A topological Proof of the Bott Periodicity Theorems.

Memoria di E. DYER and R. LASHOF (1) (a Chicago in U.S.A.).

To Enrico Bompiani on his scientific Jubilee.

Summary. - *A proof is given of the* BOTT *periodicity theorems using only well known techniques of algebrdbc topology.*

0. Introduction.

In this paper we give a proof of the BOTT periodicity theorems [4] for the infinite classical groups using only well known techniques of algebraic topology. Whereas there is some overlap with the proof given by MOORE in the CARTAN Seminar [7], the algebraic techniques are entirely different. MOORE uses homological algebra methods in dealing with spectral sequences of universal bundles. In this proof the main argument consists in showing that the BOTT maps [5] induce isomorphisms in integral homology, and this is done by showing that they induce isomorphisms mod p, p or odd prime, and mod 2. For the mod p proof (section 3) all that are used are some commuting topological diagrams (also used by MOORE) and the fact that certain fibrations may be considered as product spaces as far as mod p homology is concerned. (This last was pointed out to us by BRUNO HARRIS). The mod p result is a trivial consequence of these facts (in particular no spectral sequence erguments are needed). For the mod 2 result (section 4) (besidest he commuting diagrams referred to above), a short spectral sequence argument is used in the case of each map, based on STEENROD squaresf or the cohomology sequences and on the analogous mod 2 homology operations of ARAKI and KUDO for the homology sequences.

The proof for the unitary group (section 2) was developed in the SUMMER of 1959 at the University of Chicago and presented in a course in the FALL of 1959. The essential argument is due to R. SWAN. (We understand that MOORE's proof is very similar-but we have not seen it).

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Section 1 is devoted to the definition of the BOTT maps and their elementary properties.

1. The Bott maps (see [5]).

Let $W_{2n} = W_n \oplus W_n$ be the $2n$-dimensional quaternionic vector space with right and left symplectic inner products. Let $Sp(2n)$ be the group of symplectic transformations of W_{2n} commuting with *right* multiplication by the quaternions H. Let $(1, i, j, k)$ be the usual generators of H over the real field R. Let $C \subset H$ be the complex subfield generated by $(1, i)$. By this means we may consider W_{2n} to be a $4n$-dimensional complex vector space under rigt multiplication by C, and an $8n$-dimensional real vector space. The complex part of the right symplectic inner product becomes an hermetian inner product, and the real part of the symplectic inner product becomes an orthogonal inner product. From this structure we get the inclusions

$$Sp(n) \times Sp(n) \subset Sp(2n) \subset U(4n) \subset SO(8n), \tag{1.1}$$

where $U(4n)$ is the group of unitary transformations and $SO(8n)$ the group of special orthogonal transformations. We note that $U(4n)$ and $Sp(2n)$ are characterized as those subgroups of $SO(8n)$ acting on W_{2n} which commute with right multiplication by C and H respectively.

Now consider in the above sequence of groups, the respective subgroups that commute with *left* multiplication by H. These groups form the sequence:

$$O(n) \times O(n) \subset O(2n) \subset U(2n) \subset Sp(2n). \tag{1.2}$$

Here $Sp(2n)$ is the group of symplectic transformations with respect to the left iuner product. Further there are subspaces $R_{2n} \subset C_{2n} \subset W_{2n}$, for which the inner product becomes orthogonal and hermetian respectively. $O(2n)$ and $U(2n)$ are the subgroups of $Sp(2n)$ leaving R_{2n} and C_{2n} invariant.

Explicitly: $C_{2n} = \{w \in W_{2n} / w = iwi^{-1}\}$, $R_{2n} = \{w \in C_{2n} / w = jwj^{-1}\}$.

The inclusion of (1.2) in (1.1) induces the inclusions:

$$\Gamma_n(R) \subset \Gamma_n(H), \quad U(2n)/O(2n) \subset U(4n)/Sp(2n), \tag{1.3}$$

$$Sp(2n)/U(2n) \subset SO(8n)/U(4n), \quad Sp(2n) \subset SO(8n),$$

where $\Gamma_n(R) = O(2n)/O(n) \times O(n)$ and $\Gamma_n(H) = Sp(2n)/Sp(n) \times Sp(n)$.

The BOTT maps are as follows (where Ω_ν is the space of paths from the base point of the space in question to another fixed point as described below):

$$\varphi_0: U(4n)/U(2n) \times U(2n) \to \Omega_\nu(U(4n))$$

$$\varphi_1: \Gamma_n(H) \to \Omega_\nu(U(4n)/Sp(2n)) \qquad \varphi_4: \Gamma_n(R) \to \Omega_\nu(U(2n)/O(2n))$$

$$\varphi_2: U(4n)/Sp(2n) \to \Omega_\nu(SO(8n)/U(4n)) \qquad \varphi_5: U(2n)/O(2n) \to \Omega_\nu(Sp(2n)/U(2n))$$

$$\varphi_3: SO(8n)/U(4n) \to \Omega_\nu(SO(8n)) \qquad \varphi_6: Sp(2n)/U(2n) \to \Omega_\nu(Sp(2n)).$$

To define φ_i, i from O to 6, write G_i and K_i for the groups in the numerator of the domain and the numerator (inside the brackets) of the range of φ_i resp.; then $G_i \subset K_i$ by (1.1) and (1.2). Let $\bar{G}_i$ and $\bar{K}_i$ be the respective quotient spaces. We first define a map ξ_i: $E\bar{G}_i \to \bar{K}_i$ of the two point suspension of $\bar{G}_i$ into $\bar{K}_i$; then φ_i: $\bar{G}_i \to \Omega_\nu \bar{K}_i$ will be the naturally associated map of $\bar{G}_i$ into the space of paths in $\bar{K}_i$ going from the image under ξ_i of the south pole to the image of the north pole; i.e. $\varphi_i(\bar{T})(\theta) = \xi_i(\bar{T}, \theta)$, $\bar{T} \in \bar{G}_i$, θ the suspension parameter. To define ξ_i, let $T \in G_i \subset K_i$, then ξ_i is the map induced from

$$\text{(1.4)} \qquad \hat{\xi}_i: G_i \times [0, \pi] \to K_i, \ \hat{\xi}_i(T, \theta) = T\alpha_i(\theta)T^{-1},$$

by passage to the quotients, where for $(x_1, x_2) \in W_n \oplus W_n$

$$\alpha_0(\theta)(x_1, x_2) = (x_1 e^{i\theta}, x_2 e^{-i\theta})$$

$$\alpha_1(\theta)(x_1, x_2) = \alpha_4(\theta)(x_1, x_2) = (x_1 e^{i\theta/2}, x_2 e^{-i\theta/2})$$

$$\alpha_2(\theta)(x_1, x_2) = \alpha_5(\vartheta)(x_1, \dot{x}_2) = (x_1 e^{j\theta/2}, x_2 e^{j\theta/2})$$

$$\alpha_3(\theta)(x_1, x_2) = \alpha_6(\theta)(x_1, x_2) = (x_1 e^{i\theta}, x_2 e^{i\theta}).$$

It follows from the above, that φ_4, φ_5, φ_6 are obtained from φ_1, φ_2, φ_3 respectively, by restriction. (See (1.3). (We also note in this connection that $\alpha_i(\theta)$ commutes with left multiplication by H).

As is well known, Ω_ν is of the same homology type as the ordinary loop space Ω (the spaces in question being connected). We may obtain a map $\emptyset_i$: $\bar{G}_i \to \bar{K}_i$ which will be equivalent to (1.4) under such a homotopy equivalence (in fact a homeomorphism) by setting $\emptyset_i(T)(\theta) = \psi_i(\bar{T}, \theta)$, where

$$\text{(1.5)} \qquad \hat{\psi}_i(T, \theta) = T\alpha_i(\theta)T^{-1}\alpha_i(\theta)^{-1}.$$

On the other hand, such a homotopy equivalence may be obtained by adding any path from $\alpha_i(\pi)$ to $\alpha_i(O)$ to the paths in Ω_ν. In particular, if we add the path $\alpha_i(\pi - \theta)$, the resultant map

$$\text{(1.5)}' \qquad \psi_i'(\bar{T}, \theta) = \begin{matrix} \xi_i(\bar{T}, 2\theta), & O \leq \theta \leq \pi/2 \\ \alpha_i(2\pi - 2\theta), & \pi/2 \leq \theta \leq \pi \end{matrix}$$

defines a map Φ_i': $\bar{G}_i \to \Omega\bar{K}_i$ homotopic to $\emptyset_i$. In fact $\Phi_i'\beta_i$

$$\beta_i(\theta) = \begin{cases} \alpha_i(2\theta)^{-1}, & 0 \leq \theta \leq \pi/2 \\ \alpha_i(2\pi - 2\theta)^{-1}, & \pi/2 \leq \theta \leq \pi \end{cases}, \text{ is essentially } \Phi_i;$$

but β_i is homotopic to the trivial loop. Although (1.5) is simpler, (1.5)$'$ has the advantage that the corrrespondence with Ω_ν is natural under maps of $\bar{K}_i$ into $\bar{K}_j$, which take $\Omega_\nu\bar{K}_i$ into $\Omega_\nu\bar{K}_j$. We also note that in each case $\emptyset_i$, and hence $\emptyset_i'$ map $\bar{G}_i$ into the arc componeot of the trivial loop; and in the rest of the paper ΩX will mean the connected component of the trivial loop.

Properties of the maps $\emptyset_i$.

Choose a fixed ordered set of basis vectors b_i, $i = 1, 2, 3, \ldots$, in the countably infinite quaternionic vector space W_∞, and let W_n be the subspace spanned by the first n basis vectors. This defines inclusions of W_n in W_{n+1}, $W_n \oplus W_n$ in $W_{n+1} \oplus W_{n+1}$, and hence of all the groups and homogeneous spaces for n in the corresponding ones for $n+1$. It is clear from (1.5) that $\emptyset_i$ commutes with the inclusions, and defines a map on the direct limit. Writing $Sp = \varinjlim Sp(n)$, etc.; we get:

$$\begin{array}{ll} \Phi_0\colon B_U \to \Omega U & \\ \Phi_1\colon B_{Sp} \to \Omega(U/Sp) & \Phi_4\colon B_O \to \Omega(U/O) \\ \Phi_2\colon U/Sp \to \Omega(SO/U) & \Phi_5\colon U/O \to \Omega(Sp/U) \\ {}_3\colon SO/U \to \Omega SO & \Phi_6;\ Sp/U \to \Omega Sp. \end{array}$$

The proof of the Bott periodicity theorems then amounts to showing that all the maps Φ_i above are weak homotopy equivalences; i.e., induce isomorphisms on the homotopy groups.

