Contributions to the Spectral Theory for Nonlinear Operators in Banach Spaces (*) (**).

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Sunto. – *Si introduce una definizione di spettro $\sigma(f)$ per applicazioni continue definite in uno spazio di Banach. Tale definizione coincide con quella classica nel caso in cui f sia lineare e continua. Alcuni dei risultati più noti della teoria spettrale lineare vengono estesi al caso non lineare. In particolare si dimostra che $\sigma(f)$ è chiuso e che la sua frontiera è contenuta nello spettro puntuale approssimato $\sigma_\pi(f)$.*

1. – Introduction.

Our main task here is to give a notion of spectrum for nonlinear maps defined on Banach spaces in such a way to preserve as much properties of the spectra for linear operators as possible. In particular, we shall introduce the spectrum for nonlinear maps such that when applied to linear operators it gives exactly the usual spectrum of the linear theory.

This programm will be accomplished in several steps.

The second section of this paper contains most of the notations and definitions to be used in the sequel. It also contains some preliminary results. The majority of them are well-known and are therefore presented without proofs.

In the third section we start to buil up the machinery that will lead us eventually to the definition of spectrum for nonlinear maps. Given any continuous map $f: E \to F$ from a Banach space E into a Banach space F, we define the following three extended real numbers

$$d(x) = \liminf_{\|x\|\to\infty} \frac{\|f(x)\|}{\|x\|},$$

$$\alpha(f) = \inf\{k \geqslant 0: \alpha(f(A)) \leqslant k\alpha(A) \text{ for any bounded } A \subset E\},$$

$$\beta(f) = \sup\{k \geqslant 0: \alpha(f(A)) \geqslant k\alpha(A) \text{ for any bounded } A \subset E\},$$

where α stands for the *Kuratowski measure of noncompactness* (see Section 2). The main properties of $d(f)$, $\alpha(f)$ and $\beta(f)$ are given. It is of independent interest that a

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continuous linear operator $L: E \to F$ is Fredholm if and only if $\beta(L)$ and $\beta(L^*)$ are both positive, where L^* is the adjoint of L.

The fourth section is entirely dedicated to the study of strong surjections that are defined as follows. A continuous map f from a Banach space E into a Banach space F is called a *strong surjection* if the equation $f(x) = h(x)$ has a solution for any continuous map $h: E \to F$ such that $\overline{h(E)}$ is compact. The following result holds.

PROPOSITION 4.1.1. – *Let $f: E \to F$ be a continuous map. Assume that there exist a Banach space G and a continuous map $g: G \to E$ such that $f \circ g$ is a strong surjection. Then f is a strong surjection.*

A Continuation Principle for strong surjections is also given (see Proposition 4.1.3).

In the second part of section 4 we give two coincidence theorems. One of them in particular (see Theorem 4.2.1) will be the essential tool in proving the closedness of spectra for nonlinear operators. It should be remarked that Theorem 4.2.1 is a generalization of Schauder's classical fixed point theorem (it also extends G. DARBO's [3] and B. N. SADOVSKIJ's [22] fixed point theorems). The second coincidence theorem (see Theorem 4.2.2) is an extension of the well-known Banach Contraction Principle.

In the fifth section we introduce the notion of stably-solvable maps and investigate some of their properties. A continuous map $f: E \to F$, where E and F are Banach, is said to be *stably-solvable* if the equation

$$f(x) = h(x)$$

has a solution for any continuous and compact map $h: E \to F$ such that

$$|h| = \limsup_{\|x\| \to \infty} \frac{\|h(x)\|}{\|x\|} = 0 \, .$$

Stably-solvable maps and strong surjections are closely related. In fact, the following proposition holds

PROPOSITION 5.1.1. – *Let $f: E \to F$ be such that $d(f) > 0$. Then the following conditions are equivalent:*

(*a*) *the equation $f(x) = h(x)$ has a solution $x \in E$ for any compact map $h: E \to F$ with bounded support;*

(*b*) *f is a strong surjection;*

(*c*) *f is stably-solvable.*

In the context of linear operators, stably-salvable maps are characterized as follows.

Theorem 5.1.1. – *Let $L: E \to F$ be linear and bounded. Then L is stably-solvable if and only if it is onto.*

In the second part of Section 5 we state a Continuation Principle for stably-solvable maps.

