# On closed sets of rational functions.

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**Summary.** - *This paper contains proofs of the closure of certain sets of rational functions in various spaces. Thus, for example, conditions are derived for the closure of the sequence* $(x^2+z_\nu^2)^{-1}$ *in the space* $L_2(0, \infty)$, *and for the set* $\dfrac{ct-z_\nu}{1-ct\bar{z}_\nu}$ *in* $C(-1, +1)$. *Analogous results are proved for other related sets of rational functions. Some of these results are new; others are new proofs of known theorems. The main point is that a uniform method is used throughout this paper. For a description of the method see article 1.*

1. **Introduction.** - The following theorem is well known [see 4 and 5] (1). A necessary and sufficient condition for the closure of the sequence $\{t^{\lambda_\nu}\}$, $\nu = 1, 2, \ldots$, in $L_2(0, 1)$ is

$$\sum_{\nu=1}^{\infty} \frac{1-2R\lambda_\nu}{1+|\lambda_\nu|^2} = \infty. \qquad R\lambda_\nu > -\frac{1}{2}\,(^2).$$

The basic steps of the proof were:

*a*) Find a special sequence $\{t^{\lambda_\nu}\}$, which is closed; here we may take $\lambda_\nu = \nu - 1$, by WEIERSTRASS' s approximation theorem.

*b*) Find the minimum, $m_n$, of

$$\int_0^1 \Big|\sum_{\nu=0}^{n} u_\nu t^{\lambda_\nu}\Big|^2 dt = \sum_{\mu,\nu=0}^{n} u_\mu \bar{u}_\nu \frac{1}{\lambda_\mu+\bar{\lambda}_\nu+1}$$

where $u_0 = 1$, and $\lambda_0$ any given integer $k \geq 0$.

*c*) The set $\{t^{\lambda_\nu}\}_{1,\infty}$, is closed if and only if for each $k$ $m_n \to 0$, as $n \to \infty$.

One advantage of this direct, elementary procedure is that it applies as well to the more general case of triangular systems. We shall also employ it to discuss some sets of rational functions in the infinite interval $(0, \infty)$.

2. **Fundamental concepts.** - $L_p(0, 1)$ denotes the space of real or complex valued functions $f(t)$ in $0 < t < 1$, such that $f(t)$ and $|f(t)|^p$ are integrable; here, $p \geq 1$. Distance in this space is defined by

$$(f_1,\ f_2) = \left(\int_0^1 |f_1(t) - f_2(t)|^p dt\right)^{1/p}.$$

(*) Questo lavoro pervenne in redazione il primo giugno 1952 e l'A. rivide le prime 17 pagine delle bozze. A causa della Sua morte, improvvisamente avvenuta il 19 settembre 1952 a Montreux (Svizzera) l'ulteriore lavoro di revisione fu assunto dal Prof. H. DAVID LIPSICH.

Gli Annali di Matematica rivolgono un commosso pensiero alla memoria del loro collaboratore OTTO SZÁSZ ben noto nel mondo matematico internazionale.

(1) Numbers in square brackets refer to the list at the end of this paper.

(2) $Rz$ denotes the real part of $z$; $\bar{z}$ is the conjugate complex of $z$.

$L_\infty(0, 1)$ denotes the space of measurable, essentially bounded functions. Distance in this space is defined by

$$(f_1, f_2) = \operatorname*{ess.\,u.\,b.}_{0<t<1} |f_1(t) - f_2(t)|.$$

$C(0, 1)$ denotes the space of continuous functions. Distance in this space is defined by

$$(f_1, f_2) = \max_{0\leq t\leq 1} |f_1(t) - f_2(t)|.$$

Clearly $L_p \subset L_q$ for $p > q$; let $p' = p/(p-1)$, and $p' = \infty$ for $p = 1$; $p$ and $p'$ are called conjugate numbers.

A sequence of functions $f_n(t) \in L_p(0, 1)$ is called complete in this space, if the only function $g(t) \in L_{p'}$ which is orthogonal to all $f_n(t)$, i. e. for which

$$[f_n, g] \equiv \int_0^1 f_n(t)\overline{g(t)}dt = 0, \qquad (n = 1, 2, 3, \ldots),$$

is $g(t) \backsim 0$. Here $\overline{g(t)}$ is the conjugate complex to $g(t)$; $g(t) \backsim 0$ means $g(t) = 0$ almost everywhere (a. e.).

The sequence $f_n(t) \in L_p(0, 1)$, (or in $C$) is called closed in this space, if to any function $\psi(t) \in L_p$ there exists a sequence of linear aggregates

$$l_n(t) = \sum_{\nu=1}^{n} c_{\nu n} f_\nu(t)$$

such that

$$\lim_{n\to\infty} (\psi, l_n) = 0.$$

It is known that completeness in $L_p$ and closure in $L_p$ $(1 \leq p \leq \infty)$, are equivalent. Furthermore completeness in $L_p$ implies completeness in $L_q$ for any $q < p \leq \infty$. (Cf. [1], pp. 73-74).

The same properties extend easily to any finite interval, by a linear transformation of the variable $t$.

Finally a sequence of functions in $C(0, 1)$ is called complete in this space if for any function $g(t)$ of bounded variation the infinitely many equations

$$\int_0^1 f_n(t)d\overline{g(t)} = 0, \qquad (n = 1, 2, 3, \ldots).$$

imply $g(t) = g(1) = g(0)$, except at an enumerable set of points. Again closure and completeness in $C(0, 1)$ are equivalent. (Cf. [1], p. 73).

Similar definitions and properties hold for an infinite interval.

We also introduce the space $H_p$ of functions $f(z)$, regular in $|z| < 1$, and such that

$$M_p\{f;\, r\} = \left(1/2\pi \int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p} = O(1) \text{ as } r \uparrow 1;$$

$H_\infty$ is the class of functions $f(z)$ regular and bounded in $|z| < 1$.

It is known that if $f(z) \in H_p$, then $\lim\limits_{r \uparrow 1} f(re^{i\theta}) = f(e^{i\theta})$ exists a. e., and

$$\int_0^{2\pi} |f(e^{i\theta})|^p d\theta$$

is finite.

A sequence of functions $u_n(z) \in H_p$ is said to be complete in $H_p$ if for any $g(z) \in H_{p'}$ the equations

$$[u_n,\, g] = \int_0^{2\pi} u_n(e^{i\theta}) \bar{g}(e^{i\theta}) d\theta = 0, \qquad (n = 1,\, 2, \ldots),$$

imply $g(z) = 0$. Here $1 \leq p \leq \infty$. For $p = 2$, see Malmquist [3].

The sequence $u_n(z)$ is called closed in $H_p$ $(1 \leq p \leq \infty)$ if to any function $f(z) \in H_p$ there exist a sequence of linear aggregates

$$\sum_{\nu=1}^{n} C_{n,\nu} u_\nu(z) \equiv l_n(z),$$

such that

$$(2.1) \qquad (f,\, l_n) = \left(1/2\pi \int_0^{2\pi} |f(e^{i\theta}) - l_n(e^{i\theta})|^p d\theta\right)^{1/p} \to 0 \text{ as } n \to \infty.$$

It can be shown by using a standard device, that completeness and closure in some $H_p$ are equivalent.

