-1}.$$

The (B^2)-summability of the series $\Sigma\, n!\,(tu_k)^{-n-1}$, yields, as in the example $F(-tu_k)$, for $\mathrm{Re}\,(tu_k) \leq o$; $t \neq o$, where $F(t)$ is defined by (41). Now, by the uniform convergence of the double integral (34), with tu_k instead of $-t$, in the region $\mathrm{Re}\,(tu_k) \leq o$, $|t| \geq R > o$, it is clear that

$$(B^2)\, \Sigma\, n!\, D(tu_k)^{-n-1} = D(B^2)\, \Sigma\, n!\,(tu_k)^{-n-1}$$

and the differentiation may be repeated any number or times. Hence we get, from (45)

$$(46)\quad (B^2) \sum_{n=0}^{\infty} d_n q_n(t) = P(-D) \sum_{k=0}^{r} \frac{F(-tu_k)}{P'(u_k)}; \qquad t \in \mathfrak{A}(P); \qquad |t| \geq R > 0,$$

Let us seek a solution of the differential equation

$$(47)\qquad P(-D)y = t^{-1},$$

in the form of an ordinary LAPLACE transform

$$y = \int_0^{\infty} e^{-st} Y(s) ds = \mathfrak{L}Y(s),$$

The law $D\mathfrak{L}Y(s) = -\mathfrak{L}sY(s)$, gives us, with $\mathfrak{L}1 = t^{-1}$ and (47),

$$P(-D)y = P(-D)\mathfrak{L}Y(s) = \mathfrak{L}P(s)Y(s) = \mathfrak{L}1.$$

Hence, by the uniqueness of the LAPLACE transform

$$Y(s) = P^{-1}(s);$$

so that, by (42),

$$y(t) = \mathfrak{L}P^{-1}(s) = \int_0^{\infty} e^{-st} \left\{ \sum_{k=1}^{r} [P'(u_k)(s-u_k)]^{-1} \right\} ds; \qquad \mathrm{Re}\, t > 0.$$

Making the substitution $st = v$, we obtain by (34),

$$y(t) = \sum_{k=1}^{r} 1/P'(u_k) \cdot \int_0^{\infty} e^{-v}/(v-u_k t) \cdot dv = \sum_{k=1}^{r} F(-tu_k)/P'(u_k).$$

This with (46) and (47) proves that the FOURIER series of t^{-1} is (B^2)-summable to t^{-1} for all values of t lying in $\mathfrak{A}(P)$ outside any circle $|t| = R > 0$.

2. Our second step is to consider the series

$$\sum_{n=0}^{\infty} q_n(t) p_n(s), \tag{48}$$

which arises formally from the FOURIER series of $f(t)$ defined by (36) and (37) by introducing (37) in (36) and inverting summation and integration. The (B^2)-summation of (48) leads, by (39), to the double integral

$$\iint_Q e^{-x-y} \left\{ \sum_{n=0}^{\infty} (n!)^{-2} q_n(t) p_n(s) x^n y^n \right\} dx dy. \tag{49}$$

We have [9; p. 64; (31)]

$$|p_n(s)| < a k^{-n} e^{k|s|}; \qquad k < |u_1|, \qquad (n = 0, 1, 2, \ldots).$$

On the other hand, by (44), there exists a positive number b such that, for $t \neq 0$,

$$|q^{(n)}(t)| < b(n+r)!\, |t|^{-n-1}; \qquad (n = 0, 1, 2, \ldots).$$

These two last sequences of inequalities show that the series of (49) converges for all $t \neq o$, s, x, y. By (49) and (44) a study of the entire function of s and z

$$E(s, z) = \sum_{n=0}^{\infty} \frac{p_n(s) z^n}{n!} \tag{50}$$

is necessary regarding its asymptotic behaviour, to decide for which s and t the double integral is convergent. From (50) we have

$$E(s, z) = \sum_{n=0}^{\infty} p_n(s) \frac{1}{2\pi i} \oint e^{zv} v^{-n-1} dv, \tag{51}$$

the path being any circle $|v| = c$. But if we take $c > |u_1|^{-1}$, then the inversion of summation and integration in (51) will be correct, since the series $\Sigma\, v^{-n} p_n(s)$ will be uniformly convergent on $|v| = c$; in fact, this series represents $A(v^{-1}) e^{s/v}$ for $|v| > |u_1|^{-1}$. From (51) follows now

$$E(s, z) = \frac{1}{2\pi i} \oint_{|v| = c} e^{zv + s/v} v^{+1} A(v^{-1}) dv. \tag{52}$$

Within $|v| = c > |u_1|^{-1}$, there are the r simple poles of:

$$v^{-1}A(v^{-1}) = v^{-1}p^{-1}(v^{-1}) = \sum_{k=1}^{r} \frac{1}{P'(u_k)(1 - vu_k)}$$

(by ((42), which are $v_k = u_k^{-1}$; and the essential singular point $v = o$. Hence, by (52)

$$E(s, z) = -\sum_{k=1}^{r} \frac{e^{z/u_k + su_k}}{u_k P'(u_k)} + \sum_{k=1}^{r} \frac{1}{P'(u_k)} \frac{1}{2\pi i} \oint_{|v|=c_k} \frac{e^{zv/sv}}{1 - u_k v} dv, \tag{53}$$

where each path $|v| = c_k$ encloses $v = o$ but not $v_k = u_k^{-1}$ hence $c_k < |u_k|^{-1}$.

