The theorem of Morera in several variables.

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Sunto. *È dato dal seguente capoverso.*

The classical theorem of MORERA has been generalized from one to several variables by VOLTERRA and SEVERI ⁽¹⁾, and SEVERI's proof is very brief and elegant. We will re-prove this theorem (our Theorem 7) by a chain of reasoning which, to the contrary, will be lengthy and elaborate, but our actual contributions are the intermediary results and the method of proof, rather than the final proposition itself.

In the Euclidean E_{2k} of k complex variables $z^j = x^j + iy^j$, $j = 1, \dots, k$, we will distinguish between functions which have power series in $z^j - z_0^j$ locally; and such ones which have power series in $x^j - x_0^j$, $y^j - y_0^j$, or, what is the same, in $z^j - z_0^j$, $\bar{z}^j - \bar{z}_0^j$ locally. The first ones we will call « holomorphic »; and the second ones we will term « real-analytic », even if their values themselves will be complex numbers, as a rule. Also, if a function has power series in $\bar{z}^j - \bar{z}_0^j$ only, we will call it « anti-holomorphic ».

1. General Cycles.

In the complex (x, y) -plane we formally introduce the variables

$$z = x + iy, \quad \bar{z} = x - iy$$

and we denote an arbitrary function $F(x, y)$ by $f(z, \bar{z})$ in accordance with a well-known symbolism ⁽²⁾. The classical theorem of MORERA then states that if such a function is defined and continuous in a domain D and if we then have

$$\int_{\bar{B}_1} f dz = 0$$

⁽¹⁾ F. SEVERI, *Sur une propriété fondamentale des fonctions analytiques de plusieurs variables complexes*, Comptes Rendus, Paris, 192, (1931).

⁽²⁾ Compare, for instance, S. BOCHNER and W. T. MARTIN, *Several complex variables*, (1948), p. 37.

over every 1-cycle B_1 , then f is holomorphic:

$$\frac{\partial f}{\partial \bar{z}} = 0,$$

and, of course, dually, if we have

$$\int_{B_1} f d\bar{z} = 0$$

then f is anti-holomorphic

$$\frac{\partial f}{\partial z} = 0.$$

Now, perhaps the most natural generalization of this theorem is as follows.

THEOREM 1 ⁽³⁾. - *In the space E_{2k} of the k complex variables $z^\alpha = x^\alpha + iy^\alpha$, $\alpha = 1, \dots, k$, if*

$$(1) \quad f = f(z^1, \dots, z^k, \bar{z}^1, \dots, \bar{z}^k)$$

is continuous in a domain D , and if we then have

$$(2) \quad \int_{B_{p+q}} f dz^{\alpha_1} \dots dz^{\alpha_p} d\bar{z}^{\beta_1} \dots d\bar{z}^{\beta_q} = 0$$

over every $(p+q)$ -dimensional cycle B_{p+q} , where

$$(3) \quad 1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_p \leq k, \quad 1 \leq \beta_1 < \beta_2 < \dots < \beta_q \leq k$$

are fixed indices with

$$(4) \quad 0 \leq p, \quad 0 \leq q, \quad 1 \leq p+q \leq 2k-1,$$

then f is holomorphic in those z^β , which are different from the given ones, and anti-holomorphic in those z^α which are different from the given ones; that is

$$(5) \quad \frac{\partial f}{\partial z^\alpha} = 0, \quad \alpha \neq \alpha_1, \dots, \alpha_p$$

$$(6) \quad \frac{\partial f}{\partial \bar{z}^\beta} = 0, \quad \beta \neq \beta_1, \dots, \beta_q.$$

⁽³⁾ For the propositions of which Theorem 1 is the converse see B. SEGREG, *Sull'estensione della formula integrale di Cauchy e sui residui degli integrali n -pli nella teoria delle funzioni di n variabili complesse*, « Atti I Congresso U. M. I. », Bologna, (1937), and E. MARTINELLI, *Formule Integrali e topologia nella teoria delle funzioni di più variabili complesse*, « Acta Pontif. Acad. Scient. », IX, p. 235-250.