Section 6 is devoted to the study of regular maps by means of which we shall define the spectrum for nonlinear maps. Let $f: E \to F$ be continuous, then f is said to be *regular* if it is stably-solvable and, moreover, $d(f)$ and $\beta(f)$ are both positive. For linear operators we have

Proposition 6.1.2. – *Let $L: E \to F$ be linear and bounded. Then L is regular if and only if L is an isomorphism.*

The following result will be of importance in the nonlinear spectral theory.

Proposition 6.1.3. – *Let $f: E \to F$ be regular and let $g: E \to F$ be such that $\alpha(g) < \beta(f)$, $|g| < d(f)$. Then $f + g$ is regular.*

In the second part of Section 6 we restrict our attention to finite dimensional spaces and give a characterization of regular maps by means of homotopy classes of maps defined on spheres.

In the third part of Section 6 we take up the problem of characterizing essential compact vector fields using the concept of regular maps. We recall that a nonvanishing compact vector field $f: S \to E_0$ from the unit sphere S of a Banach space E into a subspace E_0 is called *essential* if any extension of f to a compact vector field $g: D \to E_0$ to the unit ball D of E vanishes at some point $x_0 \in D$.

In this context we have the following

Proposition 6.3.1. – *Let $f: S \to E_0$ be a nonvanishing compact vector field. Then f is essential if and only if $\bar{f}$ is regular where $\bar{f}(x) = \|x\| f(x/\|x\|)$ if $x \neq 0$ and $\bar{f}(0) = 0$.*

Section 7 represents a key step in developing our nonlinear spectral theory. In the first part we show that the set $\sigma(E, F)$ consisting of all non regular maps from a Banach space E into a Banach space F is a closed subset of the vector space $C(E, F)$ or all continuous maps from E into F endowed with a suitable topology. This extends the well-known fact that the set of all linear isomorphisms is open in $L(E, F)$.

In the second part we decompose $\sigma(E, F)$ as follows:

$$\sigma(E, F) = \sigma_\pi(E, F) \cup \sigma_\delta(E, F) ,$$

where

$$\sigma_\delta(E, F) = \{f \in C(E, F): f \text{ is not stably-solvable}\} ,$$

and

$$\sigma_\pi(E, F) = \{f \in C(E, F): d(f) = 0 \quad \text{or } \beta(f) = 0\} .$$

It turns out that $\sigma_\pi(E, F)$ is closed. Moreover, the boundary $\partial\sigma(E, F)$ of $\sigma(E, F)$ is contained in $\sigma_\pi(E, F)$ (see Theorem 7.2.1). This result turns out to be very important in nonlinear spectral theory.

The third part of Section 7 opens with a Continuation Principle for regular maps. A consequence of this principle (see Corollary 7.3.1) has, to a certain extent surprising, applications to the theory of monotone operators. More precisely, the following result holds.

PROPOSITION 7.3.2. – *Let $f: E \to E$ be monotone, coercive and proper. Assume, moreover, that f is locally α-Lipschitz sending bounded sets into bounded sets. Then f is a strong surjection.*

We recall that a map $f: E \to E$ is called *monotone* if $\operatorname{Re} < f(x) - f(y),\ z' > \geqslant 0$ for all $x, y \in E$ and some $z' \in J(x - y)$, where $J: E \to E^*$ is the duality map and $\langle \cdot, \cdot \rangle$ is the pairing between E and its dual E^*.

The following well-known result (see J. SCHAUDER [23]) is a direct consequence of Corollary 7.3.1.

PROPOSITION 7.3.3. – *Let L_0, L_1: $E \to F$ be bounded linear operators. Assume that*

(*a*) *L_0 is an isomorphism,*

(*b*) *there exists a real number $k > 0$ such that $\|x\| \leqslant k\|H(x, t)\|$ for any $x \in E$ and $t \in [0, 1]$, where $H(x, t) = L_0(x) + t\big(L_1(x) - L_0(x)\big)$.*

Then L_1 is an isomorphism.

In Section 8 we define the spectrum for continuous maps acting on Banach spaces and study some of its properties.