Substituting $v = (s/z)^{1/2} w$ for each integral of (53) we obtain

$$(2\pi i)^{-1} \oint_{|w|=w_k} \frac{\exp[(zs)^{1/2}(w + 1/w)]}{(z/s)^{1/2} - u_k w} dw = g_k(s, z) \tag{54}$$

with $w_k = |z/s|^{1/2} c_k < |z/s|^{1/2} |u_k|^{-1}$; but since we are interested in the asymptotic behaviour of these integrals for $|z| > |su_k^2|$, we can choose for w_k any positive number and consider then $|z| > w_k^2 |su_k^2|$, s being fixed.

Now, for $|w| = w_k$, one has

$$\left| \exp\left[(zs)^{1/2}\left(w + \frac{1}{w}\right)\right]\right| \leq \exp\left[|zs|^{1/2}\left(w_k + \frac{1}{w_k}\right)\right] \tag{55}$$

The minimum value of $w_k + w_k^{-1}$ is 2 which is reached for $w_k = 1$, hence we will take $w_k = 1$ for each k and in each integral (54) we consider only $|z| > |su_k^2|$. By (54), (55), with $w_k = 1$, one gets

$$|g_k(s, z)| \leq \frac{|s|^{1/2}}{|z|^{1/2} - |u_k^2 s|^{1/2}} \exp(2|zs|^{1/2}); \qquad |z| > |u_k^2 s|. \tag{56}$$

By (37) and (43), the double integral (42) takes the form

$$\frac{1}{2\pi i} \iint_Q e^{-x-h}[P(-D)t^{-1}E(s, xyt^{-1})] dx dy. \tag{57}$$

By (53) and (54) we have to study the convergeuce of the integrals

(58) $$\iint_Q \exp\left(-x-y+\frac{xy}{tu_k}\right)dxdy,$$

(59) $$\iint_Q \exp(-x-y)g_k(s,\ xyt^{-1})dxdy.$$

For (58) the restrictions $\operatorname{Re}(tu_k) \leq 0$; $|t| \geq t_0 > o$ ensure the uniform convergence for all such values of t.

For (59) let us first consider the part Q_1 of Q which is defined by $x \geq o$; $y \geq o$; $xy \leq R\ |tu_k^2|$; t beind fixed and $t \neq o$, and $R > 0$. Since $g(s, z,$ is an entire function of z and since here $z = xyt^{-1}$; one has

(60) $$|g_k(s,\ xyt^{-1})| < M_k(s); \qquad (x,\ y) \in Q_1.$$

where $M_k(s)$ depends only on s, not on x, y, t.

For the complementary part Q_2 to Q_1, of Q, which is defined by $x > o$; $y > o$; $xy > R\ |tu_k^2|$; $(t \neq o)$, $R > |s|$, the inequality (56) holds, insce $|z| = xy/t^{-1} > R\ |u_k^2| > |su_k^2|$. Let us now restrict t by the condition $|t| \geq R > |s|$; then

(61) $$a = \left|\frac{s}{t}\right|^{1/2} \leq \left(\frac{|s|}{R}\right)^{1/2} < 1.$$

Now considering (59) and (56) with $z = xyt^{-1}$, we have

(62) $$-x-y+2a(xy)^{1/2} = -(x^{1/2}-y^{1/2})^2 - 2(1-a)(uv)^{1/2}.$$

Let $o < k < 1$. For the part of Q_2 where $y \leq kx$, we obtain, by (62), (61) and setting $(1-k^{1/2})^2 = k_1 > o$, $\exp[-x-y+2a(xy)^{1/2}] < \exp[-(x^{1/2}-y^{1/2})^2] \leq \leq \exp(-k_1x)$, and for the part of Q_2 where $y \geq k^{-1}x$; one obtains, in a similar way,

$$\exp[-x-y+2a(xy)^{1/2}] < \exp[-(y^{1/2}-x^{1/2}] \leq \exp(-k_1y).$$

For the remaining part of Q_2, where $kx < y < k^{-1}x$, one has, by (62) and (61)

$$\exp[-x-y+2a(xy)^{1/2}] \leq \exp[-2(1-a)(xy)^{1/2}] \leq \exp[-k_2(xy)^{1/2}],$$

with $k_2 = 2\left(1-\left|\frac{s}{R}\right|^{1/2}\right) > 0.$

If we put $N_k(s) = |s|^{1/2}|u_k|^{-1}(R^{1/2} - |s|^{1/2})^{-1}$ and use (50), we can, from the discussion on the integrand of (59) in $Q(x \leq o;\ y \leq o)$ conclude that it can be majorised in Q by the following functions of x, y, s independent of t;

for $y \leq kx$, by $M_k \exp(-x-y) + N_k \exp(-k_1 x)$;

for $y \geq k^{-1}x$, by $M_k \exp(-x-y) + N_k \exp(-k_1 y)$;

for $kx < y < k^{-1}x$, by $M_k \exp(-x-y) + N_k \exp[-k_2(xy)^{1/2}]$.