Proof. We first note that, for instance,

$$(7) \quad \int_{B_k} f dz^1 \dots dz^k = 0$$

implies that f is holomorphic in all z^z , that is

$$\frac{\partial f}{\partial \bar{z}^\beta} = 0, \quad \beta = 1, \dots, k;$$

and that, for instance, for $k \geq 2$,

$$\int_{B_2} f dz^1 d\bar{z}^1 = 0$$

implies that f is « arbitrary » continuous in $z^1, \bar{z}^1$, that is in x^1, y^1 but that it is constant in all the other variables, that is

$$\frac{\partial f}{\partial z^\gamma} = 0 = \frac{\partial f}{\partial \bar{z}^\gamma}, \quad \gamma = 2, \dots, k,$$

and hence

$$\frac{\partial F}{\partial x^\gamma} = 0 = \frac{\partial F}{\partial y^\gamma} = 0, \quad \gamma = 2, \dots, k.$$

For the proof we note that if f has also continuous first derivatives in all x^γ, y^γ , then the theorem is a direct consequence of STOKES' theorem. In fact, if B_{p+q} bounds a chain B_{p+q+1} then (2) implies

$$(8) \quad \int_{B_{p+q+1}} df dz^{z_1} \dots dz^{z_p} d\bar{z}^{\beta_1} \dots d\bar{z}^{\beta_q} = 0,$$

and since B_{p+q+1} can be an arbitrary simplex, this implies

$$(9) \quad df dz^{z_1} \dots dz^{z_p} d\bar{z}^{\beta_1} \dots d\bar{z}^{\beta_q} = 0,$$

where

$$df = \frac{\partial f}{\partial z^1} dz^1 + \dots + \frac{\partial f}{\partial z^k} dz^k + \frac{\partial f}{\partial \bar{z}^1} d\bar{z}^1 + \dots + \frac{\partial f}{\partial \bar{z}^k} d\bar{z}^k = 0.$$

However, for fixed $\alpha_\rho, \rho = 1, \dots, p; \beta_\sigma, \sigma = 1, \dots, q;$ and variable $\alpha \neq \alpha_\rho, \beta \neq \beta_\sigma$ the $(k-p) + (k-q)$ monomials

$$dz^{z_1} dz^{z_2} \dots dz^{z_p} d\bar{z}^{\beta_1} \dots d\bar{z}^{\beta_q}, \quad d\bar{z}^{\beta_1} dz^{z_1} \dots dz^{z_p} d\bar{z}^{\beta_1} \dots d\bar{z}^{\beta_q}$$

are all linearly independent, and this implies relation (5), (6).

Next, if it is only known that f is continuous, we will have to apply an approximation process and the underlying approximation theorem is as follows ⁽⁴⁾.

In (real) Euclidean $E_n: (t^1, \dots, t^n)$ it is possible to construct a sequence of functions $\{\varphi_\mu(t^1, \dots, t^n)\}$, $\mu = 1, 2, 3, \dots$, having the following properties.

(i) Each $\varphi_\mu(t)$ is defined in all of E_n but vanishes outside a sphere

$$K_\mu: (t^1)^2 + \dots + (t^n)^2 \leq \delta_\mu^2,$$

where

$$\lim_{\mu \rightarrow \infty} \delta_\mu = 0.$$

(ii) Each $\varphi_\mu(t) \in C^\infty$ in E_n , that is, it has continuous partial derivatives of every mixed order.

(iii) $\varphi_\mu(t) \geq 0$, and

$$(iv) \quad \int_{E_n} \varphi_\mu(t) dv_t = \int_{K_\mu} \varphi_\mu(t) dv_t = 1.$$

Now, if $F(x^1, \dots, x^n)$ is defined and continuous and also bounded in a domain D , then we supplement it by values 0 outside D , and we set up the expressions

$$\begin{aligned} F_\mu(x^1, \dots, x^n) &= \int_{E_n} F(x^1 + t^1, \dots, x^n + t^n) \varphi_\mu(t^1, \dots, t^n) dv_t \\ &= \int_{K_\mu} F(x^1 + t^1, \dots, x^n + t^n) \varphi_\mu(t^1, \dots, t^n) dv_t. \end{aligned}$$

for $x = (x^v)$ in D ; $\mu = 1, 2, \dots$. These expressions are approximating functions in the following sense ⁽⁵⁾.

LEMMA 1. - (i) Each $F_\mu(x) \in C^\infty$ in E_n , and if $D' \subset D$ is a subdomain whose closure is likewise contained in D , $\bar{D}' \subset D$, then

$$\lim_{\mu \rightarrow \infty} F_\mu(x) = F(x)$$

uniformly in every D' .