In the first part of Section 8 we define the spectrum $\sigma(f)$ of a continuous map $f: E \to E$ as follows

$$\sigma(f) = \{\lambda \in \boldsymbol{K}: \lambda - f \text{ is not regular}\} .$$

This definition is obviously equivalent to

$$\sigma(f) = \{\lambda \in \boldsymbol{K}: \lambda - f \in \sigma(E, E)\} .$$

We also define

$$\sigma_\pi(f) = \{\lambda \in \boldsymbol{K}: \lambda - f \in \sigma_\pi(E, E)\}$$

and

$$\sigma_\delta(f) = \{\lambda \in \boldsymbol{K}: \lambda - f \in \sigma_\delta(E, E)\} .$$

We have $\sigma(f) = \sigma_\pi(f) \cup \delta_\delta(f)$. Moreover, in the context of bounded linear maps we get the following

THEOREM 8.1.1. – *Let $L: E \to E$ be bounded and linear. Then*

(*a*) $\sigma(L)$ *is the usual spectrum of* L.

(*b*) $\sigma_\delta(L)$ *is the approximate defect spectrum of* L.

(*c*) $\sigma_\pi(L)$ *is the approximate point spectrum of* L.

We also have the following

THEOREM 8.1.2. – *Let $f \in C(E, E)$. Then*

(*a*) $\sigma(f)$ *is closed,*

(*b*) $\sigma_\pi(f)$ *is closed,*

(*c*) $\partial\sigma(f) \subset \sigma_\pi(f)$.

Other properties of $\sigma(f)$ are contained on Proposition 8.1.2.

The second part of Section 8 deals with the problem of the nonemptiness of $\sigma(f)$. Several results in this direction are obtained. As an example we state the following result.

PROPOSITION 8.2.1. – *Let $f: E \to E$ be quasibounded and α-Lipschitz. Assume that* $\dim E = +\infty$ *and* $\alpha(f) < d(f)$. *Then* $\sigma(f) \neq \emptyset$.

The end of the second part closes with an example of a continuous map $f: \boldsymbol{C}^2 \to \boldsymbol{C}^2$ with empty spectrum.

In the third part of Section 8 we show that the multivalued map that to every $f \in C(E, E)$ associates its spectrum $\sigma(f)$ is upper semicontinuous, extending to the nonlinear context a well-known fact of the linear theory (see e.g. T. KATO [15]).

In Section 9 we give a Fredholm alternative for nonlinear maps. In the first part we observe that the classical Fredholm alternative for linear operators may be interpreted in terms of spectral theory. More precisely, we have that for a linear compact operator $K: E \to E$ the equality $\sigma_\pi(K) = \sigma(K)$ is equivalent to the following well-known Fredholm alternative.

Let $\lambda \in \boldsymbol{K}$, *assume moreover* $\lambda \neq 0$ *if* $\dim E = \infty$. *Then the equation* $\lambda x - Kx = y$ *is solvable for any* $y \in E$ *if and only if the equation* $\lambda x = Kx$ *has only the trivial solution.*

This simple observation motivates the introduction of the following notion. A continuous map $f: E \to E$ is called *alternative* if $\sigma(f) = \sigma_\pi(f)$.

Examples of alternative maps are given by the following

THEOREM 9.1.1. – *Let $f: E \to E$ be asymptotically odd and compact. Then f is alternative.*

We say that $f\colon E \to E$ is *asymptotically odd* if there exists an odd map $g\colon E \to E$ such that $\|f(x) - g(x)\|/\|x\| \to 0$ as $\|x\| \to +\infty$.

The second part of Section 9 exhibits other examples of alternative maps. A bounded linear operator $A\colon E \to E$ is called *balanced* if $\sigma_e(A) = \sigma_\beta(A)$ where

$$\sigma_e(A) = \{\lambda \in \boldsymbol{K}\colon \lambda - A \text{ is not Fredholm of index } 0\},$$

is the essential spectrum of A, and

$$\sigma_\beta(A) = \{\lambda \in \boldsymbol{K}\colon \beta(\lambda - A) = 0\}.$$

Examples of balanced operators are, among others, normal operators (therefore, self-adjoint operators), operators such that its n-th iterate is compact for some $n \in \boldsymbol{N}$, and quasinilpotent operators (*i.e.* operators whose spectral radius is zero). The following theorem motivates the introduction of balanced operators.

THEOREM 9.2.1. – *Let $A\colon E \to E$ be balanced and $h\colon E \to E$ be compact and asymptotically odd. Then $A + h$ is alternative.*

Notice that Theorem 9.2.1 extends Theorem 9.1.1 since the identically zero map is balanced.

In Section 10 we gather some topological consequences of the nonlinear spectral theory.

In the first part we show that some classical results such as the Birkoff-Kellog theorem, the Hopf theorem on spheres, the Borsuk-Ulam theorem and others may be proved by means of purely spectral tools.