It is easily verified that these three majorizing functions are integrable in their respective parts of Q. Hence the integral

$$\iint_Q e^{-x-y} E\left(s,\ \frac{xy}{t}\right) dxdy$$

converges absolutely and uniformly for all values of t lying in $\mathfrak{A}(P)$ outside any circle $|t| = R > |s|$, for a fixed s. Reverting to (57) and noting that, by (54) $\partial^k E(s, z)/\partial z^k$ is analogous in its asymptotic behaviour to $E(s, z)$, we conclude that the double integral (67) or (49) represents an analytic function of t in the mentioned domain. If we now keep t fixed, $t \neq o$, and consider s such that $|s| \leq R < |t|$, a similar reasoning would yield that this integral (57) represents an analytic function of s in this circle. Let us denote this function by $H(s, t)$. Thus

(63) $$(B^2) \sum_{n=0}^{\infty} q_n(t) p_n(s) = H(s,\ t); \qquad t \in \mathfrak{A}(P); \qquad |t| > |s|.$$

3. Next we shall prove that

(64) $$H(s,\ t) = \frac{1}{2\pi i(t-s)}; \qquad t \in \mathfrak{A}(P); \qquad |t| > |s|.$$

For $P(u) = A^{-1}(u) = 1$; $p_n(s) = s^n/n!$; $q(t) = (2\pi i t)^{-1}$; it is easily verified that series (48) converges, for $|s| < |t|$, to $(2\pi i(t-s))^{-1}$, and so it is too (B^2) summable to this function.

In our general case, we prove first that $H(s, t)$ is a function of $t - s$; for that, it is sufficient to verify that the function

(65) $$2\pi i G(s,\ t) = \iint_Q e^{-x-y} t^{-1} E(s,\ xyt^{-1}) dxdy$$

satisfies the equation $\frac{\partial G}{\partial s}+\frac{\partial G}{\partial t}=0$, since, by, (50),

$$H(s,\ t)=P(-D)G(s,\ t)=P(-D)G(t-s).$$

Now, by (50) and (65)

$$-2\pi i\frac{\partial G}{\partial t}=\iint\limits_{Q} e^{-x-y}\left\{\sum_{n=0}^{\infty}\frac{n+1}{n\,!}\,\frac{p_n(s)x^n y^n}{t^{n+2}}\right\}dxdy$$

$$=\int_0^\infty e^{-y}dy\left\{\left[-e^{-x}x\,\Sigma\frac{p_n(s)(xy)^n}{n\,!\,t^{n+2}}\right]_0^\infty+\int_0^\infty e^{-x}\left[\Sigma\frac{p_n(s)x^{n+1}y^n}{n\,!\,t^{n+2}}\right]dx\right\}$$

$$=\int_0^\infty e^{-x}dx\left\{\left[e^{-y}\,\Sigma\frac{p_u(s)(xy)^{n+1}}{(n+1)\,!\,t^{n+2}}\right]_0^\infty+\int_0^\infty e^{-y}\left[\Sigma\frac{p_n(s)(xy)^{n+1}}{(n+1)\,!\,t^{n+2}}\right]dy\right\}$$

$$=\iint\limits_{Q} e^{-x-y}\left\{\Sigma\frac{p_n(s)(xy)^{n+1}}{(n+1)\,!\,t^{n+2}}\right\}dxdy$$

$$=\iint\limits_{Q} e^{-x-y}\sum_{n=1}^{\infty}\frac{p_{n-1}(s)(xy)^n}{n\,!\,t^{n+1}}\Big\}dxdy=2\pi i\frac{\partial G}{\partial s};$$

the last equality following from $p_{n-1}(s)=p'_n(s)$.

Hence $H(s,\ t)=H(t-s)$.

Let us now calculate $H(o,\ t)=H(t)$. By (65) and (50),

$$2\pi iG(o,\ t)=\iint\limits_{Q} e^{-x-y}\left\{\sum_{n=0}^{\infty}\frac{2\pi i a_n(xy)^n}{n\,!\,t^{n+1}}\right\}dxdy.$$

Hence, by (45) and by part 1 of the proof, we see that $2\pi i\ H(o,\ t)$, which is equal to $2\pi i\ P(-D)G(o,\ t)$, is just the result of the (B^2)-summation of the FOURIER series of t^{-1}. Thus, we obtain that $H(t)=1/2\pi it$, and (64) is established.