The sequence $u_n(z) \in H_2$ is called orthogonal and normal, if

$$[u_\nu,\, u_k] = \begin{cases} 0, & \text{for } \nu \neq k \\ 1, & \text{for } \nu = k \end{cases} \qquad (\nu,\, k = 1,\, 2, \ldots).$$

**3. Preliminary lemmas.**

Lemma 1. - If $Rz > 0$ [3], then

$$A(z) \equiv \int_0^\infty \frac{dx}{x^2 + z^2} = \frac{\pi}{2z}.$$

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(3) For our purpose there is no loss of generality in the restriction $Rz > 0$.

This is a well known formula and can be found in many textbooks. Note that

$$A(z) = z^{-2}A(z^{-1}); \qquad A(-z) = A(z).$$

LEMMA 2. - If $Ru > 0$, $Rz > 0$, then

$$B(u, z) \equiv \int_0^\infty \frac{dx}{(x^2+u^2)(x^2+z^2)} = \frac{\pi}{2uz(u+z)}.$$

In fact

$$\frac{1}{x^2+u^2}\cdot\frac{1}{x^2+z^2} = \left(\frac{1}{x^2+u^2} - \frac{1}{x^2+z^2}\right)\frac{1}{z^2-u^2}.$$

Hence. in view of Lemma 1,

$$B(u, z) = \frac{1}{z^2-u^2}\int_0^\infty \left(\frac{1}{x^2+u^2} - \frac{1}{x^2+z^2}\right) dx$$

$$= \frac{\pi}{2(z^2-u^2)}\left(\frac{1}{u} - \frac{1}{z}\right) = \frac{\pi}{2uz(u+z)}.$$

LEMMA 3. - If $g_\nu$, $r_\nu$ $(\nu = 0, 1, \ldots, n)$ are real or complex numbers, then the determinant

$$D_n = [(g_\nu + r_k)^{-1}]_0^n$$

has the value

$$D_n = \prod_{\nu, k=0}^{n} (g_\nu + r_k)^{-1} \cdot \prod_{\nu > k \geq 0} (g_\nu - g_k)(r_\nu - r_k).$$

For references see [5].

A consequence of this Lemma is:

$$\frac{[(g_\nu + r_k)^{-1}]_0^n}{[(g_\nu + r_k)^{-1}]_1^n} = \frac{1}{g_0 + r_0} \prod_{\nu=1}^{n} \frac{(g_\nu - g_0)(r_\nu - r_0)}{(g_\nu + r_0)(r_\nu + g_0)}.$$

LEMMA 4. - Given a sequence of real or complex numbers $z_1, z_2, \ldots, z_\nu \neq z_k$, in the halfplane $Rz > 0$. If the sequence has a finite limit point $\zeta$ with $R\zeta > 0$, then the sequence

$$\frac{1}{x^2 + z_\nu^2}, \qquad (\nu = 1, 2, \ldots) \tag{3.1}$$

is complete in $L_2(0, \infty)$.

To prove this, we wish to show that if $f(x)$ is a function in $L_2(0, \infty)$, and such that

$$\int_0^\infty \frac{f(x)dx}{x^2 + \bar{z}_\nu^2} = 0, \qquad (\nu = 1, 2, \ldots).$$

Then $f(x) = 0$ almost everywhere.

The function

$$F(z) = \int_0^\infty \frac{f(x)dx}{x^2 + z^2}$$

is analytic in the halfplane $Rz > 0$, and vanishes at infinitely many points, with a limit point in the domain of regularity. Hence

$$F(z) \equiv 0 \quad \text{in} \quad Rz > 0.$$

Now for real $z > 0$

$$\frac{1}{x^2 + z^2} = \int_0^\infty e^{-(x^2+z^2)t} dt,$$

hence,

$$F(z) = \int_0^\infty \int_0^\infty f(x) e^{-(x^2+z^2)t} dt dx,$$

and the double integral is absolutely convergent for $z > 0$. On putting

$$\Phi(t) = \int_0^\infty f(x) e^{-x^2 t} dx$$

we have now

$$F(z) = \int_0^\infty e^{-z^2 t} \Phi(t) dt \equiv 0;$$

it follows that

$$\Phi(t) = 0 \text{ a. e. in } (0, \infty),$$

hence

$$f(t) = 0 \text{ a. e.}$$

This proves the lemma. It is known that completeness and closure in $L_2(0, \infty)$ are equivalent. Hence under the assumption of Lemma 4 to any given $\varepsilon > 0$ and to any function $f(x) \varepsilon L_2(0, \infty)$ there exists a linear aggregate $\sum_1^n a_\nu (x^2 + z_\nu^2)^{-1}$ such that

(3.2) $$\int_0^\infty \left| f(x) - \sum_1^n \frac{a_\nu}{x^2 + z_\nu^2} \right|^2 dx < \varepsilon .$$

We shall employ Lemma 3 and 1 to find the necessary and sufficient condition for the closure of the sequence (3.1).

**4. Closure of the sequence $(x^2+z_\nu^2)^{-1}$ in $L_2(0,\infty)$.**

A necessary and sufficient condition for the closure of the sequence (3.1) is that any preassigned fraction $(x^2+z_0^2)^{-1}$, $Rz_0>0$, can be approximated by (3.1) in the sense of (3.2). The sufficiency follows evidently from Lemma 4. Consider now the minimum of

$$Q_n=\int_0^\infty\left|\sum_0^n\frac{u_\nu}{x^2+z_\nu^2}\right|^2dx$$

for $u_0=1$ and $u_1, u_2, \dots, u_n$, arbitrary. The approximation is possible if and only if

$$\min Q_n\equiv\mu_n(z_0)\to0\quad\text{as}\quad n\to\infty.$$

Now

$$Q_n=\sum_{\nu,k=0}^n c_{\nu k}u_\nu\bar u_k,\qquad c_{\nu k}=\int_0^\infty(x^2+z_\nu^2)^{-1}(x^2+\bar z_k^2)^{-1}dx,$$

and

$$\min Q_n=[c_{\nu k}]_0^n:[c_{\nu k}]_1^n.$$

By Lemma 2

$$c_{\nu k}=\frac{\pi}{2z_\nu\bar z_k(z_\nu+\bar z_k)};$$

thus Lemma 3 yields (with $g_\nu=z_\nu$, $r_\nu=\bar z_\nu$)

$$\mu_n=\frac{\pi}{2|z_0|^2(z_0+\bar z_0)}\prod_1^n\left|\frac{z_\nu-z_0}{z_\nu+\bar z_0}\right|^2.$$

On putting $z_n=v_n+iw_n$ and

$$\eta_n=1-\left|\frac{z_n-z_0}{\bar z_n+z_0}\right|^2, \tag{4.1}$$

we have

$$\mu_n=\frac{\pi}{4v_0|z_0|^2}\prod_1^n(1-\eta_\nu),$$

and

$$\eta_n=\frac{4v_0v_n}{|\bar z_n+z_0|^2}. \tag{4.2}$$

Thus $\mu_n\to0$ if and only if $\sum_1^\infty v_n|\bar z_n+z_0|^{-2}=\infty$. We may choose $z_0$ real and positive, so that the condition becomes

$$\sum_1^\infty\frac{v_n}{(v_n+v_0)^2+w_n^2}=\infty. \tag{4.3}$$

But

$$/z_n/^2 + v_0^2 \leq (v_n + v_0)^2 + w_n{}^2 \leq 2(/z_n/^2 + v_0^2);$$

hence (4.3) is equivalent to

$$\Sigma \frac{v_n}{/z_n/^2 + v_0^2} = \infty.$$

If this holds for some $v_0$, it holds for every positive $v$, hence we may choose $v_0 = 1$. We have established the following theorem:

THEOREM 1. - *The sequence* $(x^2 + z_v{}^2)^{-1}$, $Rz_v > 0$, $v \geq 1$, *is closed (complete) in* $L_2(0, \infty)$ *if and only if*

$$\sum_{v=1}^{\infty} \frac{Rz_v}{1 + /z_v/^2} = \infty.$$

**5. Closure of the sequence $x(x^2 + z_v{}^2)^{-1}$ in $L_2(0, \infty)$.**

The sequence $x(x^2 + z_v{}^2)^{-1}$, $Rz > 0$, can be treated in much the same way. If $f(x) \varepsilon L_2$, then

$$F(z) = \int_0^\infty \frac{xf(x)}{x^2 + z^2} dx$$

represents a regular function for $Rz > 0$. If $F(\zeta_v) = 0$ and the sequence $\{\zeta_v\}$ has a limit point inside the halfplane $Rz > 0$, then $F(z) \equiv 0$. Now for real positive $z$,

$$F(z) = \int_0^\infty \int_0^\infty xf(x) e^{-(x^2+z^2)} dt dx$$

$$= \int_0^\infty e^{-z^2 t} \Phi(t) dt,$$

where

$$\Phi(t) = \int_0^\infty xf(x) e^{-x^2 t} dx.$$

As before it follows that $f(x) \sim 0$ in $(0, \infty)$. We next consider the minimum of

$$Q_n = \int_0^\infty x^2 \left| \sum_1^n \frac{u_v}{x^2 + z_v{}^2} \right|^2 dx, \quad \text{for} \quad u_0 = 1; \quad u_1, u_2, \ldots, u_n \text{ variable.}$$

Now

$$Q_n = \sum_{v, k=0}^{n} c_{vk} u_v \bar{u}_k,$$

where

$$c_{\nu k} = \int_0^\infty \frac{x^2 dx}{(x^2 + z_\nu^2)(x^2 + \bar{z}_k^2)} = \frac{1}{\bar{z}_k^2 - z_\nu^2} \int_0^\infty \left( \frac{\bar{z}_k^2}{x^2 + \bar{z}_k^2} - \frac{z_\nu^2}{x^2 + z_\nu^2} \right) dx$$

$$= \frac{\pi(\bar{z}_k - z_\nu)}{2(\bar{z}_k^2 - z_\nu^2)} = \frac{\pi}{2(\bar{z}_k + z_\nu)} \qquad \text{(from Lemma 1).}$$

Lemma 3 yields (with $g_\nu = z_\nu$, $r_\nu = \bar{z}_\nu$)

$$\min Q_n = \frac{\pi}{2(z_0 + \bar{z}_0)} \prod_1^n \left| \frac{z_\nu - z_0}{z_\nu + \bar{z}_0} \right|^2 .$$

Thus the condition for closure remains the same as in the previous case. We have proved:

THEOREM 2. - *The sequences* $\{ x/(x^2 + z_\nu^2) \}$, $Rz_\nu > 0$, *is closed in* $L_2(0, \infty)$ *if and only if* $\sum_1^\infty Rz_\nu/(1 + |z_\nu|^2) = \infty$.

On putting

$$z_\nu = \frac{2}{1 + 2\lambda_\nu} = \frac{2(1 + 2\bar{\lambda}_\nu)}{|1 + 2\lambda_\nu|^2} \qquad R\lambda_\nu > -\frac{1}{2},$$

we get

$$\Sigma \frac{Rz_\nu}{1 + |z_\nu|^2} = 2 \Sigma \frac{1 + 2\lambda_\nu}{|1 + 2\lambda_\nu|^2},$$

and the condition $\Sigma z_\nu/(1 + |z_\nu|^2) = \infty$ becomes

$$\Sigma \frac{1 + 2\lambda_\nu}{1 + |\lambda_\nu|^2} = \infty$$

which is exactly the necessary and sufficient condition for the closure of the sequence $\{ x^{\lambda_\nu} \}$ in $L_2(0, 1)$ (cf. [4]). On writing

$$\frac{1}{z_\nu} = \frac{\bar{z}_\nu}{|z_\nu|^2} = \gamma_\nu, \qquad \frac{1}{x^2 + z_\nu^2} = \gamma_\nu^2 \cdot \frac{1}{1 + x^2\gamma_\nu^2},$$

it is seen that the sequences

$$\left\{ \frac{1}{x^2 + z_\nu^2} \right\}, \qquad \left\{ \frac{1}{1 + x^2\gamma_\nu^2} \right\}$$

are simultaneously closed or not closed. Furthermore $R\gamma_\nu > 0$ if $Rz_\nu > 0$ and conversely. Finally

$$\text{(5.1)} \qquad \Sigma \frac{Rz_\nu}{1 + |z_\nu|^2} = \Sigma \frac{\bar{z}_\nu^{-1}}{|z_\nu|^{-2} + 1} = \Sigma \frac{R\gamma_\nu}{1 + |\gamma_\nu|^2}.$$

Thus Theorem 1 is equivalent to:

THEOREM 3. - *The sequence* $(1 + x^2\gamma_\nu^2)$, $R\gamma_\nu > 0$, *is closed in* $L_2(0, \infty)$ *if and only if the series* (5.1) *is divergent.*

**6. The sequence $x^{\alpha+\nu}\, e^{-x/2}$.**

Let $f(x) \in L_2(0, \infty)$; the function

$$(6.1) \qquad F(z) = \int_0^\infty e^{-x/2} x^z f(x)\,dx$$

is regular in the halfplane $Rz > -\frac{1}{2}$. Hence if $F(z_\nu) = 0$, $\nu \geq 1$, and the sequence $z_\nu$ has a finite limit point inside the halfplane $Rz > -\frac{1}{2}$, then $F(z) \equiv 0$. Now for $c > 0$

$$1/2\pi i \int_{c-i\infty}^{c+i\infty} z^{-1} a^z dz = \begin{cases} 1, \text{ for } a > 1 \\ 0, \text{ for } 0 < a < 1. \end{cases}$$

Hence for $a > 0$, using (6.1)

$$1/2\pi i \int_{c-i\infty}^{c+i\infty} z^{-1} F(z) a^z dz = \int_{1/a}^\infty e^{-x/2} f(x) dx.$$

Thus if $F(z) \equiv 0$, then

$$\int_\alpha^\infty e^{-x/2} f(x) dx = 0 \quad \text{for all} \quad \alpha > 0;$$

it follows that $f(x) \backsim 0$ in $(0, \infty)$. We have proved:

THEOREM 4. – *The sequence* $\{ e^{-x/2} x^{z_\nu} \}$ *is complete in* $L_2\, 0, \infty)$ *if the sequence* $z_\nu$ *has a finite limit point inside the halfplane* $Rz > -\frac{1}{2}$.