In the second part we introduce the following notion. Let $X \subset E$ and $f\colon X \to E$ be continuous. Then f is called hypocompact if $\beta(\lambda + f) > 0$ for any $\lambda > 0$. The class of hypocompact maps is quite large. Indeed, compact maps, compact (α-contractive, condensing) vector fields, monotone operators, continuous maps acting on a finite dimensional space, are examples of hypocompact maps. In this context we have the following

THEOREM 10.2.1. – *Let $f\colon E \to E$ be a coercive, proper hypocompact map. Assume that*

(*a*) *f is locally α-Lipschitz, sending bounded sets into bounded sets,*

(*b*) $\liminf \beta(1 + tf) > 0$ *as* $t \to 0^+$.

Then f is a strong surjection.

We would like to remark that Theorem 10.2.1 is proved using spectral techniques. The following consequence of Theorem 10.2.1 is of independent interest.

COROLLARY 10.2.1. – *Let U be a bounded open subset of a Banach space E. Then the boundary ∂U of U is not a retract of $\overline{U}$ under a locally α-Lipschitz hypocompact map.*

Notice that Corollary 10.2.1 extends results previously known for compact (condensing) vector fields.

In Section 11 we obtain some results on asymptotic bifurcation points.

In the first part we study asymptotic bifurcation points for equations of the form $\lambda x = g(x)$, $\lambda \in \boldsymbol{K}$, where $g\colon E \to E$ is continuous. We recall that $\lambda \in \boldsymbol{K}$ is an asymptotic bifurcation point for g if there exists a sequence (λ_n, x_n) in $\boldsymbol{K}\times E$ such that $\|x_n\| \to +\infty$, $\lambda_n \to \lambda$ and $\lambda_n x_n = g(x_n)$ (see A. M. KRASNOSEL'SKIJ [16]). We show that the set of all asymptotic bifurcation points of g denoted by $B(g)$ is a subset of $\sigma(g)$. Actually, a more precise result holds (see Proposition 11.1.1).

For conditions ensuring that $B(g) \neq \emptyset$ see Theorem 11.1.1. Another result is the following

THEOREM 11.1.2. – *Let $g\colon E \to E$ be continuous. Let λ_0, $\lambda_1 \in \boldsymbol{K}\setminus\sigma_\pi(g)$ be such that $\lambda_0 \notin \sigma(g)$ and $\lambda_1 \in \sigma(g)$. Then λ_0 and λ_1 belong to different components of $\boldsymbol{K}\setminus(B(g) \cup \sigma_\beta(g))$.*

When g is defined on a finite dimensional space one can get (by means of homotopy classes on spheres) results regarding $B(g)$. For example we have the following

THEOREM 11.1.3. – *Let $g\colon \boldsymbol{R}^{2n+1} \to \boldsymbol{R}^{2n+1}$ be such that $|g| = \limsup \|g(x)\|/\|x\| < +\infty$ as $\|x\| \to +\infty$. Then $B(g) \neq \emptyset$.*

The second part of Section 11 is devoted to the more general problem of asymptotic bifurcation points for equation of the form $\varphi(x, \lambda) = 0$, where $\varphi\colon E\times \boldsymbol{K} \to F$ is continuous and such that the map $\lambda \to \varphi(\cdot, \lambda)$ is continuous. A result analogous to Theorem 11.1.2 is given (see Theorem 11.2.1).

In the first part of Section 12 we consider a definition of numerical range for nonlinear maps acting on a complex Hilbert space H. Let $f\colon H \to H$ be continuous. Define the *numerical range* $n(f)$ of f as $n(f) = \Sigma(f_N)$, where $f_N(x) = \big((f(x), x)/\|x\|^2\big)\, x$ is the normal component of f and $\Sigma(f_N)$ is the asymptotic spectrum of f_N introduced in [9], i.e.,

$$\Sigma(f_N) = \{\lambda \in \boldsymbol{C}\colon d(\lambda - f_N) = 0\}\,.$$

This definition is equivalent to that introduced in [10] and coincides with the closure of the usual numerical range in the linear case. We obtain that $\Sigma(f) \subset n(f)$ for any continuous $f\colon H \to H$. Moreover if $|f_N| < +\infty$, then $n(f)$ is nonempty connected and compact (see Theorem 12.1.2 and Theorem 12.1.3 respectively).