Further we claim we may give all the homogeneous spaces above an H-space structure; i.e. a multiplication which is homotopy associative and has a homotopy unit. To do this map W_∞ back into itself on the one hand by sending b_i to b_{2i-1} and on the other hand by sending b_i to b_{2i}. This defines a map: $W_\infty \oplus W_\infty \to W_\infty$; which in turn defines a multiplication in all the groups and homogeneous spaces involved. The desired properties are easily checked: The only thing involved is a permutation of pairs of coordinates. Since in each case the permutation matrix is homotopic to the identity (the groups being connected), conjugation by the permutation matrix is homotopic to the identity. The homotopies for each pair of coordinates are performed in succession (in half the remaining time). Since only a finite number of coordinates is involved for any given element of the direct limit, this is well defined. Also by the same argument one sees that all these H-space structures are homotopy commutative. Consequently, the PONTRJAGIN homology [2] rings are commutative, associative rings with unit.

Now for the loop space of an H-space, the addition of loops is homotopic to the multiplication on the loop space induced by the multiplication in the underlying H-space. Likewise the direct limit multiplication in the infinite groups is homotopic to the multiplication defined above - again by a permutation of coordinates argument. Thus in both cases, the PONTRJAGIN rings are the same for the two multiplications.

Finally, we note by (1.5) that the maps $\emptyset_i$ are H-maps with respect to the above multiplication.

The fact that the maps are H-maps enables us to use a refined form of the Whitehead theorem:

THEOREM 1.6 - Let $f: X \to Y$ be a map of connected topological spaces; if $f_*: \pi_i(X) \simeq \pi_i(Y)$ all i, then $f_*: H_i(X; Z) \simeq H_i(Y, Z)$ all i. Conversely, if f is an H-map of H-spaces and $f_*: H_i(X; Z) \simeq H_i(Y; Z)$ all i, then $f_*: \pi_i(X) \simeq \pi_i(Y)$ all i.

PROOF. - Let C be the mapping cylinder of f, then X may be considered as a subspace of C, and Y is a deformation retract of C. Also f is the inclusion map of X into C followed by the retraction onto Y. The first part of the theorem follows from the relative HUREWICZ theorem (see theorem, p. 166 of [10]) and the homotopy and homology sequences of the pair (C, X). To prove the converse, note that we may define an action of X on C by $(x, t)x' = (xx', t) \in C$, $0 \leq t < 1$, and $(y)x' = (yf(x'))$, since f is an H-map. It follows that $\pi_1(X)$ acts trivially on $\pi_i(C, X)$, $i \geq 2$, by a standard argument. On the other hand, $\pi_1(X)$ and $\pi_1(Y)$ are abelian (since they are H-spaces), and by the naturality of the HUREWICZ homomorphism $\pi_1(X)$ maps isomorphically onto $\pi_1(C) = \pi_1(Y)$, and hence $\pi_1(C, X) = 0$. The result now follows by the HUREWICZ theorem referred to above.

REMARK. - We note the above proof still applies if the hypothesis of the converse is weakened to read: If X is an H-space operating on Y such that f commutes with the action of X (acting on itself by right translation), $\pi_1(Y)$ is abelian, and $f_*: H_i(X; Z) \simeq H_i(Y; Z)$ all i, then $f_*: \pi_i(X) \simeq \pi_i(Y)$ all i.

Finally, a trivial application of the WHITEHEAD mapping cylinder and the universal coefficient theorem gives:

THEOREM 1.7. - Let X and Y be topological spaces with $H_i(X)$ and $H_i(Y)$ finitely generated all i. A map $f: X \to Y$ induces isomorphisms on

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(2) We use singular homology theory throughout this paper.

the integral homology groups if and only if f induces isomorphisms on the mod p homology groups, p an odd prime, and on the mod 2 homology groups (3).

2; The Unitary Group.

For purposes of this section, it is unnecessary to assume an underlying quaternionic vector space, so to simplify notation we merely assume that $U(2n)$ is acting on $C_{2n} = C_n \oplus C_n$ and $\Phi_0\colon U(2n)/U(n) \times U(n) \to \Omega\, U(2n)$, where $\widehat{\psi}_0(T, \theta) = T\alpha_0(\theta)T^{-1}\alpha_0(\theta)^{-1}$ and $\alpha_0(\theta)(x_1, x_2) = (x_1 e^{i\theta}, x_2 e^{-i\theta})$, $(x_1, x_2) \in C_n \oplus C_n$.

Now let $j\colon U(n+1) \to U(2n)$, where we consider $U(n+1)$ acting on $C_n \oplus C_1$ and $C_n \oplus C_n = C_n \oplus C_1 \oplus C_{n-1}$; i.e. $j(T) = (T, I_{n-1})$, $T \in U(n+1)$. Then Φ_0 carries $jU(n+1)$ into the image of $\Omega U(n+1)$ in $\Omega U(2n)$ under Ωj (Ωj the map on the loop spaces induced by j); i.e.

$$\begin{aligned}\widehat{\psi}_0(jT, \theta) &= (T, I_{n-1})((e_r^{i\theta}, e_r^{-i\theta}), e_r^{-i\theta})(T^{-1}, I_{n-1}^{-1})((e_r^{i\theta}, e_r^{-i\theta}), e_r^{-i\theta})^{-1}\\ &= (T(e_r^{i\theta}\cdot e_r^{-i\theta})T^{-1}(e_r^{-i\theta}\cdot e_r^{i\theta}), I_{n-1}) \in jU(n+1) \subset U(2n),\end{aligned}$$

where $e_r^{i\theta}$ means right multiplication by $e^{i\theta}$ in the given subspace. consequently, we may define Φ_0: $\dfrac{U(n+1)}{U(n) \times U(1)} \to \Omega\, U(n+1)$ by the formula

$$\widehat{\psi}_0(T, \theta) = T\alpha_0(\theta)T^{-1}\alpha_0(\theta)^{-1},\ \alpha_0(\theta)(x_1, x_2) = (x_1 e^{i\theta}, x_2 e^{-i\theta}),$$

$(x_1, x_2) \in C_n \oplus C_1$; and get the commutative diagram

$$\Delta_0 \qquad \begin{array}{ccc} U(n+1)/U(n) \times U(1) & \xrightarrow{\Phi_0} & \Omega\, U(n+1) \\ \downarrow j & & \downarrow \Omega j \\ U(2n)/U(n) \times U(n) & \xrightarrow{\Phi_0} & \Omega\, U(2n). \end{array}$$

Now $U(n+1)/U(n) \times U(1)$ is just complex projective space $CP(n)$, and we write $CP = \operatorname{Lim} CP(n)$; then $\Phi_0\colon CP \to \Omega\, U$. We wish to prove that Φ_{0*} maps $H_*(CP; Z)$ monomorphically into $H_*(\Omega\, U; Z)$, and that the image generates the Pontrjagin ring of $\Omega\, U$. It is well known that $j_*(H_*(CP; Z))$ generates the Pontrjagin ring of $H_*(B_U; Z)$ (see Prop. 2.6), and hence the fact that $\Phi_{0*}\colon H_*(B_U; Z) \simeq H_*(\Omega\, U; Z)$ will follow from the commutativity of Δ_0.

Lemma 2.1. Let $\pi\colon U(n+1) \to U(n+1)/U(n) = S_{2n+1}$ be the natural projection. The map $\pi\psi_0\colon (ECP(n), ECP(n-1)) \to (U(n+1), U(n)) \to (S_{2n+1}, p)$ induces $\pi_*\psi_*\colon H_*(ECP(n), ECP(n-1); Z) \simeq H_*((S_{2n+1}, p); Z)$.

Proof. - $\psi_0(\overline{T}, \theta)$ is homotopic to $\psi(\overline{T}, \theta) = \psi_0(\overline{T}, \theta)(I_n, e_r^{-2i\theta})$. (For $0 \le s \le 1$, set $F(s, \overline{T}, \theta) = \psi_0(\overline{T}, \theta)(I_n, e_r^{-2is\theta})$.) It follows that $\psi(\overline{T}, \theta)$ is defined

(3) In this paper, coefficients mod p will mean with respect to and *odd* prime p.

by $\widehat{\psi}(T, \theta) = \widehat{\psi}_0(T, \theta)(I_n, e_r^{2i\theta})$ by passage to the quotient. Explicitly $\widehat{\psi}(T, \theta) = T(e_r^{i\theta}, e_r^{-i\theta})T^{-1}(e_r^{-i\theta}, e_r^{i\theta})(I_n, e_r^{-2i\theta}) = T(e^{i\theta}, e_r^{-i\theta})T^{-1}(e_r^{-i\theta}, e_r^{-i\theta}) = T(I_n, e_r^{-2i\theta})T^{-1}$.

Now $\pi\psi(T, \theta) = \psi(T, \theta)\varepsilon$, $\varepsilon \in C_n \subset C_n \oplus C_1$, the unit vector left invariant by $U(n)$. Let γ be the unit vector in C_1, then $\varepsilon = T(\gamma)c + (\varepsilon - T(\gamma)c)$, $c = [\varepsilon, T(\gamma)]$ (hermetian inner product) and $T(\varepsilon) \perp (\varepsilon - T(\varepsilon)c)$. Hence

$$\pi\widehat{\psi}(T, \theta) = T(\gamma)ce^{-2i\theta} + (\varepsilon - T(\gamma)c) = \varepsilon + T(\gamma)c(e^{-2i\theta} - 1).$$

For $\theta \neq 0, \pi$; $\widehat{\psi}(T, \theta)\varepsilon = \varepsilon \leftrightarrow T(\gamma) \perp \varepsilon$.

On the other hand, for any unit vector $u \neq \varepsilon$

$$u \doteq \varepsilon + u - \varepsilon = \varepsilon + \frac{u - \varepsilon}{\| u - \varepsilon \|} \, \frac{[\varepsilon, u] - 1}{\| u - \varepsilon \|} \, \frac{\| u - \varepsilon \|^2}{[\varepsilon, u] - 1} = \varepsilon + v[\varepsilon, v](e^{-2i\theta} - 1),$$

where $v = \dfrac{u - \varepsilon}{\| u - \varepsilon \|}$ and $e^{-2i\theta} = -\dfrac{\overline{[\varepsilon, u] - 1}}{[\varepsilon, u] - 1}$ (since $\| u - \varepsilon \|^2 = [u - \varepsilon, u - \varepsilon] = 2 - [\varepsilon, u] - [\varepsilon, u[= 1 - [\varepsilon, u] + \overline{1 - [\varepsilon, u]}$). From this one sees easily that $\pi\psi$ is a relative homeomorphism, and the result follows.

Let $G \to E \xrightarrow{\pi} S_K$ be a principal fibre bundle over a K-sphere. Let D_K be the unit disc in Euclidean K-space with boundary S_{K-1}. Let f: $(D_K, S_{K-1}) \to (S_K. p)$, p a base point, be a map such that f_*: $H_*((D_K, S_{K-1}); Z) \simeq H_*((S_K, p); Z)$. We may lift f to a map g: $(D_K, S_{K-1}) \to (E, G)$, since D_K is contractible to a point. The class $\alpha \in \pi_{K-1}(G)$ of the map g/S_{K-1} is the characteristic homotopy class, and its HUREWICZ image $a \in H_{K-1}(G, Z)$ is called the characteristic homology class of the bundle.