(ii) If, moreover, $F \in C^r$ in D , $r \geq 1$, then we also have

$$\lim_{\mu \rightarrow \infty} \frac{\partial^{q_1 + \dots + q_n} F_\mu}{\partial (x^1)^{q_1} \dots \partial (x^n)^{q_n}} = \frac{\partial^{q_1 + \dots + q_n} F}{\partial (x^1)^{q_1} \dots \partial (x^n)^{q_n}}$$

for $0 \leq q_1 + \dots + q_n \leq r$; again uniformly in D' .

⁽⁴⁾ S. BOCHNER and W. T. MARTIN, loc. cit., p. 109-13.

⁽⁵⁾ BOCHNER and MARTIN, loc. cit., p. 36.

Returning to our Theorem 1, if (2) holds for « all » cycles B_{p+q} , and if we pick a fixed such cycle, then it also holds for « translated » cycles $B_{p+q} - (t)$ for small $|t|$. However in the relation (2), instead of translating the cycle, by $-t$, we can also translate the argument of the function by $+t$. That is, if we denote the function (1) for the moment by

$$F(x^1, \dots, x^n) \quad (n = 2k),$$

then (2) also holds for the function

$$F(x^1 + t^1, \dots, x^n + t^n)$$

for sufficiently small t . For given B_{p+q} , we can then multiply the resulting relation by $\varphi_\mu(t)$ and integrate over K_μ , for μ sufficiently large, and thus, we obtain

$$\int_{B_{p+q}} f_\mu(z, \bar{z}) dz^{\alpha_1} \dots dz^{\alpha_p} d\bar{z}^{\beta_1} \dots d\bar{z}^{\beta_q} = 0,$$

for $\mu \geq \mu(B_{p+q})$. But $f_\mu \in C^1$ and thus we have our conclusions

$$\frac{\partial f_\mu}{\partial z^\alpha} = 0, \quad \alpha \neq \alpha_1, \dots, \alpha_p$$

$$\frac{\partial f_\mu}{\partial \bar{z}^\beta} = 0, \quad \beta \neq \beta_1, \dots, \beta_q,$$

for $\mu \geq \mu(D')$, for every subdomain D' as before. However, in D' , f_μ converges uniformly towards f ; and by WEIERSTRASS' theorem on the uniform limit of holomorphic functions, which also holds for anti-holomorphic functions, we now obtain the relations (5), (6) themselves, q. e. d.

2. Rigidly-placed products of one-cycles.

In the z^j -plane, $j = 1, \dots, k$, we take a one-dimensional cycle B_j^j which bounds a two-chain B_2^j and we consider the products

$$(10) \quad B_{2k} = B_2^1 \times B_2^2 \times \dots \times B_2^k$$

and

$$(11) \quad B_k = B_1^1 \times B_1^2 \times \dots \times B_1^k.$$

For a given domain D , if (10) is contained in D , then we will say that (11) is a rigidly-placed products of one-cycles.

A « repeated » application of STOKES' theorem gives formally

$$\begin{aligned} \int_{B_k} f dz^1 \dots dz^k &= \int_{B_{2k}} \frac{\partial^k f}{\partial z^1 \dots \partial z^k} d\bar{z}^1 dz^1 d\bar{z}^2 dz^2 \dots d\bar{z}^k dz^k \\ &= (2i)^k \int_{B_{2k}} \frac{\partial^k f}{\partial \bar{z}^1 \dots \partial \bar{z}^k} dV_{x,y} \end{aligned}$$

where

$$dV_{x,v} = dx^1 dy^1 dx^2 dy^2 \dots dx^k dy^k$$

is the ordinary volume-element, and, again with the aid of lemma 1, the following conclusion can be drawn.

THEOREM 2. - *If $f \in C^k$, and*

$$(12) \quad \frac{\partial^k f}{\partial z^1 \dots \partial \bar{z}^k} = 0,$$

then we have

$$(13) \quad \int_{B_k} f dz^1 \dots dz^k = 0$$

for every rigidly-placed product (11).