Denote by $\zeta_1, \zeta_2, \ldots$ a sequence of real numbers such that $\zeta_\nu \to \zeta > -\frac{1}{2}$. By Theorem 4 the sequence $e^{-x/2} x^{\zeta_\nu}$ is closed in $L_2(0, \infty)$. Thus a sequence $\{ e^{-x/2} x^{z_\nu} \}$, $Rz_\nu > 0$, will be closed in $L_2(0, \infty)$ if and only if each $e^{-x/2} x^{\zeta_\nu}$ can be approximated by linear aggregates of the functions $e^{-x/2} x^{z_\nu}$ in $L_2(0, \infty)$. Consider one such $\zeta_\nu$ which is different from each $z_\nu$; denote it by $z_0$. Now write

$$Q_n = \int_0^\infty \left| \sum_{\nu=0}^n u_\nu e^{-x/2} x^{z_\nu} \right|^2 dx$$

$$= \int_0^\infty \left( \sum_{\nu, k=0}^n u_\nu \bar{u}_k e^{-x} x^{z_\nu + \bar{z}_k} \right) dx.$$

In particular let $z_0 = \alpha$, and

$$z_\nu = \alpha + \nu, \qquad \nu = 1, 2, 3, \ldots; \text{ where } \alpha > -\frac{1}{2};$$

then

$$Q_n = \sum_{\nu, k=0}^{n} u_\nu \bar{u}_k \frac{\Gamma(2\alpha + \nu + k + 1)}{\Gamma(2\alpha + 1)}.$$

Now [cf. 6, p. 46] $\min_{u_0=1} Q_n \to 0 (n \to \infty)$; if follows that the sequence $\{ e^{-x/2} x^{\alpha+\nu} \}_{1,\infty}$ is complete in $L_2(0, \infty)$. For other proofs cf. [10] and [2].

**7. Closure of the sequence**

$$\frac{ct - z_\nu}{1 - \bar{c}tz_\nu}$$

**in** $C(-1, +1)$.

Let $c$ be a real or complex constant, $0 < |c| \leq 1$; if $\psi(t)$ is of bounded variation, then

$$F(z) = \int_{-1}^{+1} \frac{ct - z}{1 - \bar{c}tz} d\psi(t)$$

is regular in $|z| < 1$, and

$$|F(z)| \leq \int_{-1}^{+1} |d\psi(t)|.$$

Assume that $F(z_\nu) = 0$ for $\nu = 1, 2, 3, \ldots$, and that $\Sigma (1 - |z_\nu|) = \infty$; it then follows from Blaschke's theorem, that $F(z) \equiv 0$. Now

$$\frac{ct - z}{1 - \bar{c}tz} = ct - z(1 - |c|^2 t^2) \sum_1^\infty (z\bar{c}t)^{n-1},$$

and termwise integration is permitted as the series is boundedly convergent to a continuos function. For, we have for $-1 \leq t \leq 1$

$$|z|(1 - |c|^2 t^2) \left| \sum_1^n (z\bar{c}t)^{\nu-1} \right| \leq \sum_1^n (1 - |c|^2 t^2) \, |ct|^{\nu-1}$$
$$= (1 + |ct|)(1 - |ct|^n) < 2.$$

Thus

$$F(z) = c \int_{-1}^{+1} t d\psi(t) - z \sum_1^\infty (1 - |c|^2 t^2)(z\bar{c}t)^{n-1} d\psi(t) \equiv 0;$$

it follows that

$$\int_{-1}^{+1} t d\psi(t) = 0, \qquad \int_{-1}^{1} (1 - |c|^2 t^2) t^{n-1} d\psi(t) = 0, \qquad (n = 1, 2, \ldots),$$

or

$$\int_{-1}^{1} t d\psi(t) = 0, \quad \int_{-1}^{1} t^{n-1} d\psi(t) = |c|^2 \int_{-1}^{1} t^{n+1} d\psi(t), \qquad (n \geq 1).$$

Thus

$$\int_{-1}^{1} t^{2k-1} d\psi(t) = 0, \qquad (k = 1, 2, 3, \ldots),$$

and

$$J_k \equiv \int_{-1}^{1} t^{2k} d\psi(t) = |c|^2 J_{k+1}, \qquad (k = 0, 1, 2, \ldots). \tag{7.1}$$

It follows that $|J_k| \leq |J_{k+1}|$ Furthermore

$$J_k = |c|^{-2k} J_0, \qquad (k = 0, 1, 2, \ldots);$$

hence if $|c| < 1$, then $|J_k| \to \infty$, unless $J_0 = 0$. But

$$|J_k| \leq \int_{-1}^{1} |d\psi(t)|,$$

hence $J_0 = 0$ and $J_k = 0$, $k = 1, 2, 3, \ldots$.

The system $\{t^n\}_0^\infty$ being closed in $C(-1, +1)$, it follows, using a well known theorem of F. Riesz on closure, that the system

$$\frac{ct - z}{1 - \bar{c}tz_\nu}, \ \nu \geq 1, \qquad \Sigma(1 - |z_\nu|) = \infty,$$

is closed in $C(-1, +1)$.

If $|c| = 1$, then write

$$\int_{-1}^{1} t^{2k} d\psi(t) = \psi(1) - \psi(-1) - 2k \int_{-1}^{1} t^{2k-1} \psi(t) dt,$$

hence

$$J_k = J_0 - 2k \int_{-1}^{1} t^{2k-1} \psi(t) dt.$$

Now from (7.1) $J_k = J_0$ hence

$$\int_{-1}^{1} t^{2k-1} \psi(t) dt = 0, \qquad (k = 1, 2, 3, \ldots),$$

or

(7.2) $$2k\int_0^1 \mid \psi(t) - \psi(-t) - J_0 \mid t^{2k-1}dt = -J_0.$$

Clearly

$$g(t) \equiv \psi(t) - \psi(-t) - J_0 \to 0 \quad \text{as} \quad t \uparrow 1.$$

Given $\varepsilon > 0$, choose $\eta$ so that $/g(t)/ < \varepsilon$ for $1 - \eta < t < 1$, and $M$ so that $/g(t)/ < M$; then from (7.2)

$$/J_0/ < 2kM\int_0^{1-\eta} t^{2k-1}dt + 2k\,\varepsilon \int_{1-\eta}^1 t^{2k-1}dt,$$

thus

$$/J_0/ < M(1-\eta)^{2k} + \varepsilon\,\{1 - (1-\eta)^{2k}\},$$

and letting $k \to \infty$

$$/J_0/ \leq \varepsilon.$$

Hence $J_0 = 0$, $J_k = 0$, $k = 1, 2, 3, \ldots$.

(For continuous $f(t)$, P. Funk proved (Math. Ann. 77, 1916, p. 146-147) that $n\int_0^1 f(t)t^n dt \to f(1)$ as $n \to \infty$).

Summarizing, we have proved:

Theorem 5. - *Let* $0 < /c/ \leq 1$, $/z_\nu/ < 1$, $\Sigma(1 - /z_\nu/) = \infty$, *then the sequence* $\dfrac{ct - z_\nu}{1 - \bar{c}tz_\nu}$ *is closed in* $C(-1, +1)$.

8. **Completeness of the sequence** $1/(1 - z_\nu z)$ **in** $H_\infty$.

If $f(z) \varepsilon H_1$, and

(8.1) $$\int_0^{2\pi} e^{-int}f(e^{it})dt = 0, \qquad (n = 0, 1, 2, \ldots),$$

then, putting $f(z) = \sum_0^\infty c_n z^n$ we have

(8.2) $$\int_0^{2\pi} e^{-int}f(e^{it})dt = 2\pi c_n = 0, \qquad (n = 0, 1, 2, \ldots),$$

hence $f(z) \equiv 0$. Thus the sequence $z^n$, $n = 0, 1, 2, \ldots$, is complete in $H_\infty$.