4. Let $f(t)$ be analytic outside $|\,t\,|=h$ and vanish at infinity.

We have for $|\,t\,|>R_0>h$.

$$f(t)=\frac{1}{2\pi i}\oint\limits_{|s|=R_0}\frac{f(s)}{t-s}\,ds$$

where the sense along the path is clockwise, leaving the domain $|t| > R_0$ to the left. If we take any $R > R_0$ and restrict t by the conditions $|t| \geq R$; $t \in \mathfrak{A}(P)$, one has, by the uniformity of the (B^2)-summability for all s on the circle $|s| = R_0$,

$$f(t) = \oint\limits_{|s|=R_0} f(s) \Big\{ (B^2) \sum_{n=0}^{\infty} q_n(t) p_n(s) \Big\} ds$$

$$= (B^2) \sum_{n=0}^{\infty} q_n(t) \oint\limits_{|s|=R_0} f(s) p_n(s) ds = (B^2) \sum_{n=0}^{\infty} d_n q_n(t).$$

This is the theorem.

IV. Fourier expansions along the real axis in terms of a sequence of derivatives.

Let $A(u)$ be an entire generating function, such that $A^{-1}(iy)$ belongs to S as a function of the real variable y and satisfying the two inequalities

$$|A(u)| < c \exp(h|u|^a + k|u|), \tag{66}$$

$$|A(iy)|^{-1} < c_1 \exp(-h|y|^a), \tag{67}$$

y being real and $c > o$, $c_1 > o$, $h > o$, $k \geq o$, $a > 1$. If $A(u)$ is an entire function satisfying (66) and such that $A^{-1}(u)$ satisfies the inequality

$$|A^{-1}(u)| < C_1 \exp(-h|u|^a)$$

in the strip $|\operatorname{Re} u| \leq u_0$, then $A^{-1}(iy)$ will belong to S and satisfy (67).

The simplest example is $A(u) = \exp(-u^2)$: here $h = 1$; $k = o$, $a = 2$; this function generates the sequence $p_n(t) = (1/n!)H_n(t/2)$, $H_n(x)$ being HERMITE's polynomials and the corresponding function $q(t)$ is, by (19),

$$q(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyt} e^{-y^2} dy = \frac{1}{2\sqrt{\pi}} e^{-t^2/4}$$

and

$$q_n(t) = (-1)^n q^{(n)}(t) = \frac{1}{2^{n+2}\sqrt{\pi}} e^{-t^2/4} H_n(t/2).$$

For expansions in terms of HERMITE functions and polynomials see [6].

More generally, if n is a positive integer and $E(u)$ is an entire function of exponential type whose modulus has a positise lower bound in the strip

$|\operatorname{Re} u| \leq u_0$, then the function $A(u) = E(u)\exp[(-u^2)^n]$ satisfies (66), (67) and $A^{-1}(iy)$ belongs to S.

The function $q(t)$ corresponding to a function $A(u)$ of the type considered here, by means of (19), is entire since $a > 1$, and belongs to S along the real axis. With every function $A(u)$ we associate a class $\mathfrak{F}_{a,k}$ containing all the functions $f(t)$ which are regular in a strip $|Imt| \leq t_0$; $t_0 > k$, and satisfying, in this strip, the inequality

$$(68) \qquad |f(t)| < C\exp(-h_0|t|^{a_0}),$$

with $a_0 > a'$; $\frac{1}{a} + \frac{1}{a'} = 1$; $h_0 > o$.

For the summation of the FOURIER series of $f(t)$,

$$(69) \qquad \sum_{n=0}^{\infty} d_n q_n(t); \qquad d_n = \int_{-\infty}^{\infty} f(s)p_n(s)ds,$$

we shall use a summability method which is a combination of the (A)-method (ABEL-POISSON) and of the (B)-method (BOREL).

$$(70) \qquad (AB)\,\Sigma\, c_n = \lim_{b \to 1+} (B)\,\Sigma\, c_n b^{-n}.$$

The (AB)-method is regular, since if $\Sigma\, c_n$ is convergent, then the series on the right hand side of (70) is absolutely convergent for every $b > 1$; hence, by the regularity of (B) and of (A) we get $(AB)\,\Sigma\, c_n = \Sigma\, c_n$.

THEOREM 9. - *Let $A(u)$ be an entire function satisfying* (66) *and* (67) *and such that $A^{-1}(iy) \in S$; let $(p_n;\, q_n)$ be the biorthonormal system generated by $A(u)$. Then the Fourier series* (69) *of any function $f(t)$ belonging to $\mathfrak{F}_{a,k}$ is (AB)-summable to $f(t)$ on the real axis.*

PROOF. - 1. We consider first

$$(71) \qquad (B)\sum_{n=0}^{\infty} \frac{d_n q_n(t)}{b^n} = \int_0^{\infty} e^{-x}dx \left\{ \sum_{n=0}^{\infty} \frac{d_n q_n(t)}{n!}\left(\frac{x}{b}\right)^n \right\}; \qquad b > 1.$$

The last series is convergent for every $x \geq o$, as will be clear from the following. Using (69) for d_n, one has, formally for the present,

$$(72) \qquad \sum_{n=0}^{\infty} \frac{d_n q_n(t)}{n!}\left(\frac{x}{b}\right)^n = \int_{-\infty}^{\infty} f(s)ds \left\{ \sum_{n=0}^{\infty} \frac{p_n(s)q_n(t)}{n!}\left(\frac{x}{b}\right)^n \right\}.$$

The last series will now be transformed by means of the integral representation of $q_n(t)$. namely

$$(73) \qquad q_n(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(iy)^n}{A(iy)} e^{-iyt} dy.$$