Conversely, if (13) holds for all such cycles in D , and if $f \in C^k$, then f satisfies the equation (12), and if $f \in C^0$, then f is a uniform limit in D' of solutions of (12)

If a function (1) is real-analytic and if we expand it in a power-series of $z^\alpha, \bar{z}^\alpha$ in the neighborhood of the origin, say, then it is a solution of (12) if and only if the expansion has the form

$$(14) \quad \sum_{j=1}^k \sum_{(n)} f_{n_1 \dots n_{j-1} n_{j+1} \dots n_k}(z)(\bar{z}^1)^{n_1} \dots (\bar{z}^k)^{n_k}$$

in which the coefficients $f_{,,}$ are holomorphic in their variables. But we do not know whether any solution of (12) must automatically be real-analytic, and thus no absolute assertion can be formulated.

3. Affinely-placed products of one-cycles.

We will say that a k -cycle B_k in D is an affinely-placed product of one-cycles if there is a non-singular holomorphic transformation with constant coefficients

$$(15) \quad z^j = \sum_{l=1}^k a_{jl} w^l + a_j,$$

such that the image of B_k in the w -space is a rigidly-placed product of one-cycles as previously defined.

THEOREM 3. - *If a function has the form*

$$(16) \quad f = \sum_{0 \leq p_1 + \dots + p_k \leq k-1} a_{p_1 \dots p_k}(z)(\bar{z}^1)^{p_1} \dots (\bar{z}^k)^{p_k}$$

with holomorphic coefficients

$$(17) \quad a_p(z) = a_{p_1 \dots p_k}(z^1, \dots, z^k),$$

then we obviously have

$$(18) \quad \frac{\partial^k f}{\partial z^{\alpha_1} \dots \partial \bar{z}^{\alpha_k}} = 0.$$

for all possible combinations

$$(19) \quad 1 \leq \alpha_1 \leq k, \quad 1 \leq \alpha_2 \leq k, \dots, \quad 1 \leq \alpha_k \leq k,$$

and we have

$$\int_{B_k} f dz^1 dz^2 \dots dz^k = 0$$

for every affinely placed B_k .

Proof. If we introduce the linear operators

$$(20) \quad L_j = \sum_{i=1}^k \bar{a}_{ij} \frac{\partial}{\partial \bar{z}^i}$$

with any constant coefficients a_{ij} then (18) implies

$$(20') \quad L_1 L_2 \dots L_k f = 0,$$

and this means that we have property (12) even after any affine transformation (15). And the conclusion now follows from Theorem 2.

THEOREM 4. - *If f continuous in D_1 and if we have*

$$(21) \quad \int_{B_k} f dz^1 \dots dz^k = 0$$

for every affinely-placed B_k , then f is real-analytic, and we have (18), and it can be represented as an expression (16).

Proof. Assume first that it is already known that f is real-analytic. It is then in particular C^k , and (21) implies by Theorem 2 that we have (20') for all possible operators (20), and hence (18). But f being also real analytic, we can expand it in the neighborhood of every point z_0^α in a power-series in the quantities $z^z - z_0^\alpha$, $\bar{z}^z - \bar{z}_0^\alpha$, and due to (18) the expansion must be of the form

$$(22) \quad \sum_{0 \leq p_1 + \dots + p_k \leq k-1} a_p(z; z_0) (\bar{z}^1 - \bar{z}_0^1)^{p_1} \dots (\bar{z}^k - \bar{z}_0^k)^{p_k}$$

where $a_p(z; z_0)$ is holomorphic in the z^z in the neighborhood of the given point z_0^z . Now, (22) is a polynomial of degree $\leq k-1$ in the $\bar{z}^z - \bar{z}_0^\alpha$, and as such it is also a polynomial in the $\bar{z}^z$ themselves:

$$\sum_{0 \leq p_1 + \dots + p_k \leq k-1} b_p(z; z_0) (\bar{z}_0^1)^{p_1} \dots (\bar{z}_0^k)^{p_k}$$

with coefficients $b_p(z; z_0)$ which are holomorphic in the z^z , in a neighborhood of z_0^z . But by a known uniqueness theorem these coefficients must be identical with one another in the intersection of two such neighborhoods, and hence there is a representation (16) jointly for the entire domain D .

But now we must eliminate the assumption that f is real-analytic, and we will perform this elimination in the following two steps, where the second step will again be based on our process of approximation as ensuing from lemma 1.

Step 1. If $f \in C^{k+1}$ and (18) holds for all combinations (19), then f is real-analytic.