DEFINITION 1. - The class of functions $f(z)$ regular in $/z/ < 1$ and continuous in $/z/ \leq 1$ shall be denoted by $K$; a sequence $u_n(z) \varepsilon K$ is called closed in $K$ if to any function $f(z) \varepsilon K$ and to any given $\varepsilon > 0$ there exists a linear aggregate $l_n = \sum_1^n c_{n\nu} u_\nu$ so that

$$/f(z) - l_n(z)/ < \varepsilon \quad \text{for} \quad /z/ \leq 1. \tag{8.3}$$

It follows from a theorem on normed vector spaces [cf. [1], p. 58, Theorem 7], that the sequence $\{u_n(z)\}$ is closed in $K$, if and only if the infinitely many equations

$$\int_0^{2\pi} \overline{u_n(e^{it})} d\psi(t) = 0, \qquad (n = 1, 2, 3, ...) \tag{8.4}$$

imply

$$\int_0^{2\pi} \overline{g(e^{it})} d\psi(t) = 0 \quad \text{for every} \quad g(z) \varepsilon K. \tag{8.5}$$

Here $\psi(t)$ is of bounded variation. On putting $g(z) = z^n$, it follows from (8.5) that

$$\int_0^{2\pi} e^{-int} d\psi(t) = 0, \qquad (n = 0, 1, 2, ...). \tag{8.6}$$

Conversely if (8.6) holds, then employing a well known theorem of FEJÉR on arithmetic means, it follows that (8.5) holds.

Thus the sequence $z^n$, $n = 0, 1, 2, ...,$ is closed in $K$ and the statements (8.5) and (8.6) are equivalent. Our aim is to prove the following theorem.

THEOREM 6. - *Let*

$$/z_\nu/ < 1, \qquad z_\nu \neq z_k \quad \text{for} \quad \nu \neq k;$$

*the sequence*

$$r_\nu(z) = \frac{1}{1 - z_\nu z}, \qquad (\nu = 1, 2, 3, ...),$$

*is complete in* $H_\infty$ *if and only if* $\Sigma (1 - /z_\nu/) = \infty$.

Consider the function

$$F(z) = \int_0^{2\pi} \frac{g(e^{-it})}{1 - ze^{-it}} dt, \qquad (g(z) \varepsilon H_1);$$

it is regular for $/z/ < 1$; by assumption $F(\bar{z}_\nu) = 0$.

Furthermore, by termwise integration

$$F(z) = \sum_0^\infty z^\nu \int_0^{2\pi} e^{-i\nu t} g(e^{it})\, dt\,;$$

hence $F(z) = g(z) \varepsilon H_1$, and by a theorem of F. Riesz, $F(z) \equiv 0$. It follows that

$$\int_0^{2\pi} e^{-i\nu t} g(e^{it}) dt = 0, \qquad (\nu = 1,\ 2,\ 3, \dots),$$

hence $g(z) \equiv 0$.

Conversely, if $\sum_0^\infty (1 - /z_\nu/) < \infty$, then the product

$$\prod_1^\infty \frac{\bar{z}_\nu - z}{\bar{z}_\nu - z/z_\nu/^2} = h(z)$$

is a function, regular in $/z/ < 1$ and belonging to $H_\infty$.

By Fatou' s theorem on the boundary function

$$h(z) = \frac{1}{2\pi i} \int_{|x|=1} \frac{h(x) dx}{x - z} = \frac{1}{2\pi} \int_0^{2\pi} \frac{h(e^{it}) e^{it} dt}{e^{it} - z}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{h(e^{it}) dt}{1 - z e^{-it}}.$$

By assumption,

$$h(\bar{z}_v) = \frac{1}{2\pi} \int_0^{2\pi} \frac{h(e^{it}) dt}{1 - \bar{z}_\nu e^{-it}} = 0\,;$$

hence the sequence $1/(1 - z_\nu z)$ is orthogonal to the function $h(z)$. Thus we have proved that the condition $\Sigma (1 - /z_\nu/) = \infty$ is sufficient for completeness of the sequence $\{ 1/(1 - z_\nu z) \}$ in $H_\infty$, and necessary for the completeness in $H_p$, $1 \leq p \leq \infty$.

In this connection, the following theorem is of interest:

Theorem 7. - *If* $u_n(z) \varepsilon H_p$ (or $K$), *and if the sequence* $\{ u_n(z) \}$ *is closed in* $H_p$ (or $K$), *then the sequence* $\{ u_n(z), \overline{u_n(z)} \}$ $z = e^{it}$ *is closed in* $L_p$ (or $C$) *for all real or complex valued functions of* $t$ *in* $(0, 2\pi)$.

For the proof, note that any real valued function $\Phi(t) \varepsilon L_p$ $(0, 2\pi)$ (or $C$) is the real part of a function $f(e^{it})$ where $f(z) \varepsilon H_p$ (or $K$). Now $f(z)$ belongs to the span of the sequence $\{ u_n(z) \}$, hence $\bar{f}(z)$ belongs to the span of the sequence $\{ \bar{u}_n(z) \}$. This yields our assertion for real valued functions. If $\Phi(t)$ is complex valued $\Phi(t) = \Phi_1(t) + i\Phi_2(t)$, then the same argument applies to $\Phi_1$ and $\Phi_2$, which proves our theorem. A corresponding theorem holds for completeness.

9. **Direct proof of closure in $H_2$.**

It is of interest to give a direct elementary discussion of the closure of the sequence $\{1/(1-z_v z)\}$ in $H_2$. It is parallel to the one used in §§ 1, 2.

Suppose that $f(z) \varepsilon H_2$, and that

$$\int_0^{2\pi} \frac{\overline{f(e^{it})}dt}{1-z_\nu e^{it}} = 0 \qquad (\nu = 1,\ 2, \ldots).$$

Consider the function

$$F(z) = \int_0^{2\pi} \frac{\overline{f(e^{it})}dt}{1-ze^{it}};$$

it is clear that $F(z)$ is regular for $/z/ < 1$, and $F(z_\nu) = 0$ for $\nu = 1, 2, \ldots$. If the sequence $\{z_v\}$ has a limit point inside the unit circle, then by an elementary theorem $F(z) \equiv 0$. But for $/z/ < 1$

$$F(z) = \sum_0^\infty z^\nu \int_0^{2\pi} e^{i\nu t}\overline{f(e^{it})}dt;$$

hence

$$\int_0^{2\pi} e^{i\nu t}\overline{f(e^{it})}dt = 0, \qquad (\nu = 0,\ 1,\ 2,\ 3, \ldots).$$

It follows that $f(e^{it}) \backsim 0$. We have thus proved the lemma:

Lemma 5. - *If $\{z_v\}$ h's a limit point inside the unit circle, then the sequence $\{1/(1-z_v z)\}$ is complete (closed) in $H_2$.*