LEMMA 2.3. - The WANG sequence

$$\to H_i(G; Z) \xrightarrow{i_*} H_i(E; Z) \xrightarrow{\lambda_*} H_{i-K}(G; Z) \xrightarrow{\sigma_*} H_{i-1}(G; Z) \to \ldots$$

is an exact sequence of $H_*(G; Z)$ modules and σ_* is left multiplication by the characteristic class a.

If $a = 0$, then $H_*(E; Z)$ is the free $H_*(G, Z)$ module with generatros 1 and any class $x_K \in H_K(E, Z)$, such that $\pi_*(x_K)$ is a basis element of $H_K(S_K; Z)$.

PROOF. - Let u: $(E, G) \times G \to (E, G)$ be right action on both factors. Then $h = u \circ (g \times 1)$: $(D_K, S_{K-1}) \times G \to (E, G) \times G \to (E, G)$ is a relative homeomorphism. It is easy to see (even if the spaces are not compact) that h_*: $H_*((D_K, S_{K-1}) \times G; Z) \simeq H_*((E, G); Z)$. Starting from the exact sequence of the pair we have:

$$\begin{array}{ccccccc}
\to H_i(G; Z) & \xrightarrow{i_*} & H_i(E; Z) & \xrightarrow{K_*} & H_i(E, G); Z) & \xrightarrow{\partial_*} & H_{i-1}(G; Z) \to \\
 & & & & \simeq \uparrow h_* & & h_* \uparrow \\
 & & & & H_i((D_K, S_{K-1}) \times G; Z) & \to & H_{i-1}(S_{K-1} \times G; Z) \\
 & & & & \simeq \updownarrow & & \\
 & & & & H_{i-K}(G; Z) & &
\end{array}$$

From this the WANG sequence and its properties listed above follow immediately.

REMARK. - The result also holds with a field of coefficients. For further details see [9].

Now for any ring R, let $\Lambda_R(x_{i_1}, x_{i_2}, \ldots)$ be the graded exterior ring over R with generators of dimension $i_1, i_2, \ldots$; and let $R[x_{i_1}, x_{i_2}, \ldots]$ be the graded polynomial ring over R with generators of dimension $i_1, i_2, \ldots$.

PROPOSITION 2.4. - $H_*(U(n+1); Z) \simeq \Lambda_Z(x_1, x_3, \ldots, x_{2n+1})$, where the x_i's, $i \neq 1$, are the images of an additive basis of $H_*(ECP(n); Z)$ under ψ_{0*}.

PROOF. - Let ρ: $(D_{2n+1}, S_{2n}) \to (ECP(n), ECP(n-1))$ be the map which attaches the top dimensional cell of $ECP(n)$. By Lemma 2.1, $\pi_*\psi_{0*}\rho_*$: $H_*((D_{2n+1}, S_{2n}); Z) \to H_*((S_{2n+1}, p); Z)$, and hence $\psi_0\rho$: $S_{2n} \to ECP(n-1) \to U(n)$ defines the characteristic class. Since $H_{2n}(ECP(n-1); Z) = 0$, $a = 0$. By 2.3 it follows that $H_*(U(n); Z)$ maps monomorphically into $H_*(U(n+1); Z)$. Since (see section 1) $H_*(U: Z)$ is commutative, $H_*(U(n); Z)$ is commutative, all n.

The result is trivial for $n = 0$, assume the result for $U(n)$; i.e., $H_*(U(n); Z) \simeq \Lambda_Z(x_1, x_3, \ldots, x_{2n-1})$. By the above paragraph and 2.3, it follows that $H_*(U(n+1); Z)$ is the free $H_*(U(n); Z)$ module with generators 1 and the class x_{2n+1}, image of the top dimensional class of $ECP(n)$ under ψ_*. Since $H_*(U(n+1); Z)$ is commutative by the above paragraph, $x_{2n+1}^2 = 0$ and $H_*(U(n+1); Z) \simeq \Lambda_Z(x_1, x_3, \ldots, x_{2n+1})$.

As $\psi_0(ECP(n)) \subset SU(n+1)$, we may conclude by exactly the same argument that:

PROPOSITION 2.5. - $H_*(SU(n+1); Z) \simeq \Lambda_Z(x_3, x_5, \ldots, x_{2n+1})$, the x_i's being the images of the additive basis of $H_*(ECP(n); Z)$.

COROLLARY 2.6. - $H_*(\Omega SU(n+1); Z) \simeq H_*(\Omega U(n+1); Z) \simeq Z[d_2, d_4, \ldots, d_{2n}]$, and the generators are the images of the additive basis of $H_*(CP(n); Z)$ under Φ_0.

PROOF. - The simply connected covering space $\tilde{U}(n+1)$ of $U(n+1)$ is homeomorphic to $R \times SU(n+1)$. Hence $H_*(\tilde{U}(n+1); Z) \simeq H_*(SU(n+1); Z)$ is transgressively generated by Prop. 2.5. By the theorem stated below, the result follows.

THEOREM 2.7. - Let X be an H-space such that $H_*(X; K)$ is a transgressively generated exterior algebra on odd generators, K a field or the integers. Then $H_*(\Omega X; K)$ is a polynomial algebra generated by their transgressions.

The proof is by a standard application of the comparison theorem for spectral sequences (see [8]). Actually, one does not need to assume $H_*(X; K)$

is transgressively generated as this follows by a slightly more difficult argument (see [8]). This theorem may be looked at as the dual of BOREL's transgression theoıem for groups whose cohomology algebra is exterior.

THEOREM 2.8. - $\Phi_{0*}\colon H_*(B_U; Z) \simeq H_*(\Omega U; Z)$.

PROOF. - From (2.6) it follows by taking the direct limit, that $H_*(\Omega U; Z) \simeq Z[d_2, d_4, .., d_{2i}, ...]$ and the generators are the images of the additive basis of $H_*(CP; Z)$ under Φ_0. Since $H_*(B_U; Z) \simeq Z[Z_2, Z_4, ..., Z_{2i}, ...]$ and the generators are the images of the additive basis of $H_*(CP; Z)$ under J_* (see Prop, 2.9 below). it follows from Δ_0 that $\Phi_{0*}\colon H_*(BU; Z) \simeq H_*(\Omega U; Z)$. (note that although $J\colon U(n+1) \to U(2n)$ does not induce the identity map in the limit, it nevertheless induces the identity map in homology in the limit, since $J_*\colon H_i(U(n+1) \simeq H_i(U(2n))$, $i < 2n+2$; and hence ΩJ_* is the identity on homology).

The following is essentially a restatement of the WHITNEY sum theorem for CHERN classes.

PROPOSITION 2.9. - $H^*(B_U; Z) \simeq Z[C_2, C_4, ..., C_{2i}, ...]$, C_{2i} the $2i$-dim CHERN class, wiʒh codiagonal map $\mu^* C_{2i} = \sum_{j+k=i} C_{2j} \otimes C_{2k}$. Dually, it follows that $H\ (B_U; Z) = Z[Z_2, Z_4, ..., Z_{2i}, ...]$, with diagonal map $d_* Z_{2i} = \sum_{j+k=i} Z_{2j} \otimes Z_{2k}$, where $Z_{2i} = \bar{C}_2^i$, $\bar{C}_2^i$ the dual class to C_2^i for the additive basis of $H^*(B_U; Z)$ consisting of the monomials in the C_{2i}.

Further, $H^*(CP; Z) = H^*(B_{U(1)}; Z) = Z[b_2]$, and the inclusion map $J\colon B_{U(1)} \to B_U$ indudes $J^*(C_{2i}) = 0$, $i \neq 1$, $J^*(C_2) = b_2$, and consequenlty, $J_*(\bar{b}_2^i) = Z_{2i}$; i.e., J^* maps the additive basis of $H_*(CP; Z)$ onto the generators of the PONTRJAGIN ring $H_*(B_U; Z)$.

Finally, the primitive subspace of $H_*(B_U; Z)$ is the free module over Z with basis the dual classes to the CHERN classes C_{2i}. The primitive basis elements p_{2i} being given by the formula

$$p_{2i} - p_{2(i-1)} \cdot Z_2 + p_{2(i-2)} \cdot Z_4 - ... \pm iZ_{2i} = 0.$$

3. Orthogonal and symplectic groups mod p homology.

The inclusion $\rho\colon O(4n) \to U(4n)$. We wish to extend the sequence of inclusions (1.1) one step further to $U(8n)$. To simplify notation, however, we halve the dimension. We will define a map ρ mapping as follows:

$$Sp(n) \subset U(2n)' \subset O(4n) \xrightarrow{\rho} Sp(n) \times Sp(n) \subset Sp(2n) \subset U(4n)$$

$$O(n) \subset U(n)' \subset Sp(n) \xrightarrow{\rho} O(n) \times O(n) \subset O(2n) \subset U(2n).$$

($U(2n)'$ is the subgroup of $O(4n)$ which commutes with right muliplication by j instead of i. Similarly for $U(n)'$).

To define ρ, first consider $(x, x') \in W_n \oplus W_n$ with the following structure:

$$(x, x')i = (-x', x), \qquad (x, x')j = (xj, -x'j).$$

Let $O(4n)$ act on W_n considered as R_{4n}. For $T \in O(4n)$, set $\psi(T) = (T, T)$. Then $\psi(T)$ commutes with the action of i; i.e. $\psi(T)$ belongs to the group $U(4n)$ associated with this structure.

We wish to convert this structure into the usual structure on $W_n \oplus W_n$; i.e.

$$(x, x')i = (xi, x'i), \qquad (x, x')j = (xj, x'j).$$

To do this set:

$$\tau(x, x') = \left(\frac{x + x'i}{2}, \frac{x' + xi}{2}\right)$$

$$\sigma_j(x, x') = (x, x'j), \qquad \sigma_i(x, x') = (x, x'i).$$

Then

$$\sigma_i\sigma_j\tau i_r\tau^{-1}\sigma_j^{-1}\sigma_i^{-1}(x, x') = (xi, x'i)$$

$$\sigma_i\sigma_j\tau j_r\tau^{-1}\sigma_j^{-1}\sigma_i^{-1}(x, x') = (xj\ \ x'j).$$

Hence define $\rho(T) = \sigma_i\sigma_j\tau\psi(T)\tau^{-1}\sigma_j^{-1}\sigma_i^{-1}$.

Besides the properties listed above, ρ has the following properties which we leave to the reader to check.

1. ρ: $O(4n) \to U(4n)$ is equivalent (under an isomosphirm of $U(4n)$) to the inclusion of $O(2n)$ in $U(2n)$ of (1.2) (in twice the dimension).