Step 2. If we are given a sequence of function f_μ in D , if each f_μ has a representation (16) with holomorphic coefficients (17) and if the sequence is uniformly convergent,

$$\lim_{\mu \rightarrow \infty} f_\mu = f,$$

then f itself can likewise be represented in the form (16).

Now Step 1 obviously follows from the following more general theorem.

THEOREM 5. - *If for some $l \geq 1$, $l \leq k$, we are given a function $f \in C^{l+1}$ and if the expressions*

$$(23) \quad g_{\alpha_1 \dots \alpha_l}(z, \bar{z}) = \frac{\partial^l f}{\partial \bar{z}^{\alpha_1} \dots \partial \bar{z}^{\alpha_l}}$$

for all possible combinations $\alpha_1, \dots, \alpha_l$ are real-analytic (for instance, if they are all 0), then the function f itself is real-analytic.

Proof. We will need the following lemma.

LEMMA 2. - *If we are given formal power series*

$$(24) \quad g_\alpha(t) = \sum_{p_j \geq 0} a_{\alpha p_1 \dots p_k} t_1^{p_1} \dots t_k^{p_k}, \quad \alpha = 1, \dots, k,$$

in symbols $t_1, \dots, t_k$ with coefficients from an arbitrary vector field over rational numbers, and if the integrability conditions

$$(25) \quad \frac{\partial g_\alpha}{\partial t_\beta} = \frac{\partial g_\beta}{\partial t_\alpha}$$

are fulfilled formally, then the series without constant term

$$(26) \quad G(t) = \sum_{p_1 + \dots + p_k > 0} a_{p_1 \dots p_k} t_1^{p_1} \dots t_k^{p_k}$$

with the coefficients

$$(27) \quad a_{p_1 \dots p_k} = \frac{\sum a_{\alpha p_1 \dots (p_\alpha - 1) \dots p_k}}{p_1 + \dots + p_k}$$

is such that we have

$$(28) \quad \frac{\partial G}{\partial t_\alpha} = g_\alpha(t), \quad (\alpha = 1, \dots, k).$$

Proof of Lemma 2 ⁽⁶⁾. If (26) is to satisfy (28) then on putting

$$(29) \quad H(\lambda_1, \dots, \lambda_k; s) = G(\lambda_1 s, \dots, \lambda_k s)$$

we must have

$$\begin{aligned} \frac{\partial H}{\partial s} &= \sum_{\alpha} \frac{\partial G}{\partial t_{\alpha}} \lambda_{\alpha} = \sum_{\alpha} g_{\alpha} \lambda_{\alpha} \\ (30) \quad &= \sum_{p_1 + \dots + p_k > 0} \frac{1}{s} (p_1 + \dots + p_k) a_{p_1 \dots p_k} (\lambda_1 s)^{p_1} \dots (\lambda_k s)^{p_k} \\ &= \sum_{p, \alpha} \lambda_{\alpha} a_{\alpha p_1, \dots, p_k} (\lambda_1 s)^{p_1} \dots (\lambda_k s)^{p_k} \end{aligned}$$

so that (27) is the only possible choice of coefficients to that purpose. Conversely, if we now set

$$w_{\alpha}(\lambda; s) = \frac{\partial H}{\partial \lambda_{\alpha}} - s h_{\alpha}$$

where

$$h_{\alpha}(\lambda; s) = g_{\alpha}(\lambda_1 s, \dots, \lambda_k s)$$

then we obtain

$$\frac{\partial w_{\alpha}}{\partial s} = \frac{\partial}{\partial \lambda_{\alpha}} \left(\frac{\partial H}{\partial s} \right) - h_{\alpha} - s \sum_{\beta} \frac{\partial g_{\alpha}}{\partial t_{\beta}} \lambda_{\beta}$$

and by (30) this is

$$h_{\alpha} + s \sum_{\beta} \lambda_{\beta} \frac{\partial g_{\beta}}{\partial t_{\alpha}} - h_{\alpha} - s \sum_{\beta} \frac{\partial g_{\alpha}}{\partial t_{\beta}} \lambda_{\beta} = 0,$$

by (25). Therefore, if (25) holds, then $w_{\alpha}(\lambda; s)$ must be independent of the formal variable s . But $w_{\alpha}(\lambda; 0) = 0$, and therefore

$$\frac{\partial H}{\partial \lambda_{\alpha}} = s h_{\alpha},$$

or what is the same

$$\frac{\partial G}{\partial t_{\alpha}} = g_{\alpha},$$

as claimed.