Hence,

$$(74) \qquad 2\pi \sum_{n=0}^{\infty} \frac{p_n(s) q_n(t)}{n!}\left(\frac{x}{b}\right)^n = \sum_{n=0}^{\infty} \frac{p_n(s)}{n!}\left(\frac{x}{b}\right)^n \int_{-\infty}^{\infty} \frac{(iy)^n}{A(iy)} e^{-iyt} dy$$

$$= \int_{-\infty}^{\infty} \frac{e^{-iyt}}{A(iy)} dy \left\{ \sum_{n=0}^{\infty} \frac{p_n(s)}{n!}\left(\frac{iyx}{b}\right)^n \right\}.$$

The inversion of integration and summation in the last equality is justified in the following way.

From the equality

$$(75) \qquad p_n(s) = \frac{1}{2\pi i}\oint \frac{A(u)e^{us}}{u^{n+1}} du; \qquad |u| = R.$$

we have

$$(76) \qquad |p_n(s)| < \frac{M}{R^n} e^{R|s|}; \qquad (R > o),$$

so that

$$(77) \qquad \sum_{n=0}^{\infty} \left| \frac{p_n(s)}{n!}\left(\frac{ixy}{b}\right)^n \right| < M \exp\left(R\,|\,s\,| + \frac{x\,|\,y\,|}{R}\right); \qquad (x \geq o).$$

Hence, by (67), the las integral in (74) converges when the modulus of each term and factor is taken. Since, furthermore, the last series in (74) converges uniformly in every finite interval of y, the inversion is justified and the convergence of the last series in (72) is immediately proved.

Let us now transform the last series in (74):

$$(78) \qquad \sum_{n=0}^{\infty} \frac{p_n(s)}{n!}\left(\frac{ixy}{b}\right)^n = \sum_{n=0}^{\infty} p_n(s)\left(\frac{iy}{b}\right)^n \frac{1}{2\pi i}\oint_{|z|=r} \frac{e^{xz}}{z^{n+1}} dz$$

$$= \frac{1}{2\pi i}\oint_{|z|=r} \frac{1}{z} \exp\left(xz + \frac{iys}{bz}\right) A\left(\frac{iy}{bz}\right) dz.$$

Using (66), this integral representation gives the estimation

$$\left|\sum_{n=0}^{\infty}\frac{p_n(s)}{n!}\left(\frac{ixy}{b}\right)^n\right| < c\exp\left(xr + h\left|\frac{y}{br}\right|^a + \frac{|ys|}{br} + k\frac{|y|}{br}\right). \tag{79}$$

By YOUNG's inequality

$$XY \leq \frac{X^{a_1}}{a_1} + \frac{Y^{a_1'}}{a_1'}; \qquad \left(X,\ Y,\ a_1,\ a_1' > o;\ \frac{1}{a_1} + \frac{1}{a_1'} = 1\right),$$

with $X = \frac{|y|}{br}$; $Y = |s|$, the right-hand member of (78) is not greater than $c\exp\left(xr + h\left|\frac{y}{br}\right|^a + \frac{1}{a_1}\left|\frac{y}{br}\right|^{a_1} + \frac{1}{a_1'}|s|^{a_1'} + k\frac{|y|}{br}\right)$. We choose now r and a_1' such that $b^{-1} < r < 1$ and $1 < a' < a_1' < a_0$; hence $br > 1$ and $a_1 < a$. Reverting to (74) with $|Im\, t| \leq T$, T being arbitrary and positive for the present, and using (67) we obtain the following approximation:

$$\begin{aligned}\left|\sum_{n=0}^{\infty}\frac{p_n(s)q_n(t)}{n!}\left(\frac{x}{b}\right)^n\right| &< c_2\exp\left(xr+\frac{|s|^{a_1'}}{a_1'}\right)\int_{-\infty}^{\infty}\exp\left\{-h\left[1-\frac{1}{(br)^a}\right]|y|^a+\frac{1}{a_1}\left|\frac{y}{br}\right|^{a_1}+\left(\frac{y}{br}+T\right)|y|\right\}dy\\ &= c_3\exp\left(xr+\frac{|s|^{a_1'}}{a_1'}\right).\end{aligned} \tag{80}$$

Reverting to (72), we obtain, by (68) and (80).

$$\begin{aligned}\left|\sum_{n=0}^{\infty}\frac{d_nq_n(t)}{n!}\left(\frac{x}{b}\right)^n\right| &< c_4\int_{-\infty}^{\infty}\exp\left(xr+\frac{|s|^{a_1'}}{a_1'}-h_0|s|^{a_0}\right)ds\\ &= c_5e^{xr}; \qquad (r<1;\ x\geq 0),\end{aligned} \tag{81}$$

the last integral being convergent since $a_1' < a_0$.

Since $r < 1$, the integral (71) converges, that is to say, the series $\Sigma\, b^{-n}d_nq_n(t)$ is (B)-summable for any complex t and $b > 1$. Let us put

$$f(t,\ b) = (B)\sum_{n=0}^{\infty}\frac{d_nq_n(t)}{b^n}. \tag{82}$$

2. We shall now deduce an integral representation for $f(t,\ b)$, which will be, in fact, an integral transform of the type considered in Section I.

