Functions with Generalized Gradient and Generalized Surfaces (*).

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1. **Introduction.** - DE GIORGI [3a] has generalized the notion of gradient for functions $f(x)$ of k real variables by associating with each f for which a certain number $I(f)$ is finite a vector measure Φ. I call such a vector measure exact, and also say that Φ is the generalized gradient of f. In terms of distribution theory, f is a function whose first order derivatives are measures; see KRICKEBERG [5b].

In [4a] L. C. YOUNG and the author defined a general notion of boundary and closed surface, within the framework of YOUNG's theory of generalized surfaces. A $k-1$ dimensional generalized surface L in k-space defines a vector measure Φ, termed closed if L is closed. It is shown that for measures Φ with compact support closed is equivalent to exact (Theorem 2) (1). I then investigate the extreme points of the set of all f vanishing outside a fixed cube K for which $I(f) \leq N$. Each extreme point is a multiple of a characteristic function, but not conversely (Theorems 3 and 4). Finally, any f with $I(f)$ finite is represented as a mixture of characteristic functions. The proofs of all these theorems rely on machinery developed in [4].

Functions with generalized gradient are closely related not only to distribution theory (and DE RHAM's theory of currents as well) but also with work of CESARI, GOFFMAN, and many other mathematicians. See the succeeding papers by KRICKEBERG [5b] and PAUC [6b]. For functions with compact support (the only case considered here) the condition $I(f)$ finite is equivalent to the requirement that f be of bounded variation in CESARI's sense on any cube whose interior contains the support of f. For continuous functions this notion of bounded variation agrees with TONELLI's.

2. Functions with generalized gradient.

NOTATION AND DEFINITIONS. - $x = (x_1, \ldots, x_k)$ denotes a generic point in euclidean k-space $R_k (k \geq 2)$, and K a given cube in R_k. K remains *fixed* throughout. m denotes k-dimensional LEBESGUE measure: $f(x)$ an integrable

(*) This research was supported by the United States Air Force through the Office of Scientific Researchof Air Research and Development Command.

(1) This result has been obtained independently by KRICKEBERG [5b] using distribution theory.

function which *vanishes outside* K. $f_E(x)$ is the characteristic function of the set E. $\Phi = (\Phi_1, \ldots, \Phi_k)$ denotes a vector valued RADON measure, with support in K and finite total variation $V(\Phi)$. $|\Phi|$ is the total variation measure associated with Φ.

I say that $f(x)$ has *generalized gradient* Φ if, for every continuously differentiable function $g(x)$:

$$\int_K f(x) \operatorname{grad} g(x) \, dm = -\int_K g(x) \, d\Phi, \tag{2.1}$$

Let us define, with DE GIORGI [3a],

$$I(f) = \lim_{\lambda \to 0} \int_{R_k} |\operatorname{grad} f_\lambda(x)| \, dm, \tag{2.2}$$

where $f_\lambda(x)$ is an appropriate gaussian average of $f(x)$ defined in [3a]. DE GIORGI showed [3a, Theorems 1 and 3] under different assumptions on f (bounded, not necessarily vanishing outside a cube) that f has a generalized gradiend if and only if $I(f)$ is finite; moreover, if Φ is the generalized gradient of f, then $I(f) = V(\Phi)$. The proofs of these results also apply with the present assumptions. Let $\mathfrak{F}$ denote the set of all f with $I(f)$ finite, and $\mathfrak{F}_N$ the subset for which $I(f) \leq N$.

I call the measurable set $E \subset K$ a CACCIOPPOLI *set* if $I(f_E)$ is finite. CACCIOPPOLI [1] considered the case when E is an open set whose frontier has measure 0, and assumed that E is the limit in measure of polyhedra π_n whose frontiers have bounded areas. In [3a] DE GIORGI extended these ideas to include arbitrary measurable sets, and showed that the approximation property assumed by CACCIOPPOLI is equivalent to the condition that $I(f_E)$ be finite. Following DE GIORGI, I write $P(E)$, and say *perimeter* of E, to stand for $I(f_E)$. Let $\mathfrak{E}$ denote the set of all CACCIOPPOLI sets, embedded in $\mathfrak{F}$ in the obvious way, and $\mathfrak{E}_N = \mathfrak{F}_N \cap \mathfrak{E}$.