Turning now to Theorem 5, it suffices to prove the theorem in the neighborhood of a point, and we will prove it by induction on l . For $l=1$, we have $f \in C^2$, and (23) is

$$g_{\alpha}(z, \bar{z}) = \frac{\partial f}{\partial z^{\alpha}}, \quad (\alpha = 1, \dots, k).$$

⁽⁶⁾ Compare our paper: *Formal Lie groups*, « Annals of Math. », 47, (1946), 192-201.

Differentiating once more we obtain

$$\frac{\partial g_\alpha}{\partial \bar{z}^\beta} = \frac{\partial g_\beta}{\partial \bar{z}^\alpha},$$

and by lemma 2 there exists locally a real-analytic function $G(z, \bar{z})$ such that

$$\frac{\partial G}{\partial \bar{z}^\alpha} = g_\alpha(z, \bar{z}).$$

For the difference $\varphi = f - G$ we obtain

$$\frac{\partial \varphi}{\partial \bar{z}^\alpha} = 0,$$

thus φ is holomorphic, and in particular real-analytic, and $f = \varphi + G$ is likewise so.

For $l > 1$ if we put

$$h = \frac{\partial^{l-1} f}{\partial \bar{z}^{\alpha_1} \dots \partial \bar{z}^{\alpha_{l-1}}}$$

for any fixed combination $\alpha_1, \dots, \alpha_{l-1}$, then by assumption of theorem 5

$$\frac{\partial h}{\partial \bar{z}^\alpha} = g_{\alpha_1 \dots \alpha_{l-1} \alpha}, \quad (\alpha = 1, \dots, k),$$

is real-analytic, and by what we have just proven, the function h is real-analytic itself. Therefore the expressions

$$h_{\alpha_1 \dots \alpha_{l-1}} = \frac{\partial^{l-1} f}{\partial \bar{z}^{\alpha_1} \dots \partial \bar{z}^{\alpha_{l-1}}}$$

are real-analytic and if the theorem was known to hold now for $l-1$, then f is real-analytic, and thus theorem holds for l itself, as claimed.

We will make a remark to Theorem 5 for its own sake.

THEOREM 6. - *In Theorem 5, instead of assuming $f \in C^{l+1}$, it is also sufficient to assume $f \in C^l$ and to add the explicit assumption that the integrability condition*

$$(31) \quad \frac{\partial g_{\alpha_1 \dots \alpha_{l-1} \alpha}}{\partial \bar{z}^\beta} = \frac{\partial g_{\alpha_1 \dots \alpha_{l-1} \beta}}{\partial \bar{z}^\alpha}$$

shall be valid.

Now, (31) is certainly valid if $g_{\alpha_1 \dots \alpha_{l-1} \alpha} = 0$, as is especially the case for equations (18) with $l = k$, so that in step 1 it would have been sufficient to assume $f \in C^k$ instead of $f \in C^{k+1}$. But for our context itself this is of no difference.

Next, for the proof of step 2, it again suffices to deal with a neighborhood of a point, and if we expand each $a_p(z)$ in a power series, then it is not hard to see that the following lemma, of some interest in itself, is sufficient for the purpose. We will again in this lemma, as in lemma 2, place all indices as subscripts, instead of as superscripts.

LEMMA 3. - *In the domain*

$$(32) \quad 0 \leq |z_j| \leq 2, \quad (j = 1, \dots, k),$$

we consider the absolutely convergent expansion

$$f(z, \bar{z}) = \sum_{p, q} a_{p_1 \dots p_k; q_1 \dots q_k} z_1^{p_1} \dots z_k^{p_k} \bar{z}_1^{q_1} \dots \bar{z}_k^{q_k},$$

in which

$$(33) \quad 0 \leq p_j < \infty, \quad 0 \leq q_j \leq l,$$

for some finite l , and we assume that we have

$$(34) \quad |f(z, \bar{z})| < 1$$

for all points in (32).