If we substitute (72) in (71), then the obtained double integral will converge absolutely by (80) and (68). We thus may invert the integrations and so obtain

$$(83)\qquad f(t,\ b) = \int_{-\infty}^{\infty} f(s)ds \int_{\infty}^{\infty} e^{-x}dx \left\{ \sum_{n=0}^{\infty} \frac{p_n(s)q_n(t)}{n\,!} \left(\frac{x}{b}\right)^n \right\}.$$

The inner integral is

$$(B) \sum_{n=0}^{\infty} \frac{p_n(s)q_n(t)}{b^n} = K(s,\ t).$$

By (74), we have

$$2\pi K(s,\ t) = \int_{0}^{\infty} e^{-x}dx \int_{-\infty}^{\infty} \frac{e^{-iyt}}{A(iy)}\, dy \left\{ \sum_{n=0}^{\infty} \frac{p_n(s)}{n\,!} \left(\frac{ixy}{b}\right)^n \right\}.$$

By (78) and the result of YOUNG's inequality we see that the last double integral converges absolutely, so that

$$2\pi K(s,\ t) = \int_{-\infty}^{\infty} \frac{e^{-iyt}}{A(ix)}\, dy \int_{0}^{\infty} e^{-y}dx \left\{ \sum_{n=0}^{\infty} \frac{p_n(s)}{n\,!} \left(\frac{ixy}{b}\right)^n \right\}.$$

For any fixed y, we can invert the sum and the inner integral since, in (77) we may take $R > |\, y\,|$. Finally, we get

$$(84)\qquad K(s,\ t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iyt}}{A(iy)}\, dy \left\{ \sum_{n=0}^{\infty} p_n(s) \left(\frac{iy}{b}\right)^n \right\}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\,[iy(s/b - t)] A^{-1}(iy) A(iy/b) dy = K(s/b - t)$$

with

$$(85)\qquad K(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iyz} A^{-1}(iy) A(iy/b) dy.$$

Hence

$$(86)\qquad f(t,\ b) = \int_{-\infty}^{\infty} K\left(\frac{s}{b} - t\right) f(s) ds.$$

3. Finally, we have to prove that, for real t,

$$\lim_{b \to 1+} f(t, b) = f(t).$$

Putting

$$(87) \qquad \widehat{K}(y) = \int_{-\infty}^{\infty} e^{-iyz} K(z) dz; \qquad \widehat{f}(y) = \int_{-\infty}^{\infty} e^{iyz} f(z) dz,$$

we have, from (85) $\widehat{K}(y) = A(iy/b)A^{-1}(iy)$, so that, by (59) and (67), the inequalities

$$(88) \qquad |\widehat{K}(y)| < cc_1 \exp(-h(1-b^{-a})|y|^a + k|y|) < cc_1 e^{k|y|}$$

hold. Substituting (84) in (86), we have, by the notations of (87)

$$(89) \qquad f(t, b) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s) ds \int_{-\infty}^{\infty} e^{iy(s/b - t)} \widehat{K}(y) dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyt} \widehat{K}(y) \widehat{f}(y/b) dy$$

the inversion being justified by absolute integrability which follows from (68) and the first inequality (88). Bearing in mind that $f(t)$ is analylic in the strip $|Im\, t| \leq t_0$, $t_0 > k$ and vanishing at infinity in this strip more rapidly than $\exp(-r|t|)$ for any r, we conclude that $\widehat{f}(y)$ is an entire function of y satisfying the inequality

$$(90) \qquad |\widehat{f}(y)| < c_r \exp(-t_1 |y|)$$

for any $t_1 < t_0$, in every strip $|Im\, y| \leq r$.

Let us fix a number b_1 such that $1 < b_0 < t_0/h$, and choose t_1 such that $kb_0 < t_1 < t_0$; we then have, for every b such that $1 \leq b \leq b_0$, the inequalities $t_1/b \geq t_1/b_0 > k$. By (90) and the second inequality of (88), we have, taking $r = o$,

$$(91) \qquad |\widehat{K}(y)\widehat{f}(y/b)| < cc_0 c_1 \exp[-(t_1/b_0 - k)|y|],$$

uniformly for $1 \leq b \leq b_0$. Since the function on the right-hand side of (91) is integrable and independent of b, the integral (89) converges uniformly for $1 \leq b \leq b$.

Thus, by the relations

$$\lim_{b \to 1+} \widehat{K}(y) = 1\,; \qquad \lim_{b \to 1+} \widehat{f}(y/b) = \widehat{f}(y),$$

and by (89), follows

$$\lim_{b \to 1+} f(t, b) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iyt}\widehat{f}(y)dy = f(t).$$

COROLLARY 1. - *If $k = o$, the theorem holds for every function $f(t)$ of the real variable t satisfyng the conditions,*

(a) *the inequality* (68) *holds for real t,*

(b) *$f(t)$ is twice differentiable,*

(c) *$f''(t)$ belongs to $L(-\infty, \infty)$.*

PROOF. - From condition c follows that $f'(t) \to o$, $(t \to +\infty)$; by (68), $f(t) \to o$, $(t \to \pm\infty)$; hence, integrating the second equality (87) twice by parts one get $\widehat{f}(y) = 0\,(\,|\,y\,|^{-2})$; $(y \to \pm\infty)$.