CONTINUITY AND COMPACTNESS. - I give $\mathfrak{F}$ the ordinary l_1 topology. $\mathfrak{E}$ is a closed subset of $\mathfrak{F}$, and on $\mathfrak{E}$ the l_1 topology reduces to convergence in measure of sets E. The space of measures Φ is given the weak topology. Let u denote the mapping carrying f onto the negative of its generalized gradient Φ. It is immediate from (2.1) that u is one-one between $\mathfrak{F}$ and $u\mathfrak{F}$, functions f differing in a null set being identified (from its definition $I(f)$ is not affected by changes of $f(x)$ on null sets). u is linear and continuous. We shall need the fact that u^{-1} is continuous on $u\mathfrak{F}_N$ for every N. This follows from biuniqueness and the following:

LEMMA 1. - *$\mathfrak{F}_N$ is compact in the l_1 topology for every $N > 0$.*

PROOF. - DE GIORGI showed that $\mathfrak{E}_N$ is compact [3b, Theorem 1]. We shall modify his reasoning to cover the present case. It is easy to show from the definition of $I(f)$ that $\mathfrak{F}_N$ is complete. To prove compactness it will then suffice to show that $\mathfrak{F}_N$ is totally bounded; i. e., given any $\varepsilon > 0$, $\mathfrak{F}_N$ can be covered by finitely many spheres of radius ε. Choose $\rho < \varepsilon/2k^{\frac{1}{2}}N$, and let $T_1, \ldots, T_h, \ldots$ be disjoint half-open cubes of side ρ covering k-space. Given $f \in \mathfrak{F}_N$ define $\bar{f}$ by the elementary averaging process:

$$\bar{f}(x) = \rho^{-k} \int_{T_h} f(t)dm, \quad x \in T_h, \; h = 1, 2, \ldots . \tag{2.3}$$

$\bar{f}$ is constant in T_h for each h and vanishes if T_h does not intersect K. The l_1 distance of f and $\bar{f}$ does not exceed $\varepsilon/2$. The proof of this follows DE GIORGI's reasoning [3b, Lemmas 1 and 2]. In it the analogous estimate for the gaussian averages f_λ is obtained first, and then λ tends to 0.

Let a denote the side length of K. By an easy calculation involving f_λ and passage to the limit on λ we find, since $f(x) = 0$ outside K and $I(f) \leq N$, that

$$\int_K | f(x) | dm \leq aN; \tag{2.4}$$

and therefore that the same inequality holds for $\bar{f}$. The set of functions which take a constant value in each cube T_h, vanish for cubes not meeting K, and have l_1 norm not exceeding aN is totally bounded; hence, this set is covered by a finite number of spheres of radius $\varepsilon/2$. Thus, $\mathfrak{F}_N$ is covered by a finite number of spheres of radius ε.

3. **Generalized Surfaces.** – In this section we collect some known fundamental properties of generalized surfaces, together with useful consequences of them. K is the same fixed cube as above. Let $F(x, \Theta)$ denote a continuous function of (x, Θ), $x = (x_1, \ldots, x_k) \in K$, $\Theta = (\Theta_1, \ldots, \Theta_k) \in R_k$ satisfying the homogeneity condition:

$$F(x, p\Theta) = pF(x, \Theta), \; p \geq 0. \tag{3.1}$$

Ω denotes the space of all such F. A ($k-1$ dimensional) *generalized surface L* is simply any non-negative linear functional on the space Ω (2). A generalized surface L is *closed* if $L(F) = 0$ for all exact F[4a]. For L to be closed it suffices that $L(F) = 0$ for all continuously differentiable exact F, and every such F is the form:

$$F(x, \Theta) = G(x) \cdot \Theta = \sum_{i=1}^{k} g_i(x)\Theta_i, \text{ where}$$
$$G(x) = [g_1(x), \ldots, g_k(x)] \text{ and } \operatorname{div} G(x) = 0. \tag{3.2}$$

(2) What we have defined would be called in the usage of [4] or [7a] generalized surface situated in the cube K.