2. ρ: $Sp(n) \to U(2n)$ is equivalent to the inclusion of $Sp(2n)$ in $U(4n)$ of (1.1).

3. ρ: $U(2n)' \to Sp(2n)$ is equivalent to the inclusion of $U(2n)$ in $Sp(2n)$ of (1.2).

4. ρ: $U(n)' \to O(2n)$ is equivalent to the inclusion of $U(4n)$ in $O(8n)$ of (1.1).

5. Taking the usual $U(2n) \subset O(4n)$ we have [4] for $T \in U(2n)$, $\rho(T) = (T, j_rTj_r^{-1})$. We now consider some fibrations with mod p cross-sections.

(1) $\quad Sp(2n) \xrightarrow{i_1} U(4n) \xrightarrow{p_1} U(4n)/Sp(2n)$

(2) $\quad SO(2n) \xrightarrow{i_2} U(2n) \xrightarrow{p_2} U(2n)/SO(2n)$

(3) $\quad SO(8n)/U(4n) \xrightarrow{i_3} B_{U(4n)} \xrightarrow{p_3} B_{SO(8n)}$

(4) $\quad Sp(2n)/U(2n) \xrightarrow{i_4} B_{U(2n)} \xrightarrow{p_4} B_{Sp(2n)}$.

(4) j_r means right multiplication by j.

(All maps are induced by the inclusions (1.1) or (1.2)). In the direct limit of these fibrations we will see that the mod p homology splits and that this split is given by a map of the base space into the total space. With this in mind we define maps:

$$\lambda_1 : U(4n)/Sp(2n) \to U(4n) \qquad \lambda_2 : U(2n)/O(2n) \to U(2n)$$
$$\lambda_3 : B_{O(4n)} \to B_{U(4n)} \qquad \lambda_4 : B_{Sp(n)} \to B_{U(2n)}$$

λ_1 is defined by $\widehat{\lambda}_1 : U(4n) \to U(4n)$, $\widehat{\lambda}_1(T) = Tj_r T^{-1} j_r^{-1}$, by passage to the quotient.

λ_2 is defined by restriction of λ_1.

λ_3 is the map induced by ρ (see above).

λ_4 is the map induced by the restriction of ρ.

We note that in all the above fibrations except (2). the fibre is totally non-homologous to zero mod p (see [2]). In the case (2), the difficulty is the non-stable class in dimension $n-1$; and in fact in the fibration $SO(2n+1) \to U(2n+1) \to U(2n+1)/SO(2n+1)$, the fibre is totally non-homologous to zero mod p. Consequently we have:

LEMMA 3.1. - In the fibrations

$$(1) \quad Sp \xrightarrow{i_1} U \xrightarrow{p_1} U/SO \qquad (2) \quad SO \xrightarrow{i_2} U \xrightarrow{p_2} U/SO$$
$$(3) \quad SO/U \xrightarrow{i_3} B_U \xrightarrow{p_3} B_{SO} \qquad (4) \quad Sp/U \xrightarrow{i_4} B_U \xrightarrow{p_4} B_{Sp}$$

the fibres are totally non-homologous to zero mod p. Consequently, the cohomology of the total space is (additively only in cases (3) and (4)) the tensor product of the cohomology of the base and fibre, with mod p coefficients [2].

We now prove:

LEMMA 3.2.

$$\lambda_1^* p_1^* : H^*(U/Sp;\ Zp) \simeq H^*(U/Sp;\ Zp)$$
$$\lambda_2^* p_2^* : H^*(U/SO;Zp) \simeq H^*(U/O;\ Zp)$$
$$\lambda_3^* p_3^* : H^*(B_{SO};\quad Zp) \simeq H^*(B_O;\ Zp)$$
$$\lambda_4^* p_4^* : H^*(B_{Sp};\quad Zp) \simeq H^*(B_{Sp};\ Zp).$$

REMARK. - We have used O in place of SO on the right. This does not effect the mod p cohomology, and will be useful later.

PROOF.

(1) It follows from (3.1) that $H^*(U/Sp;\ Zp) \simeq \Lambda_p(v_1, v_5, \ldots, v_{4i-3}, \ldots)$ and

$p_1{}^* v_{4i-3}(x_{4i-3}) = 1$, x_{4i-3} the generator of $H_*(U; Zp)$. Now consider the commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\widehat{\lambda}_1} & U \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ U/Sp & \xrightarrow{p_1\lambda_1} & U/Sp. \end{array}$$

We clain $\widehat{\lambda}_{1*}(x_{4i-3}) = 2x_{4i-3}$, and hence that $\widehat{\lambda}_1 p_1{}^*$ is monomorphism mod p, and by the commutativity of the diagram that $\lambda_1{}^* p_1{}^*$ is a monomorphism and consequently an isomorphism (by equality of the ranks of the cohomology groups).

To compute $\widehat{\lambda}_{1*}$ it is only necessary to see what conjugation $C(j_r)$ by j_r does to $H_*(U(4n); Z)$. For this, take the subgroup $U(2n)$ of $U(4n)$ invariant under left multiplication by H, then $U(2n)$ acts on $C_{2n} \subset W_{2n}$ (see (1.2)). Choosing a fixed basis for W_{2n} and hence C_{2n}, $T \in U(2n)$ is represented by a complex matrix M; and it is easy to see that then $j_r T j_r^{-1}$ is represented by $\bar{M}$, the complex conjugate of M. To see what complex conjugate does to $H_*(U(2n); Z)$ we use the characterization of the generators given in the proof of Prop. 2.4; i.e. that they project onto the generator of the corresponding sphere. From this one gets easily that $C(j_r)_* x_{4i-3} = -x_{4i-3}$ and $C(j_r)_* x_{4i-1} = x_{4i-1}$. Since the inclusion of $U(4n)$ into $U(8n)$, $U(8n)$ acting on W_{4n}, commutes with $C(j_r)$, the same formulas hold for $C(j_r)^*$ in $H^*(U(4n); Z)$. From this the formula for $\widehat{\lambda}_{1*}$ follows, and the first part of the lemma is proved.

(2) It follows from (3.1) that
$H^*(U/SO; Zp) \simeq \Lambda_p(w_1, w_5, \ldots, w_{4i-3}, \ldots)$ and $p_1{}^* w_{4i-3}(x_{4i-3}) = 1$, x_{4i-3} the generator of $H_*(U; Zp)$. Since U/O is an H-space with U/SO as double covering, it follows from the Cartan-Leray spectral sequence that $H^*(U/O; Zp) \simeq H^*(U/SO; Zp)$. The result now follows as in (1) from the commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\widehat{\lambda}_2} & U \\ \downarrow & & \downarrow{\scriptstyle p_2} \\ U/O & \xrightarrow{p_2\lambda_2} & U/SO. \end{array}$$

(3) To prove (3) consider the map:

$$SO(4n) \xrightarrow{\rho} U(4n) \to SO(8n).$$

By definition $\rho(T) = \sigma_i \sigma_j \tau \psi(T) \tau^{-1} \sigma_j^{-1} \sigma_i^{-1}$, and σ_i, σ_j, $\tau \in SO(8n)$. Hence ρ is homotopic to ψ in $SO(8n)$. Since $\psi(T) = (T, T)$, ψ is homotopic to the map $T \to T^2$. It follows that the map $H^*(SO; Zp) \to H^*(SO; Zp) \to H^*(SO; Zp)$ sends the primitive generators onto twice themselves, and hence is an isomorphism. Since the transgressions of these primitive generators, generate

$H^*(B_{SO}; Zp)$ it follows that the map: $H^*(B_{SO}; Zp) \xrightarrow{p_3^*} H^*(B_O; Zp) \xrightarrow{\lambda_3^*} H^*(B_O; Zp) \rightarrow H^*(B_{SO}; Zp)$ is an isomorphism. Since $H^*(B_O; Zp) \simeq H^*(B_{SO}; Zp)$, $\lambda_3^* p_3^*$ is an isomorphism.

(4) The argument for (4) is just the same as in (3), since the map $Sp(n) \rightarrow U(2n) \rightarrow Sp(2n)$ is just $T \rightarrow (T, T)$, by property 5 of ρ.

From (3.1) and (3.2) we get:

THEOREM 3.3 - Let γ: $U \times U \rightarrow U$ be the multiplication, then (additively)

$$(i_1 \times \lambda_1)^*\gamma^*: H^*(U; Zp) \simeq H^*(Sp; Zp) \otimes H^*(U/Sp; Zp)$$

$$(i_2 \times \lambda_2)^*\gamma^*: H^*(U; Zp) \simeq H^*(SO; Zp) \otimes H^*(U/O; Zp)$$

$$(i_3 \times \lambda_3)^*\gamma^*: H^*(B_U; Zp) \simeq H^*(SO/U; Zp) \otimes H^*(B_O; Zp)$$

$$(i_4 \times \lambda_4)^*\gamma^*: H^*(B_U; Zp) \simeq H^*(Sp/U; Zp) \otimes H^*(B_{Sp}; Zp)$$

$$(i_1 \times \lambda_2)^*\gamma^*: H^*(U; Zp) \simeq H^*(Sp; Zp) \otimes H^*(U/O; Zp)$$

$$(i_2 \times \lambda_1)^*\gamma^*: H^*(U; Zp) \simeq H^*(SO; Zp) \otimes H^*(U/Sp; Zp)$$

COROLLARY 3.4.

$$(\Omega i_4 \times \Omega\lambda_4)^*(\Omega\gamma)^*: H^*(\Omega B_U; Zp) \simeq H^*(\Omega(Sp/U); Zp) \otimes H^*(\Omega B_{Sp}; Zp)$$

$$(\Omega i_1 \times \Omega\lambda_1)^*(\Omega\gamma)^*: H^*(\Omega U; Zp) \simeq H^*(\Omega Sp; Zp) \otimes H^*(\Omega(U/Sp); Zp)$$

$$(\Omega i_3 \times \Omega\lambda_3)^*(\Omega\gamma)^*: H^*(\Omega B_U; Zp) \simeq H^*(\Omega(SO/U); Zp) \otimes H^*(\Omega B_O; Zp)$$

$$(\Omega i_2 \times \Omega\lambda_2)^*(\Omega\gamma)^*: H^*(\Omega U; Zp) \simeq H^*(\Omega SP; Zp) \otimes H^*(\Omega(U/O); Zp).$$

PROOF. - The first four isomorphisms of the theorem are immediate. For the last two isomorphisms it is sufficient to note from the proof of Lemma 3.2, that i_1^* and i_2^* have the same kernel.

To prove the corollary, it is easy to see that the isomorphisms of the theorem persist if we replace all the spaces involved by their simply connected covering spaces. From this it follows (by the comparison theorem argument) that the isomorphisms persist if we replace each space by its loop space. (We did not list the other two isomorphisms in the corollary, or other combinations in the theorem since we will have no need of them).