From this and from (88) with $k = o$, one obtains, for all b in the interval $1 \leq b \leq 2$,

$$|\,\widehat{K}(y)\widehat{f}(y/b)\,| < C(1 + |\,y\,|)^{-2},$$

where C is independent of b. This last equality replaces (91), and the proof is now arrived at as in the theorem.

COROLLARY 2. - *The function $f(t, b)$ defined by* (82), *converges in the square mean to $f(t)$, on the real axis.*

PROOF. - From (89), one has

$$\int_{-\infty}^{\infty} e^{iyt}f(t, b)dt = \widehat{K}(y)\widehat{f}(y/b).$$

Hence, by PARSEVAL's equality, one can write

$$2\pi\int_{-\infty}^{\infty} |\,f(t, b) - f(t)\,|^2 dt = \int_{-\infty}^{\infty} |\,\widehat{K}(y)\widehat{f}(y/b) - \widehat{f}(y)\,|^2 dy.$$

By (90) and (91) the last integrand is dominated by an integrable function of y, independenl of b for all b in the interval $1 \leq b \leq b_0$. Since this integrand tends to zero as $b \to 1^+$, one obtains, by uniform convergence, that the last integral, as well as the foregoing, tend to zero as $b \to 1^+$.

As a further consequence of the proof of the Theorem 9 the completeness of the sequence $q_n(t)$ in $L_2(-\infty, \infty)$ can be demonstrated. For this aim, we give first a lemma of GELFAND and SILOV [4; p. 243], and because of the simplicity of the proof given by these authors, we reproduce it here.

LEMMA. - *The class* $\mathfrak{F}_{a,k}$, $(a > 1;\ k \geq o)$ *is dense in* $L_2(-\infty, \infty)$.

PROOF. - It is to be shown that to every $F(t)$ belonging to L_2 and to each $\varepsilon > o$, there exists a function $f(t)$ belonging to $\mathfrak{F}_{a,k}$. such that

$$\int_{-\infty}^{\infty} |f(t) - F(t)|^2 dt < \varepsilon. \tag{92}$$

For this, it is sufficient to show that the only functions $G(t)$ of L_2 which are orthogonal to each function $f(t)$ are functions which are zero almost everywhere. Suppose, in fact, that for all $f(t)$ of $\mathfrak{F}_{a,k}$ one has

$$\int_{-\infty}^{\infty} f(t)G(t)dt = o.$$

Consider

$$H(y) = \int_{-\infty}^{\infty} e^{iyt} f_0(t) G(t) dt ; \tag{93}$$

where $f_0(t)$ is one fixed function of $\mathfrak{F}_{a,k}$. Then all functions $t^n f_0(t)$ belong to $\mathfrak{F}_{a,k}$ and so we get

$$H^{(n)}(o) = \int_{-\infty}^{\infty} t^n f_0(t) G(t) dt = o, \qquad (n = 0,\ 1,\ 2, \ldots). \tag{94}$$

But by (69), and by SCHWARZ's inequality, the integral (93) converges uniformly in every strip $|Im\, y| \leq Y$, and so $H(y)$ is an entire function, which is identically zero, by (94).

Hence $f_5(t)G(t)$ vanishes almost everywhere, and the same is true for $G(t)$ since $f_0(t)$, as an analytic function not identically zero, vanishes in an enumerable set.

THEOREM 10. - *Let $q(t)$ be defined by* (19) *where $A(u)$ is an entire function satisfying inequalities* (66) *and* (67) *and such that $A^{-1}(iy) \in S$; then the sequence of derivatives of $q(t)$ is complete in L_2 $(-\infty, \infty)$.*

PROOF. - 1. By the lemma, to a given function $F(t)$ of L_2 and a given $\varepsilon > o$, there exists a function $f(t)$ of $\mathfrak{F}_{a,k}$ such that (92) holds. By corollary 2 of Theorem 9, there exists a number $b > 1$, such that,

$$\int_{-\infty}^{\infty} | f(t, b) - f(t) |^2 dt < \varepsilon. \tag{95}$$

2. We shall prove that the series $\Sigma\, d_n b^{-n} q_n(t)$, which is (B)-summable to $f(t, b)$, (this was the definition of this function, by (75)), converges in the square mean to $f(t, b)$ along the real axis, b being fixed $(b > 1)$.