Let S_k denote the surface of the unit sphere in Θ-space. By the RIESZ representation theorem, every generalized surface L is represented by a RADON measure γ on $K \times S_k$, and conversely. Let $\mu(A) = \gamma(A \times S_k)$, all BOREL sets $A \subset K$. By known theorems there exists, for μ-almost all x, a measure α_x on S_k with $\alpha_x(S_k) = 1$ (conditional distribution for fixed x) such that $\alpha_x(B)$ is BOREL measurable in x for every BOREL set $B \subset S_k$ and:

$$(3.3) \qquad L(F) = \int_K \int_{S_k} F(x, \Theta) d\alpha_x d\mu, \quad \text{all} \quad F \in \Omega.$$

α_x is unique up to a μ-null set. The *norm* $a(L)$ equals $\mu(K)$, finite. $a(L)$ is additive and weakly continuous in the space of generalized surfaces. Let $\Theta(x)$ denote the center of gravity of α_x. Then $|\Theta(x)| \leq 1$ with equality if and only if α_x is carried by the single point $\Theta(x) \in S_k$.

Let t denote the mapping carryinm L into the vector measure Φ defined by:

$$(3.4) \qquad \Phi(A) = \int_A \Theta(x) d\mu, \quad \text{all BOREL sets } A.$$

Φ is the *track* (measure) of L. By known theorems:

$$(3.5) \qquad |\Phi|(A) = \int_A |\Theta(x)| \, d\mu, \quad \text{all BOREL sets } A;$$

in particular,

$$(3.6) \qquad V(\Phi) = |\Phi|(K) = \int_K |\Theta(x)| \, d\mu.$$

Given Φ, the set $t^{-1} \Phi$ is infinite. However, there is just one element of this set of least norm. I call it $\tau\Phi$. Using the RADON-NIKODYM theorem there is a function $\Theta_0(x)$, uniquely determined and of absolute value 1 except for an $|\Phi|$-null set, such that:

$$(3.7) \qquad \Phi(A) = \int_A \Theta_0(x) d|\Phi|, \quad \text{all BOREL sets } A.$$

Then $\tau\Phi$ has the representation:

$$(3.8) \qquad \tau\Phi(F) = \int_K F[x, \Theta_0(x)] d|\Phi|, \quad \text{all } F \in \Omega.$$

From these remarks follows:

LEMMA 2. – *If* $\Phi = tL$ *then* $V(\Phi) \leq a(L)$ *with equality if and only if* $L = \tau\Phi$.

If $\Phi = tL$ then $\Theta_0(x) = |\Theta(x)|^{-1}\Theta(x)$, $|\Phi|$-almost everywhere. Using the homogeneity condition (3.1) and (3.5):

$$\tau t L(F) = \int_K F[x, \Theta(x)] d\mu, \text{ all } F \in \Omega \text{ (}^3\text{)}. \tag{3.9}$$

The space of generalized surfaces, as well as the space of vector measures Φ, is given the weak topology. This topology for generalized surfaces can be metrized by the McSHANE distance [7a]. If $tL = \Phi$ and F has the form $G(x) \cdot \Theta$:

$$L(F) = \int_K G(x) \cdot d\Phi. \tag{3.10}$$

Therefore, the operation t is linear and continuous. Although τ is neither linear nor continuous, the following statements hold:

LEMMA 3. - *Let $L_0 = pL_1 + qL_2$, where L_0 has the least norm property and p, q are real positive. Then L_1 and L_2 have the least norm property.*

PROOF. - Let $\Phi_i = tL_i$, $i = 0, 1, 2$. Then $\Phi_0 = p\Phi_1 + q\Phi_2$, $V(\Phi_i) \leq a(L_i)$, and

$$a(L_0) = pa(L_1) + qa(L_2) \geq pV(\Phi_1) + qV(\Phi_2) \geq V(\Phi_0) = a(L_0).$$

Hence, $a(L_i) = V(\Phi_i)$ for $i = 1, 2$.