In particular we may choose $z_\nu = (v+1)^{-1}$, $v = 1, 2, 3, \ldots$. Now a sequence $\{(1-z_v z)^{-1}\}$ is closed in $H_2$ if each element of the sequence $\{(1-v^{-1}z)^{-1}\}$ can be approximated in the sense of (2.1) ($p = 2$) by linear aggregates $\Sigma\, c_\nu(1-z_\nu z)^{-1}$. Denote a term of $\{(1-(v+1)^{-1}z)^{-1}\}$, if it is not contained in the sequence $(1-z_\nu z)^{-1}$, by $(1-z_0 z)^{-1}$. Consider the minimum of the Hermitian form

$$Q\{u\} = \int_0^{2\pi} \left| \sum_{\nu=0}^{n} u_\nu (1-z_\nu z)^{-1} \right|^2 dt, \quad \text{for} \quad z = e^{it},\ u_0 = 1;$$

we have

$$Q\{u\} = \sum_{\nu,k=0}^{n} u_\nu \bar{u}_k \int_0^{2\pi} \frac{dt}{(1-z_\nu e^{it})(1-\bar{z}_k e^{-it})} = \Sigma\, u_\nu \bar{u}_k c_{\nu k}, \quad \text{say.}$$

Here

$$c_{\nu k} = \int_0^{2\pi} \left(\sum_0^\infty z_\nu{}^n e^{int}\right)\left(\sum_0^\infty \bar{z}_k{}^m e^{-imt}\right) dt$$

$$= \frac{2\pi}{1 - \bar{z}_\nu z_k} = \frac{2\pi}{z_\nu(z_\nu{}^{-1} - \bar{z}_k)}$$

and

(9.1) $$\min Q\,|\,u\,| = \frac{[c_{\nu k}]_0^n}{[c_{\nu k}]_1^n} = m_n, \quad \text{say.}$$

Now (cf. Lemma 3 with $g_\nu = z_\nu{}^{-1}$, $r_\nu = -\bar{z}_\nu$ for $\alpha = 0$ or 1)

$$[c_{\nu k}]_\alpha^n = \frac{(2\pi)^{n-\alpha+1}}{\prod_\alpha^n z_\nu} \frac{\prod_{\alpha \le k < \nu \le n} (z_\nu{}^{-1} - z_k{}^{-1})(\bar{z}_k - \bar{z}_\nu)}{\prod_{\nu, k=\alpha}^n (z_\nu{}^{-1} - \bar{z}_k)}.$$

Thus

$$m_n = \frac{2\pi}{z_0} \cdot \frac{1}{z_0{}^{-1} - \bar{z}_0} \frac{\prod_{\nu=1}^n (z_\nu{}^{-1} - z_0{}^{-1})(\bar{z}_0 - \bar{z}_\nu)}{\prod_1^n (z_\nu{}^{-1} - \bar{z}_0)(z_0{}^{-1} - \bar{z}_\nu)}$$

and this espression reduces easily to

$$m_n = \frac{2\pi}{1 - |z_0|^2} \prod_1^n \left|\frac{z_0 - z_\nu}{1 - \bar{z}_0 z_\nu}\right|^2$$

(9.2) $$= \frac{2\pi}{1 - |z_0|^2} \prod_1^n \left\{1 - \frac{(1 - |z_0|^2)(1 - |z_\nu|^2)}{|1 - \bar{z}_0 z_\nu|^2}\right\}.$$

To have closure in $H_2$ it is evidently necessary and sufficient that $m_n \to 0$, and this is the case if and only if

$$\Sigma \frac{1 - |z_\nu|^2}{|1 - \bar{z}_0 z_\nu|^2} = \infty, \quad \text{or}$$

(9.3) $$\Sigma (1 - |z_\nu|) = \infty.$$

Thus the sequence $\{1/(1 - \bar{z}_\nu z)\}$ is closed in $H_2$ if and only if (9.3) holds.

10. **Orthogonalization of the sequence** $1/(1 - \bar{z}_\nu z)$.
If $z_\nu \neq z_k$, for $\nu \neq k$, then

$$D_n = [c_{\nu k}] \neq 0,$$

hence the functions $r_\nu(z) = 1/(1 - z_\nu z)$, $\nu = 1, 2, \ldots$, are linearly independent. Orthogonalization and normalization yields the sequence $\rho_n(z)$ where

$$\rho_n(z) = M_n \begin{vmatrix} r_1(z), & \ldots, & r_n(z) \\ c_{11}, & \ldots, & c_{n1} \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ c_{1,n-1} & \ldots, & c_{n,n-1} \end{vmatrix}$$

and

$$M_n^{-1} = \delta_n \{ [c_{\nu k}]_1^n \cdot [c_{\nu k}]_1^{n-1} \}^{\frac{1}{2}}, \quad /\delta_n/ = 1;$$

employing again Lemma 3, yields

$$\rho_n(z) = \eta_n \cdot \frac{1}{1 - z_n z} \prod_1^{n-1} /z_\nu/ \frac{1 - \frac{z}{\bar{z}_\nu}}{1 - z z_\nu}$$

where $\eta_n$ is the normalizing factor.

In terms of the orthogonal sequence $\{\rho_n(z)\}$ the linear aggregate of best approximation in $H_2$ to a given function $f(z)$ can now be found by the standard method. We have

$$Q_n = \int_0^{2\pi} | f(e^{it}) - \sum_1^n c_\nu \rho_\nu(e^{it}) |^2 dt$$

$$= \int_0^{2\pi} | f(e^{it}) |^2 dt + \sum_1^n /c_\nu/^2 - 2R \sum_1^n c_\nu \int_0^{2\pi} \rho_\nu(e^{it}) \overline{f(e^{it})} dt \geq 0.$$

Now put

$$\int_0^{2\pi} \rho_\nu(e^{it}) \overline{f(e^{it})} dt = \gamma_\nu,$$

then

$$\sum_1^n /c_\nu - \gamma_\nu/^2 = \sum_1^n /c_\nu/^2 + \sum_1^n /\gamma_\nu/^2 - 2R \sum_1^n c_\nu \gamma_\nu,$$

and

$$Q_n = \sum_1^n /c_\nu - \gamma_\nu/^2 + \int_0^{2\pi} /f(e^{it} / dt - \sum_1^n /\gamma_\nu/^2$$

Hence $Q_n$ becomes minimal for variable $c_1, \ldots, c_n$, if

$$\bar{c}_\nu = \gamma_\nu, \qquad (\nu = 1, 2, \ldots, n).$$

We get

$$\min Q_n = \int_0^{2\pi} /f(e^{it})/^2 dt - \sum_1^n /\gamma_\nu/^2.$$

The development of $f(z)$ in terms of the sequence $\{\rho_n(z)\}$ is

$$f(z) \sim \sum_0^\infty \bar{\gamma}_n \rho_n(z). \tag{10.1}$$

If $\Sigma(1 - /z_n/) = \infty$ then the development converges uniformly to $f(z)$ in any circle $/z/ \leq r < 1$, and in the mean on $/z/ = 1$, whenever $f(z) \in H_2$, (cf. [3]). If $z = \bar{z}_k$, $k$ an integer, then

$$\rho_n(\bar{z}_k) = 0, \quad \text{for} \quad n = k+1,\ k+2, \ldots,$$

and

$$f(\bar{z}_k) = \sum_{\nu=1}^k \bar{\gamma}_\nu \rho_\nu(\bar{z}_k) = \sum_{\nu=1}^n \bar{\gamma}_\nu \rho_\nu(\bar{z}_k), \quad \text{for} \quad n \geq k.$$

It follows that the partial sum $S_n(z)$ of (10.1) takes on the values $f(\bar{z}_k)$ for $z = \bar{z}_k$, $k = 1, 2, \ldots, n$, hence it represents the interpolation for $f(z)$ at the given points $\bar{z}_1, \ldots, \bar{z}_n$.