We now make some remarks on path spaces which we will need shortly;

Let (E, B, F) be a fibre space with total space E, base B, fibre F, and projection p: $E \rightarrow B$. Let b_0 be the base point in B, and let $e_0 \in p^{-1}(b_0) = F$.

We will assume B is connected and simply connected, and F is connected. Let

$$\begin{aligned} P(E) &= \text{paths in } E \text{ starting at } e_0 \\ P_F(E) &= \text{paths in } E \text{ starting at } e_0 \text{ and ending in } F = p^{-1}(b_0) \\ P(B) &= \text{paths in } B \text{ starting at } b_0 \\ \Omega(B) &= \text{paths in } B \text{ starting and ending at } b_0. \end{aligned}$$

Let

$\mu = \mu_E \colon P(E) \to E$, be the endpoint projection in E

$\mu = \mu_B \colon P(B) \to B$, be the endpoint projection in B.

Consider the commutative diagram:

$$\begin{array}{ccc} P(E) & \xrightarrow{P(p)} & P(B) \\ {\scriptstyle p\mu_E}\downarrow & & \downarrow{\scriptstyle \mu_B} \\ B & \longleftrightarrow & B \end{array}$$

$p\mu_E \colon P(E) \to B$ is a fibre space map with fibre $P_F(E)$

$\mu_B \colon P(B) \to B$ is a fibre space map with fibre $\Omega(B)$

$P(p) \colon P(E) \to P(B)$ is a fibre preserving map.

Since the total spaces are contractible it follows that $P(p)\colon P_F(E) \to \Omega(B)$ is a *weak homotopy equivalence*.

Now consider another map of fibre spaces:

$$\begin{array}{ccc} P(E) & \xrightarrow{\mu E} & E \\ {\scriptstyle p\mu_E}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longleftrightarrow & B \end{array}$$

If $\pi_i(E) = 0$, $i \leq n$, then $\mu\colon P_F(E) \to F$ will induce isomorphisms on homotopy, $i < n$, and consequently in homology, $i < n$ (see [10], p. 167).

Some commuting diagrams:

$$\begin{array}{ccc} \dfrac{Sp(2n)}{Sp(n) \times Sp(n)} & \xrightarrow{i_1} & \dfrac{U(4n)}{U(2n) \times U(2n)} \\ {\scriptstyle \Phi_1'}\downarrow & & \downarrow{\scriptstyle \Phi_0'} \\ \Omega(U(4n)/Sp(2n)) & \xrightarrow{\Omega(\lambda_1)} & \Omega(4n)) \end{array} \qquad \begin{array}{ccc} \dfrac{O(2n)}{O(n) \times O(n)} & \xrightarrow{i_2} & \dfrac{U(2n)}{U(n) \times U(n)} \\ {\scriptstyle \Phi_4'}\downarrow & & \downarrow{\scriptstyle \Phi_0'} \\ \Omega\left(\dfrac{U(2n)}{O(2n)}\right) & \xrightarrow{\Omega(\lambda_2)} & \Omega(U(2n)) \end{array}$$

$$\begin{array}{ccc}
\dfrac{U(4n)}{Sp(2n)} & \xrightarrow{\lambda_1} & U(4n) \\
{\scriptstyle \widehat{\Phi}_2}\big\downarrow & & \big\uparrow{\scriptstyle \mu} \\
P_{U(4n)}(SO(8n)) & \xrightarrow{p(\rho)} & P_{U(4n)}\left(\dfrac{U(8n)}{U(4n)}\right) \\
{\scriptstyle P(p)}\big\downarrow & & \big\downarrow{\scriptstyle P(p)} \\
\Omega\left(\dfrac{SO(8n)}{U(4n)}\right) & \xrightarrow{\Omega(\rho)} & \Omega\left(\dfrac{U(8n)}{U(4n)\times U(4n)}\right)
\end{array}
\qquad
\begin{array}{ccc}
\dfrac{U(2n)}{O(2n)} & \xrightarrow{\lambda_2} & U(2n) \\
{\scriptstyle \widehat{\Phi}_5}\big\downarrow & & \big\uparrow{\scriptstyle \mu} \\
P_{U(2n)}(Sp(2n)) & \xrightarrow{P(\rho)} & P_{U(2n)}\left(\dfrac{U(4n)}{U(2n)}\right) \\
{\scriptstyle P(p)}\big\downarrow & & \big\downarrow{\scriptstyle P(p)} \\
\Omega\left(\dfrac{Sp(2n)}{U(2n)}\right) & \xrightarrow{\Omega(\rho)} & \Omega\left(\dfrac{U(4n)}{U(2n)\times U(2n)}\right)
\end{array}$$

$$\begin{array}{ccc}
\dfrac{SO(8n)}{U(4u)} & \xrightarrow{\rho} & \dfrac{U(8n)}{U(4n)\times U(4n)} \\
{\scriptstyle \Phi_3'}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_0'} \\
\Omega\, SO(8n) & \xrightarrow{\Omega(\rho)} & \Omega\, U(8n)
\end{array}
\qquad
\begin{array}{ccc}
\dfrac{Sp(2n)}{U(2n)} & \xrightarrow{\rho} & \dfrac{U(4n)}{U(2n)\times U(2n)} \\
{\scriptstyle \Phi_0'}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_0'} \\
\Omega(Sp(2n)) & \xrightarrow{\Omega(\rho)} & \Omega\, U(4n).
\end{array}$$

The maps $\widehat{\Phi}_2$ and $\widehat{\Phi}_5$ are defined by $\widehat{\psi}_2$ and $\widehat{\psi}_5$ (1.5) before passing to the quotient, i.e., $\Phi_2 = P(p)\widehat{\Phi}_2$, $\Phi_5 = P(p)\widehat{\Phi}_5$. The map i_1, i_2 are those induced by (1.1) and (1.2).

All the maps in the diagrams for Φ_4, Φ_5, Φ_6 are the restrictions of the corrisponding maps in those for Φ_1, Φ_2, Φ_3 respectively. Hence the commutativity of the diagrams for Φ_4, Φ_5, Φ_6 will follow from the commutativity of those for Φ_1, Φ_2, Φ_3.

Proof of commutativity:

THE DIAGRAM FOR Φ_1. - From the discussion in section 1, it is clear that it is sufficient to prove the commutativity for the diagram with φ_1 and φ_0 in place of Φ_1' and Φ_0' and Ω_v in place of Ω.

Let $T \in Sp(2n)$, then

$$\begin{aligned}
\widehat{\lambda}_1\widehat{\xi}_1(T,\ \theta) &= \widehat{\lambda}_1(T\alpha_1(\theta)T^{-1}) = T\alpha_1(\theta)T^{-1}j_r T\alpha_1(\theta)^{-1}T^{-1}j_r^{-1} \\
&= T\alpha_1(\theta)TT^{-1}j_r\alpha_1(\theta)^{-1}j_r^{-1}T^{-1} = T\alpha_1(\theta)j_r\alpha_1(\theta)^{-1}j_r^{-1}T^{-1} \\
&= T\alpha_1(\theta)\alpha_1(\theta)T^{-1} = T\alpha_1(2\theta)T^{-1} \\
\xi_0(i_1T,\ \theta) &= T\alpha_0(\theta)T^{-1}.
\end{aligned}$$

But $\alpha_0(\theta) = \alpha_1(2\theta)$.

THE DIAGRAM FOR Φ_2. - The commutativity of the bottom square is trivial once we observe that the map $SO(8n) \xrightarrow{\rho} U(8n) \to U(8n)/U(4n)$ is fibre preserving. But for $T \in U(4n)$, $\rho(T) = (T,\ j_rTj_r^{-1}) \in U(4n) \times U(4n)$ (property

5 of ρ), hence the image $\{(T, j_r T j_r^{-1})\}$ in $U(8n)/U(4n)$ is equal to $\{(T, I)\}$.
For the top square we have: $T \in U(4n)$

$$\begin{aligned}\mu\{\rho\psi_2(T, \theta)\} &= \mu\{\rho(T\alpha_2(\theta)T^{-1}\alpha_2(\theta)^{-1})\} = \\ &= \mu\{(T, j_r T j_r^{-1})(\alpha_2(\theta), j_r\alpha_2(\theta)j_r^{-1})(T^{-1}, j_r T^{-1} j_r^{-1})(\alpha_2(\theta)^{-1}, j_r\alpha_2(\theta)^{-1}j_r^{-1})\} \\ &= \mu\{(T\alpha_2(\theta)T^{-1}\alpha_2(\theta)^{-1}, j_r T\alpha_2(\theta)T^{-1}\alpha_2(\theta)^{-1}j_r^{-1})\} \\ &= \mu\{(T\alpha_2(\theta)T^{-1}\alpha_2(\theta)^{-1}, I)\} = T j_r T^{-1} j_r^{-1},\end{aligned}$$

since the endpoint of $\alpha_2(\theta)$ is $j.\ \widetilde{\lambda}_1(T) = T j_r T^{-1} j_r^{-1}$.

The diagram for Φ_3. – Again we substitute φ_3 and φ_0 for Φ_3' and Φ_0'. Let $T \in SO(8n)$

$$\begin{aligned}\rho\widetilde{\psi}_3(T, \theta) &= \rho(T\alpha_3(\theta)T^{-1}) = \rho(T)\rho(\alpha_3(\theta))\rho(T)^{-1} \\ \widetilde{\psi}_0(\rho(T), \theta) &= \rho(T)\alpha_0(\theta)\rho(T)^{-1}.\end{aligned}$$

But $\rho(\alpha_3(\theta)) = (\alpha_3(\theta), j_r\alpha_3(\theta)j_r^{-1}) = (\alpha_3(\theta), \alpha_3(\theta)^{-1}) = \alpha_0(\theta)$.