$F(x, \Theta)$ is called *positive semi-regular* if F is convex in Θ for each fixed x.

THEOREM 1. - *Let Φ_n be a sequence of vector measures supported in K, such that $V(\Phi_n)$ is bounded and Φ_n tends weakly to a limit Φ_0. Let $L_n = \tau\Phi_n$, $n = 0, 1, 2, \ldots$.*

(a) *For every positive semi-regular $F \in \Omega$, $L_0(F) \leq \liminf_{n \to \infty} L_n(F)$.*

(b) *If in addition $V(\Phi_n)$ tends to $V(\Phi_0)$, then L_n tends weakly to L_0; i. e., $L_0(F) = \lim_{n \to \infty} L_n(F)$ for every $F \in \Omega$.*

PROOF. - It suffices to establish that every subsequence contains a further subsequence for which the conclusions hold. Since $a(L_n) = V(\Phi_n)$ is bounded it will therefore suffice to consider the case when L_n tends weakly to a limit L^*. Let us write, according to (3.3),

$$L^*(F) = \int_K \int_{S_k} F(x, \Theta) d\alpha_x^* d\mu^*, \text{ all } F \in \Omega, \tag{3.11}$$

and $\Theta^*(x)$ the center of gravity of α_x^*. Since $\Phi_n = tL_n$ tends to tL^* by continuity of t, and also to Φ_0, $tL^* = \Phi_0$.

(3) In view of (3.9) the present definition of track differs only in form from that given in [4b].

If F is positive semi-regular, the inner integral in (3.11) is no less than $F[x, \Theta^*(x)]$; therefore, $L^*(F) \geq L_0(F)$ by (3.9). Since the norm is weakly continuous, $\lim a(L_n) = a(L^*)$. Thus, under the hypotheses of (b), $a(L^*) = V(\Phi_0) = a(L_0)$. By Lemma 2, $L^* = L_0$.

4. **Closed if and only if exact.** - I call Φ *exact* if Φ is the generalized gradient of some $f \in \mathfrak{F}$; and Φ *closed* if the integral of $G(x) \cdot d\Phi$ over K is 0 for every continuously differentiable $G(x)$ for which $\operatorname{div} G(x) = 0$. For Φ to be closed it is necessary and sufficient that Φ be the track of at least one closed generalized surface. Following [4a], a generalized surface L is an *irreducible closed polyhedron* if there is a geometric polyhedron Q in k-space R_k separating R_k into exactly two components of which Q is the common boundary, and a constant $\varepsilon = \pm 1$ called *orientation* of L, such that if $\Theta(x)$ denotes the exterior unit normal to Q (defined except on edges) and H, $k-1$ dimensional measure in R_k:

$$L(F) = \int_Q F[x, \varepsilon\Theta(x)]dH, \text{ all } F \in \Omega. \tag{4,1}$$

If π is the bounded component of $R_k - Q$, then by GAUSS' theorem:

$$uf^{\pi} = \varepsilon tL. \tag{4.2}$$

In Theorem 2 we must, strictly speaking, assume $k = 2$ or 3, since the proof depends on an approximation theorem for closed generalized surfaces proved only for these values of k[7b] [4a]. On the other hand, there seems to be no difficulty in extending the approximation theorem to arbitrary values of k. When we refer to [4a] or [4b] below, it is understood that if $k = 2$ the corresponding result of [7b] is used instead.