For another proof cf. Malmquist [3], Takenaka [11] and Walsh [12], [13]; Walsh gave some more general results.

**11. Closure of triangular sequences.**

We can use the device of the previous sections to answer the following question:

Given a triangular system of constants in the unit circle $z_{n,1}, z_{n,2}, \ldots, z_{n,n}$, $n = 1, 2, 3, \ldots$; $/z_{n\nu}/ < 1$; under what condition can every function in $H_2$ be approximated by linear aggregates $\sum_{\nu=1}^n c_\nu/(1 - z_{n,\nu}\, z)$? We assume $z_{n,\nu} \neq z_{n,k}$ for $\nu \neq k$, $\nu, k = 1, 2, \ldots, n$; if $z_{n,\nu} = z_\nu$, indipendently of $n$, we get the previous case. We introduce auxillary constants $z_\mu$, $\mu = 1, 2, \ldots$, so that $/z_\mu/ < 1$, $z_\mu \neq z_{n\nu}$ for all $u, n, \nu$, and assume that the sequence $1/(1 - z_k z)$ is closed in $H_2$.

We now seek the minimum of the Hermitian form ($z_{n,0}$ stands for a fixed $z_\mu$)

$$Q\{u\} = \int_0^{2\pi} \left| \sum_{v=0}^n \frac{u_v}{1 - z z_{n\nu}} \right|^2 dt, \quad z = e^{it},\ u_0 = 1;$$

the minimum is

$$m_n = \frac{[c_{\nu k}]_0^n}{[c_{\nu k}]_1^n}$$

where now

$$c_{\nu k} = \frac{2\pi}{z_{n,\nu}(\bar{z}_{n,\nu}^{-1} - \bar{z}_{n,k})}.$$

We find, replacing in (9.2) $z_\nu$ by $z_{n,\nu}$,

$$m_n = \frac{2\pi}{1 - |z_\mu|^2} \prod_1^{\pi} \left\{ 1 - \frac{(1 - |z_\mu|^2)(1 - |z_{n,\nu}|^2)}{|1 - z_{n,\nu} \bar{z}_\mu|^2} \right\}.$$

We now must have $\lim_{n \to \infty} m_n = 0$, which is equivalent to $\sum_{\nu=1}^{n} (1 - |z_{n,\nu}|) \to \infty$ as $n \to \infty$. Our result is:

THEOREM 8 - *Given a triangular system* $\{z_{n,\nu}\}$ $\nu = 1, 2, \ldots, n$, $n = 1, 2, 3, \ldots$, $|z_{n,\nu}| < 1$, $z_{n,\nu} \neq z_{nk}$ *for* $\nu \neq k$. *In order that every function in* $H_2$ *can be approximated by linear aggregates* $\sum_{\nu=1}^{n} c_\nu/(1 - z_{n,\nu} z)$, *it is necessary and sufficient that*

$$\lim_{n \to \infty} \left\{ n - \sum_{\nu=1}^{n} |z_{n,\nu}| \right\} = \infty.$$

On writing $z_{n,\nu} = 1/\zeta_{n,\nu}$ the basic rational functions become $\zeta_{n,\nu}/(\zeta_{n,\nu} - z)$; this form was used by WALSH, who discussed more general approximation problems for such systems. (cf. [13], chapters VIII and IX; see also TAKENAKA, [11]).

**12. Closure in $C(0, \infty)$.**

We now discuss the closure of the sequence $\{1/(x^2 + z_\nu^2)\}_{1,\infty}$ in $C(0, \infty)$; we may replace $z_\nu^2$ by $\zeta_\nu$ and $x^2$ by $(1-t)/t$, $0 < t < 1$; we thus consider the sequence $t/(1 - t + \zeta_\nu t)$ for $0 \leq t \leq 1$. Replacing $1 - \zeta_\nu$ by $\gamma_\nu$ we get the sequence $t/(1 - \gamma_\nu t) = r_\nu(t)$, say. Note that $r_\nu(0) = 0$, $\nu = 1, 2, 3, \ldots$. Adjoining the constant $r_0(t) \equiv 1$, the sequence $\{r_\nu(t)\}_{0,\infty}$ will be closed in $C(0, 1)$ under certain conditions for $\gamma_\nu$. We employ the following criterion: the sequence $\{r_\nu(t)\}$ is closed in $C(0, 1)$ if the infinitely many equations

$$\int_0^1 r_\nu(t) d\psi(t) = 0, \qquad (\nu = 0, 1, 2, \ldots), \tag{12.1}$$

imply $\psi(t) \equiv 0$. Here $\psi(t)$ is any normalized function of bounded variation, $\psi(0) = 0$, $\psi(t) = \frac{1}{2}\{\psi(t+0) + \psi(t-0)\}$, $0 < t < 1$. Consider the function

$$F(z) = \int_0^1 \frac{t}{1 - zt} d\psi(t);$$

it is regular in the complex plane outside a cut along the real axis from 1 to $+\infty$. By (12.1).

$$\int_0^1 d\psi(t) = 0, \quad \text{thus} \quad \psi(1) = \psi(0) = 0, \tag{12.2}$$

and $F(\gamma_\nu) = 0$, $\nu = 1, 2, 3, \dots$. To employ an elementary theorem assume that the sequence $\{\gamma_\nu\}$ has a finite limit point outside the line $z \geq 1$; then $F(z) \equiv 0$.

For $/z/ < 1$

$$F(z) = \sum_0^\infty z^\nu \int_0^1 t^{\nu+1} d\psi(t),$$

hence in view of (12.2)

$$\int_0^1 t^\nu d\psi(t) = 0, \quad \text{for} \quad \nu = 0, 1, 2, \dots .$$

But the sequence $\{t^\nu\}_{0,\infty}$ is closed in $C(0, 1)$ hence $\psi(t) \equiv 0$. This yields the theorem (for a related result see [7]):

THEOREM 9. - *If the sequence* $\{\gamma_\nu\}$ *has a finite limit point, not on the line* $z \geq 1$, *then the sequence* $\{t/(1 - \gamma_\nu t\}$ *is a base for all continuous functions in* $[0, 1]$, *vanishing at* $t = 0$.

As a corollary we have the result:

THEOREM 9'. - *If the sequence* $\zeta_\nu$ *has a finite limit point outside the negative real axis, then the sequence* $\{(x^2 + \zeta_\nu)^{-1}\}$ *is a base of all continuous functions in* $[0, \infty]$, *vanishing at* $x = \infty$.