Theorem 3.5. – From the commutativity of the above diagrams for every n, we get the following commutative diagrams in cohomology (or homology); arbitrary coefficients:

$$\begin{array}{ccc} H^*(B_{Sp}) & \xleftarrow{\lambda_4^*} & H^*(B_U) \\ \uparrow{\scriptstyle \Phi_1^*} & & \uparrow{\scriptstyle \Phi_0^*} \\ H^*(\Omega(U/Sp)) & \xleftarrow{(\Omega\lambda_1)^*} & H^*(\Omega U) \end{array} \qquad \begin{array}{ccc} H^*(B_O) & \xleftarrow{\lambda_3^*} & H^*(B_U) \\ \uparrow{\scriptstyle \Phi_4^*} & & \uparrow{\scriptstyle \Phi_0^*} \\ H^*(\Omega(U/O)) & \xleftarrow{(\Omega\lambda_2)^*} & H^*(\Omega U) \end{array}$$

$$\begin{array}{ccc} H^*(U/Sp) & \xleftarrow{\lambda_1^*} & H^*(U) \\ \uparrow{\scriptstyle \Phi_2^*} & & \updownarrow \\ H^*(\Omega(SO/U)) & \xleftarrow{(\Omega i_3)^*} & H^*(\Omega B_U) \end{array} \qquad \begin{array}{ccc} H^*(U/O) & \xleftarrow{\lambda_2^*} & H^*(U) \\ \uparrow{\scriptstyle \Phi_5^*} & & \updownarrow \\ H^*(\Omega(Sp/U)) & \xleftarrow{(\Omega i_4)^*} & H^*(\Omega B_U) \end{array}$$

$$\begin{array}{ccc} H^*(SO/U) & \xleftarrow{i_3^*} & H^*(B_U) \\ \uparrow{\scriptstyle \Phi_3^*} & & \uparrow{\scriptstyle \Phi_0^*} \\ H^*(\Omega SO) & \xleftarrow{\Omega i_2^*} & H^*(\Omega U) \end{array} \qquad \begin{array}{ccc} H^*(Sp/U) & \xleftarrow{i_4^*} & H^*(B_U) \\ \uparrow{\scriptstyle \Phi_6^*} & & \uparrow{\scriptstyle \Phi_0^*} \\ H^*(\Omega Sp) & \xleftarrow{i_1^*} & H^*(\Omega U). \end{array}$$

For the changes in the notation for the maps, see the propreties 1 to 4 of ρ and the definition of the maps λ_i, $i = 1, 2, 3, 4$.

Combining these diagrams with Theorem 3.3 and Corollary, we get:

THEOREM 3.6. – The following diagrams commute; *mod p* cofficients

$$
\begin{array}{ccccc}
H^*(Sp) \otimes H^*(U/O) & \longleftrightarrow & H^*(U) \\
\mu^* \updownarrow \quad \uparrow \Phi_5^* & & \updownarrow \mu^* \\
H^*(\Omega B_{Sp}) \otimes H^*(\Omega Sp/U) & \longleftrightarrow & H^*(\Omega B_U)
\end{array}
\qquad
\begin{array}{ccc}
H^*(SO) \otimes H^*(U/Sp) & \longleftrightarrow & H^*(U) \\
\mu^* \updownarrow \quad \uparrow \Phi_2^* & & \updownarrow \mu^* \\
H^*(\Omega B_{SO}) \otimes H^*(\Omega SO/O) & \longleftrightarrow & H^*(\Omega B_U)
\end{array}
$$

$$
\begin{array}{ccc}
H^*(B_{Sp}) \otimes H^*(Sp/U) & \longleftrightarrow & H^*(B_U) \\
\Phi_1^* \uparrow \quad \uparrow \Phi_6^* & & \updownarrow \Phi_0^* \\
H^*(\Omega U/Sp) \otimes H^*(\Omega Sp) & \longleftrightarrow & H^*(\Omega U)
\end{array}
\qquad
\begin{array}{ccc}
H^*(B_O) \otimes H^*(SO/U) & \longleftrightarrow & H^*(B_U) \\
\uparrow \Phi_4^* \quad \uparrow \Phi_3^* & & \updownarrow \Phi_0^* \\
H^*(\Omega U/O) \otimes H^*(\Omega SO) & \longleftrightarrow & H^*(\Omega U)
\end{array}
$$

The double arrows indicate isomorphisms proved in previous theorems.

Immediately from Theorem 3.6 we have:

THEOREM 3.7. – Φ_1, Φ_2, Φ_3, Φ_4, Φ_5, Φ_6 induce isomorphisms on mod p cohomology (and hence in homology mod p).

4. Orthogonal and symplectic groups mod 2 homology.

THE MAP Φ_1. – In the fibration $Sp \xrightarrow{i_1} U \xrightarrow{p_1} U/Sp$ (section 3), both i_1, p_1 are weak H-maps, and Sp is totally non-homologous to zero integrally [2]. It follows that $p_1{}^*$: $H_*(U;\ Z) \to H_*(U/Sp;\ Z)$ is onto, and $H_*(U/Sp;\ Z) \simeq H_*(U;\ Z)//(H_*(Sp;\ Z)$ as HOPF algebras. Thus

$$
(4.1) \qquad H_*(U/Sp;\ Z) \simeq \Lambda_Z(u_1,\ u_5,\ \ldots,\ u_{4u-8},\ \ldots).
$$

The simply connected covering space of U/Sp is $\tilde{U}/Sp = \mathrm{R} \times (SU/Sp)$. Using the fibration $Sp \to \tilde{U} \to \tilde{U}/Sp$, we get as above:

$$
(4.2) \qquad H_*(SU/Sp;\ Z) \simeq \Lambda_Z(u_5,\ u_9,\ \ldots,\ u_{4n+1},\ \ldots).
$$

Since $H_*(\tilde{U};\ Z)$ is transgressively generated (see (2.5) and corollary), $H_*(\tilde{U}/Sp;\ Z)$ is transgressively generated and hence, (by (2.7))

$$
(4.3) \qquad H_*(\Omega(U/Sp);\ Z) \simeq Z[Y_4,\ Y_8,\ \ldots,\ Y_{4n},\ \ldots].
$$

Now looking at diagram 1 of (3.5) (in homology), λ_{4*}: $H_*(B_{Sp};\ Z_2) \to H_*(B_U;\ Z_2)$ is a monomorphism (since the dual map in cohomology is onto). Since Φ_{0*}

is an isomorphism, $\Phi_{1*}: H_*(B_{Sp}; Z_2) \to H_*(\Omega(U/Sp); Z_2)$ is a monomorphism. Since the ranks of the corresponding mod 2 homology groups are the same (see Prop. 4.4 below), Φ_{1*} is an isomorphism mod 2 and hence integrally.

We have used, see (2.9):

PROPOSITION 4.4. - $H^*(B_{Sp}; Z) \simeq Z[p_4, p_8, \ldots, p_{4n}, \ldots]$, with codiagonal $\mu\,(p_{4i}) = \sum_{j+K=i} p_{4j} \otimes p_{4K}$. Dually, it follows that $H_*(B_{Sp}; Z) \simeq Z[Z_4, \ldots, Z_{4n}, \ldots]$, and $d_*(Z_{4i}) = \sum_{j+K=i} Z_{4j} \otimes Z_{4K}$, where $Z_{4i} = \overline{p_4^{\,i}}$, $\overline{p_4^{\,i}}$ the dual class to $p_4^{\,i}$.

Further, $H^*(HP; Z) = H^*(B_{Sp(1)}; Z) \simeq Z[b_4]$, HP the infinite quaternionic projective space, and the map $j_*: B_{Sp(1)} \to B_{Sp}$ induced by the inclusion $Sp(1) \subset Sp$, satisfies $j^*(p_{4i}) = 0$, $i > 1$, $j^*(p_4) = b_4$. Consequently, $j_*(\overline{b_4^{\,i}}) = Z_{4i}$; i.e., j_* maps the additive basis of $H_*(HP; Z)$ onto the generators of the PONTRJAGIN ring $H_*(B_{Sp}; Z)$.

THE MAP Φ_2. - In the fibration $U \xrightarrow{i} SO \xrightarrow{p} SO/U$, the direct limit of the fibration $U(n) \to SO(2n) \to SO(2n)/U(n)$, U is totally non-homologous to zero mod 2 ([2]). Hence $H_*(SO/U; Z_2) \simeq H_*(SO; Z_2)//H_*(U; Z_2)$ as HOPF algebras. Now $H^*(SO; Z_2) \simeq Z_2[h_1, h_3, \ldots, h_{2n-1}, \ldots]$, h_{2n-1} primitive, all n: and hence $H_*(SO; Z_2) = \Lambda_2(a_1, a_2, \ldots, a_n, \ldots)$, with a_{2n-1} primitive (see [2]). Since $H_*(U; Z_2) \simeq \Lambda_2(x_1, x_3, \ldots, x_{2n-1}, \ldots)$, $H_*(SO/U; Z_2) \simeq \Lambda_2(d_2, d_4, \ldots, d_{2n}, \ldots)$, and hence the classes dual to d_{2n}, in particular the classes g_{4n-2} dual to d_{4n-2}, are primitive. Thus $p^*(g_{4n-2}) = h^2_{2n-1}$ (the only primitive class in dim $4n-2$). it follows that the subalgebra of $H^*(SO/U; Z_2)$ generated by the g_{4n-2} is a polynomial algebra, and must be all of $H^*(SO/U; Z_2)$ by comparison of ranks with the homology groups. Thus

$$H^*(SO/U; Z_2) \simeq Z_2[g_2, g_6, \ldots], \; g_{4n-2} \text{ primitive.} \tag{4.5}$$

From diagram 2, section 3, we get the commuting diagram:

$$\begin{array}{ccc} H_*(E(U/Sp); Z) & \xrightarrow{E\lambda_{1*}} & H_*(EU; Z) \\ {\scriptstyle \psi_{2*}}\downarrow & & \downarrow{\scriptstyle \nu_*} \\ H_*(SO/U; Z) & \xrightarrow{i_{3*}} & H_*(B_U; Z) \end{array}$$

where $\nu_*: H_*(EU; Z) \overset{(E\mu)_*^{-}}{\simeq} H_*(EP_U(E_U); Z) \to H_*(E\Omega B_U; Z) \to H_*(B_U; Z)$. The suspension $\bar{u}_{4n-2}$ of the generator u_{4n-3} of $H_*(U/Sp; Z)$ maps onto $2\bar{x}_{4n-2}$, $\bar{x}_{4n-2}$ the suspension of the primitive generator x_{4n-3} of $H_*(U; Z)$, (see proof of (3.2)) under $E\lambda_{1*}$. Since $C_{4n-2}(\nu_*\bar{x}_{4n-2}) = 1$, C_{4n-2} the CHERN class, $\nu_*(E\lambda_1)_*\bar{u}_{4n-2}$ is divisible by exactly 2. On the other hand, since $i_{3*}: H_*(SO/U; Z_2) \to H_*(B_U; Z_2)$ is trivial (in the fibration $SO/U \xrightarrow{i_3} B_U \xrightarrow{p_3} B_{SO}$;

$p_3{}^*: H^*(B_{SO}; Z_2) \to H^*(B_U; Z_2)$ is onto [3]), the image under i_3 of any integral class in SO/U must be divisible by at least 2 in $H_*(B_U; Z)$. Consequently, $\psi_{2*}\bar{u}_{4n-2}$ cannot be divisible in $H_*(SO/U; Z)$, and $\psi_{2*}\bar{u}_{4n-2}$ reduced mod 2 is not zero. Since $\psi_{2*}\bar{u}_{4n-2}$ is primitive, $\psi_{2*}\bar{u}_{4n-2}$ reduced mod 2 is dual to g_{4n-4}, and g_{4n-2} suspends non-trivially onto a class e_{4n-3} in $H^*(\Omega SO/U; Z_2)$, with $\Phi_2{}^*e_{4n-3} = u_{4n-3}$ (reduced mod 2).