THEOREM 2. - *Φ is closed if and only if Φ is exact.*

PROOF. - Exact implies closed is immediate. Let Φ be closed and $L_0 = \tau\Phi$. By (4.2) $tL \in u\mathfrak{F}$ for every irreducible closed polyhedron L. Since t is linear and $u\mathfrak{F}$ is a vector space the same is true if $L = \Sigma p_i L_i$ (finite sum), where p_i is real positive and L_i is irreducible closed. By [4a, Theorem (1.1)] the closed generalized surface L_0 is the weak limit of L_n, where L_n is such a positive linear combination of irreducible closed polyhedra. Let N be an upper bound for $a(L_n)$. Then $tL_n \in u\mathfrak{F}_N$, since $I(u^{-1}tL_n) = V(tL_n) \leq a(L_n) \leq N$. $u\mathfrak{F}_N$ is compact by Lemma 1 and continuity of u; and tL_n tends to tL_0 by continuity of t. Therefore, $tL_0 \in u\mathfrak{F}_N \subset u\mathfrak{F}$.

From Theorem 2 and earlier remarks it is easy to show:

COROLLARY 1. - *The operation $u^{-1}t$ is a linear, continuous mapping of the space of closed generalized surfaces onto $\mathfrak{F}$.*

COROLLARY 2. - *Every $f \in \mathfrak{F}$ can be written $f = f^+ - f^-$ where f^+, f^- are non-negative and $I(f) = I(f^+) + I(f^-)$.*

PROOF. - Let $L = \tau uf$ (closed generalized surface of least norm corresponding to f). As above, L is the weak limit of L_n, where $L_n = \sum_i p_{in} L_{in}$, $p_{in} > 0$, and L_{in} is irreducible closed. Let L_n^+, L_n^- be the sum of those terms for which L_{in} is positively, negatively oriented, respectively. The functions $f_n^+ = u^{-1} t L_n^+$ and $f_n^- = -u^{-1} t L_n^-$ are non-negative. For a subsequence of n, L_n^+ and L_n^- tend weakly to limits L^+ and L^-; and $L = L^+ + L^-$. By Corollary 1, f_n^+, f_n^- tend in l_1, to $f^+ = u^{-1} t L^+$, $f^- = u^{-1} t L^-$, respectively, and $f_n^+ - f_n^-$ tends to f, as n describes this subsequence. Thus $f = f^+ - f^-$. By Lemma 3, L^+ and L^- have the least norm property. Therefore

$$I(f) = a(L) = a(L^+) + a(L^-) = I(f^+) + I(f^-).$$

5. **Caccioppoli sets and extreme points.** - The sets $\mathfrak{F}_N$ are convex and compact in l_1 for every $N > 0$. Therefore, by the KREIN-MILMAN theorem each $\mathfrak{F}_N$ is the convex closure of its extreme points. In fact, using a recent beautiful result of CHOQUET [2] every element of $\mathfrak{F}_N$ has an integral representation in terms of a measure carried by the set of extreme points. We shall return to this question in § 6; at present we seek to identify the extreme points. A complete answer is given only for $k = 2$. As in § 4 we shall use theorems for generalized surfaces proved only for $k = 2$, 3, but which presumably remain true for higher values of k.

LEMMA 4. - *f is an extreme point of $\mathfrak{F}_N$ if and only if $L = \tau uf$ is an extreme point of the set $\mathcal{G}_N$ of all closed generalized surfaces of norm N.*

PROOF. - Necessity is an easy consequence of the definitions and Lemma 3. To prove sufficiency, suppose $L = \tau uf$ is an extreme point of $\mathcal{G}_N$. Let $f = qf_1 + (1-q)f_2$ with $0 < q < 1$ and $I(f_i) \leq N$, $i = 1, 2$. Since $I(f) = a(L) = N$, we have by convexity of I, $I(f_i) = N$ for $i = 1, 2$. Let $L_i = \tau uf_i$. Then $qL_1 + (1-q)L_2$ has track uf and norm equal to N. Therefore $qL_1 + (1-q)L_2 = L$, since L has track uf and least norm property. This implies $L_i = L$, and hence $f_i = f$, $i = 1, 2$.