We now put $x^2 = u$, and use the formula

$$\frac{1}{u + \zeta} = \int_0^1 t^{u+\zeta-1} dt, \quad u \geq 0, \quad R\zeta > 0;$$

we employ Theorem 9' with a particular sequence $\{\zeta_\nu\}$, e. g. $\zeta_\nu = (\nu + 1)/\nu$, $\nu = 1, 2, 3, \dots$, and denote an arbitrary term of this sequence by $\zeta_0$; let $c_0 = 1$, and $c_1, c_2, \dots$, be arbitrary constants. Then

$$\sum_0^n \frac{c_\nu}{u + \zeta_\nu} = \int_0^1 \left( \sum_1^n c_\nu t^{u+\zeta_\nu-1} \right) dt,$$

hence

$$\sum_0^n \frac{c_v}{u + \zeta_\nu} \leq \int_0^1 \left| \sum_0^n c_\nu t^{\zeta_\nu - 1} \right| dt.$$

We have proved [5], that the sequence $\{t^{\zeta_\nu - 1}\}$ is closed in $L_2(0, 1)$ and *a fortiori* in $L_1(0, 1)$, if

$$R\zeta_\nu > \frac{1}{2} \quad \text{and} \quad \Sigma \frac{2R\zeta_\nu - 1}{1 + |\zeta_\nu - 1|^2} = \infty. \tag{12.3}$$

Thus, under this assumption the function $(u + \zeta_0)^{-1}$ belongs to the span of the sequence $\{(u + \zeta_\nu)^{-1}\}$. In view of Theorem 9′ we have proved the theorem :

THEOREM 9″ - *If* (12.3) *holds, then the sequence* $\{x^2 + \zeta_\nu\}_{1,\infty}$ *is a base of all continuous functions in* $[0, \infty]$, *vanishing at* $x = \infty$.

In view of Theorem 9′ the case of main interest is when $|\zeta_\nu| \to \infty$. In this case the condition (12 3) reduces to

$$R\zeta_\nu > \frac{1}{2} \quad \text{and} \quad \Sigma \frac{R\zeta_\nu}{|\zeta_\nu|^2} = \infty, \quad \text{or} \quad \Sigma \frac{1}{\zeta_\nu} = \infty.$$

The sequence $\{x/(x^2 + z_\nu)\}$ can be discussed in a similar, though less simple manner.

We replace $x^2$ by $(1 - t)/t$, and $z_\nu^2$ by $1 - \zeta_\nu$; then

$$\frac{x}{x^2 + z_\nu^2} = \sqrt{\frac{1 - t}{t}} \frac{1}{\left(\frac{1 - t}{t}\right) + 1 - \zeta_\nu} = \frac{\sqrt{t(1 - t)}}{1 - \zeta_\nu t}.$$

The function

$$F(z) = \int_0^1 \frac{\sqrt{t(1 - t)}}{1 - zt} d\psi(t)$$

is regular if $z$ is not on the line $z \geq 1$. For $|z| < 1$

$$F(z) = \sum_0^\infty z \int_0^1 t^\nu \sqrt{t(1 - t)} d\psi(t).$$

If

$$\int_0^1 \frac{\sqrt{t(1 - t)}}{1 - \zeta_\nu t} d\psi(t) = F(\zeta_\nu) = 0,$$

and if the sequence $\{\zeta_\nu\}$ has a finite limit point outside the line $z \geq 1$, then $F(z) \equiv 0$, hence

$$\int_0^1 t^{\nu + \frac{1}{2}} \sqrt{1 - t}\, d\psi(t) = 0, \quad \nu = 0, 1, 2, \ldots .$$

But the sequence 1, $\sqrt{1-t}$, $t^{\nu+\frac{1}{2}}\sqrt{1-t}$, $\nu=0, 1, 2, \dots$, is closed in $C(0, 1)$ ([9], § 8); hence the sequence

$$1, \quad \frac{x}{\sqrt{1+x^2}}, \quad \left\{\frac{x}{x^2+1-\zeta_\nu}\right\}_{1,\infty}$$

is closed in $C(0, \infty)$. We have now proved

THEOREM 10. - *If the sequence $z_\nu$ has a finite limit point outside the negative real axis, then the sequence*

$$1, \quad \frac{x}{\sqrt{1+x^2}}, \quad \left\{\frac{x}{x^2+z_\nu}\right\}_{1,\infty} \text{ is closed in } C(0, \infty).$$

To eliminate the condition of a finite limit point, we employ the formula

$$\int_0^\infty te^{-zt}\sin xt dt = x/(x^2+z^2), \quad Rz>0, \quad x\geq 0.$$

Thus

$$x\sum_0^n c_\nu/(x^2+z_\nu^2) = \int_0^\infty t\sin xt\left(\sum_0^n c_\nu e^{-z_\nu t}\right)dt,$$

and

$$x\left|\sum_0^n c_\nu/(x^2+z_\nu^2)\right| \leq \int_0^\infty t\left|\sum_0^n c_\nu e^{-z_\nu t}\right|dt \leq \int_0^1 \left|\Sigma\, c_\nu{}^{z_\nu-1}\right| dt.$$

Concluding as before we get the theorem:

THEOREM 10'. - *If $Rz_\nu > \frac{1}{2}$ and*

$$\Sigma\,\frac{2Rz_\nu-1}{1+(z_\nu-1)^2}=\infty,$$

*then the sequence*

$$1, \quad \frac{x}{1+x^2}, \quad \left\{\frac{x}{x^2+z_\nu^2}\right\}_{1,\infty}$$

*is closed in $C(0, \infty)$.*

We finally remark that closure propertiers of the sequence $x/(x^2+z_\nu^2)^2$ can be discussed in a similar manner.

*Closing note.* - In a recent paper H. Kober (A note on approximation by rational functions, « Proc. of the Edinburgh Mathematical Society », Series 2, vol. 7, 123-133) gave a proof of Theorem 6; moreover, he proved closure in $K$, on employing and extending a result due to J. E. Littlewood. Here is an account of his proof:

Consider the function

$$F(z) = \int_0^{2\pi} \frac{d\psi(t)}{1 - ze^{-it}},$$

and assume at first that $\psi(t)$ is real and monotone. Then

$$F(z) = \int_0^{2\pi} \frac{1 - R\bar{z}e^{-it}}{/1 - ze^{-it}/^2} d\psi(t) > 0, \quad \text{for } /z/ < 1;$$

thus the function $w = F(z)$ maps the circular region $/z/ < 1$ onto the halfplane $Rw > 0$, or on part of it, while the function

$$y(z) = \frac{c(1+z)}{1-z}, \qquad c = \int_0^{2\pi} d\psi(t),$$

maps $/z/ < 1$ on the whole halfplane. Moreover,

$$F(0) = \int_0^{2\pi} d\psi(t) = c = y(0);$$

hence $F(z)$ is subordinate to $y(z)$, (cf. W. ROGOSINSKI, « Mathematische Zeitschrift », 17, p. 262). It folloows from LITTLEWOOD' s theorem (« Proc. London Mathematical Society » (2), vol. 23, Theorem 2) that

$$M_\lambda(\rho;\ F) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |\ F(\rho e^{it})\ |^\lambda dt \right\}^{1/\lambda} \leq M_\lambda(\rho\ ;\ y),$$

where $0 < \rho < 1$, $0 < \lambda < 1$; but $M_\lambda(\rho;\ y) < A_\lambda$, where $A_\lambda$ is a constant, depending only on $\lambda$. A similar inequality now follows for any $\psi(t)$ of bounded variation.

Let now $F(\bar{z}_\nu) = 0$, $\nu = 1, 2, 3, \ldots$, and $\Sigma\,(1 - /z_\nu/) = \infty$, then by a theoreom of F. RIESZ (« Mathematische Zeitschrift », 18, p. 87-95), $F(z) \equiv 0$. But

$$F(z) = \sum_0^\infty z^\nu \int_0^{2\lambda} e^{-i\nu t} d\psi(t),$$

hence

$$\int_0^{2\pi} e^{-i\nu t} d\psi(t) = 0, \quad \nu = 0,\ 1,\ 2, \ldots.$$

The theorem now follows from the closure of the sequence $\{ e^{i\nu t} \}$ in $K$.

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