There exist a constant $L = L(k, l)$ such that

$$\sum_{p, q} |a_{p, q}| |z_1|^{p_1+q_1} \dots |z_k|^{p_k+q_k} \leq L$$

in

$$(35) \quad 0 \leq |z_j| \leq \frac{1}{4}.$$

Proof. If we put $z_j = \rho_j e^{i\varphi_j}$, $0 \leq \varphi_j < 2\pi$, $\rho_j > 0$, then

$$f = \sum_{p, q} a_{p, q} \rho_1^{p_1+q_1} \dots \rho_k^{p_k+q_k} e^{i[(p_1-q_1)\varphi_1 + \dots + (p_k-q_k)\varphi_k]},$$

and if we put $p_j = q_j + r_j$ then this is

$$\sum_r \left(\sum_q a_{q+r, q} \rho_1^{2q_1+r_1} \dots \rho_k^{2q_k+r_k} \right) e^{i(r_1\varphi_1 + \dots + r_k\varphi_k)}.$$

The latter expansion is a multiple FOURIER series, and in such one each coefficient is maximized by the value of the function, so that, due to (34), we have

$$\left| \sum_q a_{q+r, q} \rho_1^{2q_1} \dots \rho_k^{2q_k} \right| \leq \rho_1^{-r_1} \dots \rho_k^{-r_k}$$

for $0 < \rho_j < 2$. Therefore, for

$$(36) \quad \frac{1}{2} \leq \rho_j \leq 2$$

we have

$$\left| \sum_q a_{q+r, q} \rho_1^{2q_1} \dots \rho_k^{2q_k} \right| \leq 2^{(r_1 + \dots + r_k)}.$$

Next, there exist a number $L_0 = L_0(k, l)$ such that if we have

$$(37) \quad \left| \sum_{q_j=0}^l b_{q_1 \dots q_k} \rho_1^{2q_1} \dots \rho_k^{2q_k} \right| \leq 1$$

in (36), then

$$\sum_{q_j=0}^l |b_{q_1 \dots q_k}| \leq L_0.$$

In fact, if such a constant would not exist, there would be a sequence of systems of coefficients $\{c_{q_1 \dots q_k}^\mu\}$, $\mu = 1, \dots$, $0 \leq q_j \leq l$, such that

$$\sum_{q_j=0}^l |c_{q_1 \dots q_k}^\mu| = 1,$$

and yet

$$\left| \sum_{q_j=0}^l c_{q_1 \dots q_k}^\mu \rho_1^{2q_1} \dots \rho_k^{2q_k} \right| \leq \varepsilon_\mu$$

for $\rho_1, \dots, \rho_k$ in (36), with $\varepsilon_\mu \rightarrow 0$. Due to the finiteness of l , we would then have coefficients $c_{q_1 \dots q_k}$ not all 0, for which

$$\sum_{q_j=0}^l c_{q_1 \dots q_k} \rho_1^{2q_1} \dots \rho_k^{2q_k} = 0$$

for ρ_j in (36), which for polynomials in ρ_j is impossible.

Therefore, (37) implies

$$|a_{q+r, q}| \leq 2^{(r_1 + \dots + r_k)} L_0.$$

and therefore for $p_j \geq 0$, $q_j \geq 0$ we have

$$|a_{p, q}| \leq 2^{p_1 + \dots + p_k} L_0$$

and hence

$$\sum_{p, q} |a_{p, q}| 4^{-(p_1 + q_1 + \dots + p_k + q_k)} \leq k^l L_0(k, l) = L_1(k, l),$$

which proves the lemma.

4. Holomorphically-placed products of one-cycles.

A k -cycle B_k in D will be called a holomorphically-placed product if it is contained in some domain D' in D such that a suitable one-one holomorphic mapping

$$z^j = \varphi_j(w^1, \dots, w^k)$$

of D' transforms B_k into a cycle which is rigidly-placed in the w -space.

THEOREM 7. - *If f is continuous and if we have (21) for all holomorphically-placed cycles, then f is holomorphic.*

Proof. First of all, by theorem 4, f can be represented by an expression (16) and uniquely so. Now, we claim that (16) must be independent of z^a . In fact, if it would depend on $\bar{z}^1$, say, then the holomorphic transformation

$$z^1 = w^1 + (w^1)^k, \quad z^2 = w^2, \dots, z^k = w^k$$

would transform (16) into a polynomial

$$b_{p_1 \dots p_k}(w)(\bar{w}^1)^{p_1} \dots (\bar{w}^k)^{p_k}$$

which is of a degree $\geq k$, but by the uniqueness of (16) in every system of holomorphic variables, this is impossible. Hence the theorem.