REMARK. - The requirement that L be an extreme point of $\mathcal{G}_N$ is equivalent to the statement that L is of norm N and basic closed in the sense of [4a]. L is basic closed if L is closed and $L = L_1 + L_2$ with L_1, L_2 closed implies $L_1 = pL$ where $0 \leq p \leq 1$.

THEOREM 3. - *If f is an extreme point of $\mathfrak{F}_N$, then there exists a Caccioppoli set E of positive perimeter $P(E)$ such that*

$$f_E = \pm N^{-1} P(E) f. \tag{5.1}$$

PROOF. - Let $L = \tau uf$. A generalized surface is *singular* if its track is 0.

Since $V(tL) = a(L) = N > 0$ L is not singular. Therefore, by Lemma 4 and a slight refinement of [4a, Theorem (1.3)] proved in [4a, pp. 476-477], see also [4b, (5.2)], there exist $p > 0$ and a sequence L_n of irreducible closed polyhedra such that pL is the weak limit of L_n. Let Q_n be the carrier of L_n and π_n the bounded component of $R_k - Q_n$. By (4.2)

$$(5.2) \qquad f_{\pi_n} = \varepsilon u^{-1} t L_n ;$$

we may assume either $\varepsilon = -1$ for all n or $\varepsilon = -1$ for all n. The right side of (5.2) tends in l_1 to εpf. Since $\mathfrak{S}$ is a closed subset of $\mathfrak{F}$, this implies $\varepsilon pf = f_E$ for some CACCIOPPOLI set E. Since $I(f) = N$ we conclude that $p = N^{-1}P(E)$.

THEOREM 4. *If $k = 2$, a necessary and sufficient condition that f be an extreme point of $\mathfrak{F}_N$ is that there exist a rectifiable simple closed curve Q contained in K such that, if E denotes the interior of Q and λ the length of Q, then*

$$(5.3) \qquad f_E = \pm N^{-1} \lambda f.$$

PROOF. - To show necessity let up proceed as immediately above. Q_n is now a simple closed polygonal curve, with bounded length. The weak limit $L_0 (= pL)$ of L_n is a closed generalized curve [7c] with the least norm property. Hence, by the representation theorem for generalized curves [7c]; L_0 is a rectifiable closed curve: i. e., there is a lipschitzian vector function $x(t)$ on $(0, 1)$ with $x(0) = x(1)$ and $|x'(t)| > 0$ almost everywhere, such that:

$$(5.4) \qquad L_0(F) = \int_0^1 F[x(t), x'(t)]dt, \text{ all } F \in \Omega.$$

Consider any pair t_1, t_2 with $x(t_1) = x(t_2)$, and let L', L'' denote respectively the closed curves represented on the interval (t_1, t_2) and the complementary part of $(0, 1)$. $L_0 = L' + L''$; since L_0 is basic closed, both L' and L'' are scalar multiples of L_0. It follows that $x(t)$ represents on $(0, 1)$ a simple closed curve Q described a certain number n times in the same sense. Now $x(t)$ is the uniform limit of representations of the polygons Q_n [7c]. Since Q_n is simple closed the topological index of any point $x \notin Q_n$ with respect to Q_n is 0, 1, or -1. Passing to the limit, the same is true for any point $x \notin Q$ with respect to Q. This implies $n = 1$; moreover the index of every point in the interior E of Q is ± 1, which implies that π_n tends to E in measure. Since $P(E) = \lambda$ we find as in the proof of (5.1) that $f_E = \pm N^{-1} \lambda f$.

To prove the converse, assume for definitenesses that the $+$ sign holds in (5.3) and that $N = 1$. Let $x(t)$ be a parametric representation of the simple closed curve Q, positively oriented. Then $L_0 = \tau u f_E$ is given by (5.4). Let Γ denote the set of all closed L situated in Q (i. e., such that the corresponding measure μ has support in Q). $\mathfrak{S}_\lambda \cap \Gamma$ is convex and weakly compact.