The subalgebra generated by the e_{4n-3} maps onto $H^*(U/Sp; Z_2)$ under $\Phi_2{}^*$. We will show that this subalgebra is exterior; i.e., that $e^2_{4n-3} = 0$. But $e^2_{4-n4} = Sq^{4n-3}e_{4n-3} =$ suspension of $Sq^{4n-3}g_{4n-2} = 0$ (no odd classes in $H^*(SO/U; Z_2)$. Consequently, the subalgebra is a transgressively generated exterior algebra. Let S be the canonical spectral sequence with trivial E_∞ term, for such an exterior algebra, then $E_2^{*,0}$ is a polynomial algebra generated by the transgression of the exterior generators. Let S' be the spectral sequence (mod 2) for the fibration $\Omega(SO/U) \to P(SO/U) \to SO/U$. The fact that the e_{4n-3} are transgressive implies that there is a spectral sequence map $S \to S'$ sending exterior generators in $E_2^{0,*}$ into exterior generators in $E_2'^{0,*}$. Since then the map $E_2^{*,0} \to E_2'^{*,0}$ is an isomorphism and both E_∞ terms are trivial, the map $E_2^{0,*} \to E_2'^{0,*}$ is an isomorphism by the comparison theorem [8]. Consequently, the subalgebra generated by the e_{4n-3} is the whole algebra $H^*(\Omega SO/U; Z_2)$ and hence $\Phi_2{}^*$ is an isomorphism mod 2, and therefore integrally.

It now follows that $H^*(\Omega SO/U; Z) \simeq H^*(U/Sp; Z)$ is an exterior algebra (in fact transgressively generated), and hence $H^*(SO/U; Z)$ is a polynomial algebra (by BOREL's theorem):

$$H^*(SO/U; Z) \simeq Z[g_2, g_6, \ldots, g_{4n-2}, \ldots]. \tag{4.6}$$

We remark that it follows from the above that $i_3{}^*(C_{4n-2})$ is divisible by 2 and in fact must be twice a generator (since its value on $\psi_{2*}\bar{u}_{4n-2}$ is 2), and hence we may choose g_{4n-2} so that $i_3{}^*(C_{4n-2}) = 2g_{4n-2}$. From this and the fibration $SO/U \to B_U \to B_{SO}$ being totally non-homologous to zero mod p; the mod p HOPF algebra structure, in particular the diagonal map in cohomology mod p of the g_{4n-2} is determined. Since the g^{4n-1} are primitive mod 2 $(\mu i_3{}^*(C_{4n-2}) = i_3{}^*\mu C_{4n-2} = i_3{}^* \sum_{i+j=2n-1} C_{2i} \otimes C_{2j} \equiv 2g_{4n-2} \equiv 2g_{4n-2} \otimes 1 + 1 \otimes 2g_{4n-2} (\text{mod } 4)$, since $i_3{}^*(C_{2i})$ and $i_3{}^*(C_{2j})$ are both divisiblè by 2), and there is no torsion, the integral codiagonal maps is uniquely determined.

THE MAP Φ_3. - From the map $U(n) \to SO(2n)$ we get a map $\tilde{U}(n) \to Spin(2n)$. Now $H^*(Spin; Z_2) = Z_2[\tilde{h}_3, \ldots, \tilde{h}_{2n+1}, \ldots]$, where the $\tilde{h}_{2n+1}$ are primitive (use for example, the CARTAN-LERAY spectral sequence for the double covering of SO, the $\tilde{h}_{2n+1}$ are the images of the h_{2n+1} in $H^*(SO; Z_2)$). The map $H_*(\tilde{U}; S_2) \to H_*(Spin; Z_2)$ sends the primitive generators onto the classes dual to $\tilde{h}_{2n+2}$. Since the generators of $H_*(\tilde{U}; Z_2)$ transgress (section 2), it follows that the $\tilde{h}_{2n+2}$ suspend non-trivially onto primitive classes f_{2n} in $H^*(\Omega SO; Z_2)$. Since $Sq^i h_j = \binom{i}{j} h_{i+j}$ (where we set $h_{2i} = h_i^2$, [2], in $H^*(SO; Z_2)$, $Sq^{2^i(4n-2)}\tilde{h}_{2^i(4n-2)+1} = \tilde{h}_{2^{i+1}(4n-2)+1}$ and hence $Sq^{2^i(4n-2)} = f_{2^i(4n-2)}\ f_{2^{i+1}(4n-2)}$; i.e. $f_{2^i(4n-2)} = f^{2^i}_{(4n-2)}$. Also since the f_{4n-2} map non trivially into $H^*(\Omega U; Z_2)$ and this last is a polynomial algebra, it follows that the powers of f_{4n-2} are non-zero. The subalgebra generated by the f_{4n-2} is a sub-HOPF algebra of $H^*(\Omega SO; Z_2)$ and must be a polynomial algebra since the primitive generators are of infinite height. Since the base space in the canonical spectral sequence ($E_\infty = 0$) of a polynomial algebra whose generators and their 2^{ith} powers transgress is the polynomial algebra generated by the transgression of these classes, it follows from the comparison theorem (see similar argument for the map Φ_2) that the subalgebra is the whole algebra; i.e.,

$$H^*(\Omega SO; Z_2) \simeq Z_2[f_2, f_6, \dots, f_{4n-2}, \dots].$$

Since the f_{4n-2} are primitive, the dual algebra is an exterior algebra generated by the classes dual to the f_{4n-2} and their 2^{ith} powers. Since the images of the generators of $H_*(\Omega U; Z_2)$ have value 1 on the $f^{2i}_{(4n-2)}$, it follows that $H_*(\Omega U; Z_2) \to H_*(\Omega SO; Z_2)$ is onto. On the other hand, the map $H_*(\Omega U; Z_2) \to H_*(\Omega SO; Z_2) \to H_*(\Omega U; Z_2)$ is trivial as we shall see below, so that Ωi_{2*} is trivial. In fact $U(4n) \to SO(8n) \to U(8n)$ send $T \to (T, j_r T j_r^{-1})$. Now the map $T \to j_r T j_r^{-1}$ induces an automorphism of the HOPF algebra $H_*(U; Z)$ and hence sends each primitive generator onto $\pm$ itself. The induced map on $H_*(\Omega U; Z)$ sends each generator d_{2i} into $\mp d_{2i} + p_{2i}$, where p_{2i} is some primitive class (since diagonal must be preserved); but p_{2i} suspends non-trivially into $H_*(U; Z)$, unless it is zero (Prop. 2.9). It now follows that the map $H_*(\Omega U; Z) \to H_*(\Omega SO; Z) \to H_*(\Omega U; Z)$ sends d_{2i} into $2d_{2i}$ or 0, and the corresponding map mod 2 is trivial.

From the proof that Φ_{2*} is an isomorphism we have $i_{3*}\psi_{2*}\bar{u}_{4n-2} = 2\nu\bar{x}_{4n-2}$. Since from the above, the image of any integral class in $H_*(\Omega SO; Z)$ under Ωi_{2*} is divisible by at least 2, we see from the diagram for Φ_3, that $\Phi_{3*}\psi_{2*}\bar{u}_{4n-2}$ is not divisible, and hence is non-zero mod 2. Since these classes are primitive, they are dual to the f_{4n-2}, and hence Φ_3^* maps f_{4n-2} non-trivially. Since f_{4n-2} is primitive $\Phi_3 f_{4n-2} = g_{4n-2}$, and Φ_3^* is an isomorphism mod 2, and hence integrally.

THE MAP Φ_6. - In the fibration $Sp/U \xrightarrow{i_4} B_U \xrightarrow{p_4} B_{Sp}$, the fibre is totally non-homologous to zero integrally [2], so that $H_*(Sp/U; Z) \simeq H^*(B_U; Z) / H^*(B_{Sp}; Z)$ as HOPF algebras. In particular, since $H^*(B_U) \to H_*(B_{Sp})$ has an algebraic cross-section, $H_*(B_U; Z) \simeq H_*(Sp/U; Z) \otimes H_*(B_{Sp}; Z)$ as rings. Hence

$$H_*(Sp/U; Z) \simeq Z[x_2, x_6, \dots, x_{4n-2}, \dots]. \tag{4.8}$$

Now looking at diagram for Φ_6 in the form

$$\begin{array}{ccc} H_*(ESp/U) & \xrightarrow{i_4^*} & H_*(EB_U) \\ \psi_6 \downarrow & & \downarrow \psi_0 \\ H_*(Sp) & \xrightarrow{i_1^*} & H_*(U) \end{array}$$

we see that ψ_{6*} takes the suspension $\bar{x}_{4n-1}$ of the generators x_{4n-2} of $H_*(Sp/U;\, Z)$ onto the primitive generators of $H_*(Sp;\, Z)$, since $\psi_0 i_{4*}(\bar{x}_{4n-1})$ in the primitive generator of $H_*(U;\, Z)$. It follows that $H_*(Sp;\, Z)$ is transgressively generated and hence by (2.7),

$$(4.9) \qquad H_*(\Omega Sp;\, Z) \simeq Z[y_2,\, y_6, \ldots,\, y_{4n-2}, \ldots].$$

Further $\Phi_{6*} x_{4n-2} = y_{4n-2}$, and Φ_{6*} is an isomorphism integrally.

THE MAP Φ_5. – In the fibration $U(n)/O(n) \to B_{O(n)} \to B_{U(n)}$, $H_*(B_{U(n)};\, Z_2) \to H^*(B_{O(n)};\, Z_2)$ is a monomorphism, and hence $H_*(B_{O(n)};\, Z_2) \to H_*(B_{U(n)};\, Z_2)$ is onto. Since $U/O \to B_O$ is a weak H-map, the differential in the homology spectral sequence for the fibration $U/O \to B_O \to B_U$ is a derivation, and consequently by the above the differential is trivial. Further since the map $H_*(B_O;\, Z_2) \to H_*(B_U;\, Z_2)$ has an algebraic cross-section, $H_*(B_O;\, Z_2) \simeq H_*(B_U;\, Z_2) \otimes H_*(U/O;\, Z_2)$ as algebras. Hence since $H_*(B_O;\, Z_2) = Z_2[v_1,\, v_2, \ldots,\, v_n, \ldots]$ see (Prop. 4.15),

$$(4.10) \qquad H_*(U/O;\, Z_2) \simeq Z_2[u_1,\, u_3, \ldots,\, u_{2n-1}, \ldots].$$

Now looking at the diagram for Φ_4 in the form

$$\begin{array}{ccc} H_*(EB_O;\, Z_2) & \xrightarrow{(Ei_2)_*} & H_*(EB_U;\, Z_2) \\ \psi_4{}^* \downarrow & & \downarrow \psi_0{}^* \\ H_*(U/O;\, Z_2) & \xrightarrow{\lambda_{2*}} & H_*(U;\, Z_2). \end{array}$$

We see that ψ_{4*} takes the suspension $\bar{v}_{2n+1}$ of the generators v_{2n} of $H_*(B_O;\, Z_2)$ onto primitive classes in $H_*(U/O;\, Z_2)$ whose image under λ_{2*} are the primitive generators x_{2n+1} of $H_*(U;\, Z_2)$. It follows that $\psi_* \bar{v}_{2n+1}$ is not decomposable and hence must be a generator. Consequently, we may assume $\psi_* \bar{v}_{2n-1} = u_{2n-1}$, $n > 1$, and hence that $H_*(U/O;\, Z_2)$ is primitively generated and $\lambda_{2*} u_{2n-1} = x_{2n-1}$, $n > 1$. Further $\pi_1(U/O) = Z$ (from the homotopy sequence for $O \to U \xrightarrow{p} U/O$, we have either Z or $Z \oplus Z_2$, but from (4.10) it must be Z, and p_* : $\pi_1(U) \to \pi_1(U/O)$ sends the generator onto twice a generator) and λ_{2*} : $\pi_1(U/O) \simeq \pi_1(U)$. Hence $\lambda_{2*} u_1 = x_1$.