For the partial sums of this series, one has, by (73),

$$\sum_{n=0}^{N} d_n b^n q_n(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iyt}}{A(iy)} dy \left\{ \sum_{n=0}^{N} d_n \left(\frac{iy}{b}\right)^n \right\}. \tag{96}$$

Formally, one has, by (69) and (87)

$$\sum_{n=0}^{\infty} d_n \left(\frac{iy}{b}\right)^n = \int_{-\infty}^{\infty} f(s) \left\{ \sum_{n=0}^{\infty} p_n(s) \left(\frac{iy}{b}\right)^n \right\} ds = A(iy/b)\widehat{f}(y/b). \tag{97}$$

Now, by (76), with $R > | y |$, y being fixed, one has

$$\sum_{n=0}^{\infty} \left| p_n(s) \left(\frac{iy}{b}\right)^n \right| < \frac{bRMe^{R|s|}}{bR - |y|}.$$

By (68), the integral of (97) converges absolutely when modulus of each term and factor is taken, and so one may invert integration and summation, so that the series at the left hand side of (97) converges for every y, and equals the right-hand member. By the equality $A(iy/b)A^{-1}(iy) = \widehat{K}(y)$, one, obtains, from (97)

$$A^{-1}(iy) \sum_{n=0}^{\infty} d_n (iy/b)^n = \widehat{K}(y)\widehat{f}(y/b).$$

Hence, by (89)

$$f(t,\ b)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-iyt}}{A(iy)}\,dy\left\{\sum_{n=0}^{\infty}d_n\left(\frac{iy}{b}\right)^n\right\}. \tag{98}$$

By (96), (98) and PARSEVAL's equality, we obtain

$$2\pi\int_{-\infty}^{\infty}\left|f(t,\ b)-\sum_{n=0}^{N}d_n b^{-n}q_n(t)\right|^2dt=\int_{-\infty}^{\infty}|A(iy)|^{-2}\left|\sum_{n=N+1}^{\infty}d_n\left(\frac{iy}{b}\right)^n\right|^2dy. \tag{99}$$

We shall now estimate the last series with respect to N. By (69)

$$\sum_{n=N+1}^{\infty}d_n\left(\frac{iy}{b}\right)^n=\int_{-\infty}^{\infty}f(s)\left\{\sum_{n=N+1}^{\infty}p_n(s)\left(\frac{iy}{b}\right)^n\right\}ds, \tag{100}$$

the proof being the same as for (97). By (75),

$$\sum_{n=N+1}^{\infty}p_n(s)\left(\frac{iy}{b}\right)^n=\frac{1}{2\pi i}\oint_{|u|=R}\frac{A(u)e^{us}}{u-iy/b}\left(\frac{iy}{bu}\right)^{N+1}du. \tag{101}$$

Let us fix a number b_1 such that $1<b_1<b$; and let us put $R=b_1^{-1}|y|$ for $|y|>1$. Then (101) and (66) yield the approximation

$$\left|\sum_{n=N+1}^{\infty}p_n(s)\left(\frac{iy}{b}\right)^n\right|<C\frac{(b_1/b)^{N+1}}{|y|(b_1^{-1}-b^{-1})}\exp\left(h\left|\frac{y}{b_1}\right|^a+k\left|\frac{y}{b_1}\right|+\frac{|sy|}{b_1}\right). \tag{102}$$

By YOUNG's inequality. $|sy|\leq|y|^{a_1}/a_1+|s|^{a_1'}/a_1'$, where $1<a_1<a$; $1<a'<a_1'<a_0$, (see the first part of the proof of theorem 9). By this last inequality and by (102) and (68) there follows the approximation:

$$\left|\sum_{n=N+1}^{\infty}d_n\left(\frac{iy}{b}\right)^n\right|\leq C_1|y|^{-1}\left(\frac{b_1}{b}\right)^{N+1}\exp\left(h\left|\frac{y}{b_1}\right|^a+\frac{|y|^{a_1}}{a_1b_1}+k\frac{|y|}{b_1}\right), \tag{103}$$

for $|y|>1$.

For $|y|\leq 1$, we choose, in (101), $R=1$, and this yields

$$\left|\sum_{n=N+1}^{\infty}p_n(s)\left(\frac{iy}{b}\right)^n\right|<C_2\frac{b^{-N-1}}{1-|y|b^{-1}}e^{|s|},$$

and, substituting this inequality in (100), one has

$$\left|\sum_{n=N+1}^{\infty}d_n\left(\frac{iy}{b}\right)^n\right|<C_3\frac{b^{-N-1}}{1-|y|b^{-1}}\qquad(|y|\leq 1). \tag{104}$$

Since $1 < b_1 < b$, we see, from (103) and (104), that the right-hand integral of (99) tends to zero as $N \to \infty$. This proves our assertion on the series $\Sigma\, d_n b^{-n} q_n(t)$. Having fixed $b > 1$ such that (95) holds, we determine N such that,

$$\int_{-\infty}^{\infty} \left| f(t,\, b) - \sum_{n=0}^{N} d_n b^{-n} q_n(t) \right|^2 dt < \varepsilon. \tag{105}$$

3. Considering the integral

$$\int_{-\infty}^{\infty} \left| F(t) - \sum_{n=0}^{N} d_n b^{-n} q_n(t) \right|^2 dt ;$$

Substituting in the integrand the functions $\pm f(t)$ and $\pm f(t,\, b)$ and using SCHARWZ's ineqnality and (92), (95), (105) we easily find that this integral is less than 9ε. This completes the proof of the theorem.

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