Let L be an extreme point of $\mathfrak{S}_\lambda \cap \Gamma$. If $L = qL_1 + (1-q)L_2$ with $0 < q < 1$, then L_1 and L_2 are situated in Q. Therefore, L is also an extreme point of $\mathfrak{S}_\lambda$. Suppose L is not singular. Then, as above, pL is the limit of simple closed polygons L_n of bounded length for some $p > 0$. By results of [7c] pL is a closed generalized curve with the same track as an ordinary curve L' having Lipschitzian representation $y(t)$ which is the uniform limit of representations $y_n(t)$ of L_n. Since $y(t) \in Q$ for all t and every curve $y_n(t)$ is simple closed, there exists ε with value 0, 1, or -1 such that the topological index of every point of E with respect to the curve $y(t)$ is ε. By GREEN's theorem and passage to the limit, $tL' = \varepsilon tL_0$. Since $a(L) = a(L_0) = \lambda$ and L_0 has least norm property, $p \geq 1$ ($\varepsilon = \pm 1$ since $tL' = ptL \neq 0$).

Thus, every extreme point of $\mathfrak{S}_\lambda \cap \Gamma$, and consequently every element of $\mathfrak{S}_\lambda \cap \Gamma$, has track rtL_0 where $|r| \leq 1$. It follows from this and least norm property for L_0 that L_0 is an extreme point of $\mathfrak{S}_\lambda \cap \Gamma$, and so of $\mathfrak{S}_\lambda$. By Lemma 4 this completes the proof.

REMARK. - Considerable information is available about the structure of CACCIOPPOLI sets; see [3b], [4b, Theorems 3 and 5], [4c, Theorem 4]. For $k \geq 3$ it is an open question what stronger statements can be made in case the characteristic function of a CACCIOPPOLI set defines an extreme point.

6. **Mixtures.** - In this section we represent any function f with generalized gradient by an integral over the set $\mathfrak{E}$ of CACCIOPPOLI sets; in the terminology of [7b] and [4b], f is a *mixture* of $\mathfrak{E}$. The result obtained is similar to [4b, Theorem 2], but is stronger in a sense in that we represent f pointwise rather than merely as an abstract vector integral.

For $\nu = 1, 2, \ldots$ cover K by a net N_ν of half-open cubes T of side ν^{-1}. For each ν, $T \in N_\nu$, $x \in T \cap K$, and $E \in \mathfrak{E}$, let

$$(6.1) \qquad w_\nu(x, E) = \frac{m(E \cap T)}{m(T)}.$$

Now w_ν is continuous on $T \times \mathfrak{E}$ for each T, consequently BOREL measurable in (x, E). Then

$$(6.2) \qquad W(x, E) = \limsup_{\nu \to \infty} w_\nu(x, E)$$

is also BOREL measurable in (x, E). For given E, $W(x, E) = f_E(x)$ almost everywhere in K. $P(E)$ is lower semi-continuous.

THEOREM 5. - *Given $f_0 \in \mathfrak{F}$ there exists a signed* RADON *measure ω_0 on $\mathfrak{E}$ of total variation equal to $I(f_0)$ such that, for almost all $x \in K$:*

$$(6.3) \qquad f_0(x) = \int_{\mathfrak{E}} \frac{W(x, E)}{P(E)} \, d\omega_0 .$$

PROOF. - For simplicity take $I(f_0) = 1$. Let Γ be the weak closure of the set of all L of the form $[a(L')]^{-1} L'$ where L' is an irreducible closed polyhedron. Every $L \in \Gamma$ has norm 1. The set Γ_s of all singular $L \in \Gamma$ is weakly closed. By [4b, (5.2)] $\Gamma - \Gamma_s$ is the union of disjoint sets Γ^+ and Γ^- such that if $L \in \Gamma^+$ $(L \in \Gamma^-)$ then there exists $p > 0$ such that pL is the weak limit of a sequence of positively (negatively) oriented irreducible closed polyhedra. Moreover, if $L_n \in \Gamma^+$ tends to $L \in \Gamma^+$ then the corresponding numbers p_n are bounded away from 0 and ∞.