Now in the diagram for Φ_5:

$$\begin{array}{ccc} H_*(EU/O;\, Z_2) & \xrightarrow{E\lambda_{2*}} & H_*(EU;\, Z_2) \\ \downarrow{\scriptstyle \psi_5*} & & \downarrow{\scriptstyle \nu*} \\ H_*(Sp/U;\, Z_2) & \xrightarrow{i_4*} & H_*(B_U;\, Z_2) \end{array}$$

$\nu_* \bar{x}_{2i} = p_{2i}$, the primitive element reduced mod 2. Now from Prop. 2.9, p_{4i-2} mod 2 is equal to a generator plus decomposable elements, and hence the p_{4i-2} generate a polynomial subalgebra. It follows that the $\psi_{5*}\bar{u}_{4i-2}$ ($\bar{u}_{4i-2}$ the suspension of the generator u_{4i-3} of $H_*(U/O;\, Z_2)$ generate a polynomial subalgebra of $H_*(Sp/U;\, Z_2)$, since $i_{4*}\psi_{5*}(\bar{u}_{4i-2} = \nu_* E\lambda_{2*}(\bar{u}_{4i-2}) = p_{2i}$ (see previous paragraphs). Since the ranks of corresponding groups are equal (4.8), the subalgebra is all of $H_*(Sp/U;\, Z_2)$. Consequently $H_*(Sp/U;\, Z_2)$ is transgressively generated. By the theorem of Araki and Kudo stated below (see [1], [8]),

$$H_*(\Omega Sp/U;\, Z_2) \simeq Z_2[w_1, w_3, \ldots, w_{2n-1}, \ldots]. \tag{4.11}$$

Now by the commutativity of the diagram for Φ_5 (in its original form), Φ_{5*} maps the generators onto non-decomposable elements in $H_*(\Omega Sp/U;\, Z_2)$; and hence onto generators. Thus Φ_{5*} is an isomorphism mod 2, and hence integrally.

Theorem 4.12 (Araki and Kudo). - Let X be an associative H-space with unit, and suppose that $H_*(X;\, Z_2)$ is a transgressively generated polynomial algebra, then the 2^{ith} powers of these generators also transgress, and $H_*(\Omega X;\, Z_2)$ is a polynomial algebra generated by the transgressions of these classes.

Note. - In order to apply Theorem 4.12, we use the isomorphism Φ_{6*} already obtained, so that we may consider that we are working with ΩSp, which may be given an associative H-space structure with unit.

The map Φ_4. - The simply connected covering space of U/O is $\tilde{U}/SO = R \times SU/SO$. Consider the fibration $SU/SO \to B_{SO} \to B_{SU}$. Now $H^*(B_{SO};\, Z_2) \simeq Z_2[w_2, w_3, \ldots, w_{i+1}, \ldots]$, w_i the i^{th} Stiefel-Whitney class. Under the map $H^*(B_{SU};\, Z_2) \to H^*(B_{SO};\, Z_2)$ the Chern classes C_{2i+2}, $i > 0$ map onto w_{i+1}^2. Hence $H^*(SU/SO;\, Z_2) = H^*(B_{SO};\, Z_2)//H^*(B_{SU};\, Z_2) \simeq \Lambda_2(b_2, b_3, \ldots, b_{i+1}, \ldots]$ (the fibration is trivial in homology by the same argument as for U/O above, and hence trivial in cohomology). The duals $\bar{b}_{i+1}$ of the classes b_{i+1} are primitive. In particular, $\bar{b}_{2n-1}$ must map onto u_{2n-1} under the map $H_*(SU/SO;\, Z_2) \to H_*(U/O;\, Z_2)$, and similarly $\bar{b}_2$ maps onto u_1^2. The fact that this map is a monomorphism can be seen from the commutative diagram

$$\begin{array}{ccc} H^*(SU/SO\,;\, Z_2) & \leftarrow & H^*(U/O\,;\, Z_2) \\ \uparrow & & \uparrow \\ H^*(B_{SO}\,;\, Z_2) & \leftarrow & H^*(B_O\,;\, Z_2) \end{array}$$

i.e., $H^*(U/O\,;\, Z_2)$ is an exterior algebra with a generator in each dimension which maps onto the corresponding generator in $H^*(SU/SO\,;\, Z_2)$ (except for the first generator). Hence

$$H_*(\tilde{U}/SO\,;\, Z_2) \simeq H_*(SU/SO\,;\, Z_2) \simeq Z_2[\tilde{u}_2,\, \tilde{u}_3, \ldots,\, \tilde{u}_{2i+1}, \ldots] \tag{4.13}$$

where $\tilde{u}_{2i+1}$ maps onto u_{2i+1} and $\tilde{u}_2$ maps onto u_1^2 under the covering map.

Since the classes u_{2i+1} are in the image of $H_*(EB_O\,;\, Z_2)$ under ψ_{4*}, $H_*(\tilde{U}/SO\,;\, Z_2)$ is transgressively generated, and hence by (4.12),

$$H_*(\Omega U/O\,;\, Z_2) \simeq Z_2[r_1,\, r_2, \ldots,\, r_i\,, \ldots]. \tag{4.14}$$

It remains to prove that $H_*(B_O\,;\, Z_2)$ maps monomorphically into $H_*(\Omega U/O\,;\, Z_2)$.

Consider the commutative diagram (for definition of ρ see section 3, $U(n)' \subset Sp(n)$ is the subgroup commuting with j)

$$\begin{array}{ccc} H_*(Sp/U'\,;\, Z_2) & \xrightarrow{i_{4*}} & H_*(B_{U'}\,;\, Z_2) \\ \rho_* \downarrow & & \downarrow \\ H_*(U/O\,;\, Z_2) & \rightarrow & H_*(B_O\,;\, Z_2). \end{array}$$

The maps on top and at right are both monomorphisms, hence the map at left is a monomorphism. Since both $H_*(Sp/U\,;\, Z_2)$ and $H_*(\tilde{U}/SO\,;\, Z_2)$ are transgressively generated, it follows from (4.12) that $H_*(\Omega Sp/U\,;\, Z_2) \rightarrow H_*(\Omega U/O\,;\, Z_2)$ is a monomorphism.

Now consider the diagram

$$\begin{array}{ccc} U(n)'/O(n) & \xrightarrow{\rho} & \dfrac{O(2n)}{O(n) \times O(n)} \\ \Phi'_5 \downarrow & & \Phi'_4 \downarrow \\ \Omega(Sp(n)/U(n)') & \xrightarrow{\rho} & \Omega(U(2n)/O(2n)). \end{array}$$

To prove this is commutative, we may replace Φ'_4, Φ'_5, by φ_4, φ_5 respectively and Ω by Ω_v (see section 1). Since $U(n)'$ is taken as the subgroup commuting with j, $\tilde{\xi}_5(T,\, \theta) = Te_r^{i\theta/2}T^{-1}$ (i.e. $e_r^{\theta/2}$ instead of $e_r^{j\theta/2}$). Then

$$\begin{aligned} \rho\tilde{\xi}_5(T,\, \theta) &= \rho(T)\rho(e_r^{i\theta/2})\rho(T)^{-1} = \rho(T)(e_r^{i\theta/2},\, j_r e_r^{-i\theta/2} j_r e_r^{-i\theta/2} j_r^{-1})\rho(T)^{-1} \\ &= \rho(T)(e_r^{i\theta/2},\, e_r^{-i\theta/2})\rho(T)^{-1} \\ \tilde{\xi}_4(\rho(T),\, \theta) &= \rho(T)\alpha_4(\theta)\rho(T)^{-1} = \rho(T)(e_r^{i\theta/2},\, e_r^{-i\theta/2})\rho(T)^{-1}. \end{aligned}$$

Hence the following is commutative

$$\begin{array}{ccc} H^*(U'/O\,;\,Z_2) & \longleftarrow & H^*(B_O\,;\,Z_2) \\ \Phi_5^*\uparrow & & \uparrow\Phi_4^* \\ H^*(\Omega Sp/U') & \longleftarrow & H^*(\Omega U/O\,;\,Z_2). \end{array}$$

Since Φ_5^* is an isomorphism $\Phi_5^*\rho^*$ is onto. Consequently, there must exist a non-decomposable class Z_i in each dim in $H^*(\Omega U/O\,;\,Z_2)$ mapping onto the generators of $H^*(U'/O\,;\,Z_2)$ in each dimension. $\Phi_4^* Z_i$ must be non-decomposable in $H^*(B_O\,;\,Z_2)$ and hence a generator. Consequently, Φ_4^* is onto and Φ_{4*} is a monomorphism.

By comparison of ranks of corresponding homology groups it now follows that Φ_{4*} is an isomorphism mod 2 and hence integrally. We have used [11]:

PROPOSITION 4.15. - $H^*(B_O\,;\,Z_2) \simeq Z_2[w_1, w_2, \ldots, w_n, \ldots]$, with codiagonal $\mu(w_i) = \sum_{j+K=i} w_j \otimes w_K$. Dually, it follows that $H_*(B_O\,;\,Z_2) \simeq Z_2[v_1, v_2, \ldots, v_n, \ldots]$ and $d_*(v_i) = \sum_{j+K=i} v_j \otimes v_K$, where $v_i = \overline{w_1^i}$, $\overline{w_1^i}$ the dual class to w_1^i.

Further, $H^*(P\,;\,Z_2) = H^*(B_{O(1)}\,;\,Z_2) \simeq Z_2(u_1)$, P the infinite real projective space, and the map $j_*: B_{O(1)} \to B_O$ induced by the inclusion $O(1) \subset 0$, satisfies, $j^* w_i = 0$, $i > 1$, $j^* w_1 = u_1$. Consequently, $j_*(\overline{u_1^i}) = v_i$; i.e., j_* maps the additive basis of $H_*(P\,;\,Z_2)$ onto the generators of $H_*(B_O\,;\,Z_2)$.

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