Let $L_0 = \tau u f_0$. By the approximation theorem of [4a] cited previously L_0 is the weak limit of convex combinations of Γ. Then since Γ is compact there exists a RADON measure ω on Γ with $\omega(\Gamma) = 1$ such that:

$$L_0(F) = \int_\Gamma L(F) d\omega, \quad \text{all } F \in \Omega. \tag{6.4}$$

Since L_0 has the least norm property, so do ω-almost all L by [4b, (4.4)] (extension of Lemma 3 above). Therefore, if Γ_1^+, Γ_1^- denote the set of $L \in \Gamma^+$, Γ^- respectively with least norm property, then $\omega(\Gamma_1^+) + \omega(\Gamma_1^-) = 1$. As in the proof of Theorem 3, to each $L \in \Gamma_1^+$ corresponds a set $E \in \mathfrak{G}$ denoted by $\sigma(L)$ such that $P(E)u^{-1}tL = f_E$. The mapping σ is continuous on Γ_1^+.

Let the measure ω_0^+ on $\mathfrak{G}$ be defined by $\omega_0^+(U) = \omega(\sigma^{-1}U)$. By a known theorem we have:

$$\int_{\Gamma_1^+} \psi[\sigma(L)] d\omega = \int_{\mathfrak{G}} \psi(f_E) d\omega_0^+ \tag{6.5}$$

for every ω_0^+-measurable function $\psi(f)$ for which either integral in (6.5) exists. Let us take, for given continuously differentiable $g(x)$:

$$\psi(f) = \frac{1}{I(f)} \int_K f(x) \frac{\partial g}{\partial x_1} dm.$$

If $F(x, \theta) = g(x)\theta_1$ (6.5) becomes:

$$\int_{\Gamma_1^+} L(F) d\omega = \int_{\mathfrak{G}} \frac{1}{P(E)} \int_K W(x, E) \frac{\partial g}{\partial x_1} dm \, d\omega_0^+. \tag{6.6}$$

Taking $g(x) = x_1$ we see that $[P(E)]^{-1} W(x, E)$ is $m \times \omega_0^+$ integrable.

Similary we define a measure ω_0^- on Γ_1^- such that, for all such F:

$$\int_{\Gamma_1^+} L(F) d\omega = - \int_{\mathfrak{G}} \frac{1}{P(E)} \int_K W(x, E) \frac{\partial g}{\partial x_1} dm \, d\omega_0^-. \tag{6.7}$$

Let $\omega_0 = \omega_0^+ - \omega_0^-$. Using (6.4), (6.6), (6.7) and FUBINI's theorem we have:

$$L_0(F) = \int_K \frac{\partial g}{\partial x_1}\left(\int_{\mathcal{E}} \frac{W(x, E)}{P(E)}\, d\omega_0\right) dm. \tag{6.8}$$

The left side of (6.8) is also equal to the integral over K of $f_0\, \partial g/\partial x_1$. Since g is arbitrary, (6.3) must hold almost everywhere in K.

Clearly the total variation of ω_0 is ≤ 1 with equality if and only if ω_0^+ and ω_0^- form the JORDAN decomposition of ω_0. That ω_0^+ and ω_0^- form the JORDAN decomposition follows from the least norm property of L_0. We omit the details.

REMARK. - Using CHOQUET's theorem [2] one may require in (6.4) that the extreme points of $\mathcal{E}_1$ have ω-measure 1. With this $[P(E)]^{-1} f_E$ is an extreme point of $\mathcal{F}_1$ for ω_0-almost all E.

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