In the second part of Section 12, with the aid of this numerical range, we introduce the notion of (nonlinear) *self-adjoint* map. An α-Lipschitz map $f\colon H \to H$ is said to be *self-adjoint* if $|f_N| < +\infty$ and $n(f) \subset \boldsymbol{R}$. This extends the well-known notion of bounded linear self-adjoint operator. For self-adjoint maps we have the following.

COROLLARY 12.2.1. – *Let $A: H \to H$ be bounded, linear and $h: H \to H$ be compact. If A and h are both self-adjoint then*

(*a*) *$\sigma(A + h)$ is a compact subset of $\boldsymbol{R}$,*

(*b*) *$A + h$ is alternative, i.e., $\sigma(A + h) = \sigma_\pi(A + h)$.*

In the third part of Section 12 we obtain the following result.

THEOREM 12.3.1. – *Let $f: \boldsymbol{C}^n \to \boldsymbol{C}^n$ be continuous with $|f_N| < +\infty$. Assume that there esists a linear isomorphism $A: \boldsymbol{C}^n \to \boldsymbol{C}^n$ such that $n(A \circ f \circ A^{-1}) \subset \boldsymbol{C}_-$ where $\boldsymbol{C}_- = \{\lambda \in \boldsymbol{C}: \operatorname{Re} \lambda < 0\}$. Then all the solutions of the differential equation $\dot{z} = f(z)$ are bounded as $t \to +\infty$.*

The last result of this paper (Proposition 12.3.1) shows that Theorem 12.3.1 extends a well-known criterion for the boundedness of solutions of linear differential equations.

2. – Notations and definitions.

In this section we collect some of the notations and definitions that we shall more commonly use in this paper. Nevertheless, due to length of this work and in order to avoid a too frequent back and forth reading we shall repeat, in places scattered along the paper, some of the definitions and recall some of the results of this section.

2.1. *Generalities.*

The symbol $\boldsymbol{K}$ will stand either for the field of complex numbers $\boldsymbol{C}$ or for the field of the reals $\boldsymbol{R}$.

The capital letters E and F, unless otherwise stated, will be used to denote Banach spaces over $\boldsymbol{K}$ and H is used to denote a Hilbert space (over $\boldsymbol{K}$).

Spheres and closed balls centered at the origin and with radii $r > 0$ are denoted by

$$S_r = \{x \in E: \|x\| = r\}, \qquad D_r = \{x \in E: \|x\| \leqslant r\},$$

respectively.

Given any map $f: E \to F$ we denote the image of f by $\operatorname{Im} f$.

A continuous map $f: E \to F$ is said to be *quasibounded* if there exist two constants A, $B \geqslant 0$ such that $\|f(x)\| \leqslant A\|x\| + B$ for any $x \in E$. The infimum of those $A \geqslant 0$ for which there exists $B \geqslant 0$ such that the above inequality is satisfied is called the *quasinorm* of f and denoted by $|f|$. In other words, if f is quasibounded, then it sends bounded sets into bounded sets and

$$|f| = \limsup_{\|x\|\to+\infty} \frac{\|f(x)\|}{\|x\|} < +\infty,$$

(see A. GRANAS [13]).

We list now some of the properties of quasibounded maps. If f, $g\colon E \to F$ are quasibounded, then

(*a*) $|\lambda f| = |\lambda||f|$, $\lambda \in \boldsymbol{K}$,

(*b*) $|f + g| \leqslant |f| + |g|$,

(*c*) $|f\circ g| \leqslant |f||g|$,

(*d*) if $L\colon E \to F$ is bounded and linear, then $|L| = \|L\|$, where $\|L\|$ is the norm of L.

A continuous map $f\colon E \to F$ is said to be *asymptotically linear* (see A. M. KRASNOSEL'SKIJ [16]) if there exists a bounded linear operator $L\colon E \to F$ such that $|f - L| = 0$. The (unique) operator L is called the *asymptotic derivative* of f and is denoted by $f'(\infty)$. We have the following obvious fact.

(*e*) If $f\colon E \to F$ is asymptotically linear, then $|f| = \|f'(\infty)\|$.

A continuous map $f\colon E \to F$ is called *compact* if the closure $\overline{f(A)}$ of $f(A)$ is compact for any bounded set $A \subset E$.

2.2. *The Kuratowski measure of noncompactness.*

Here we give the definition of the Kuratowski measure of noncompactness [17] and collect its main properties.

Given any bounded subset $A \subset E$ define the *Kuratowski measure of noncompactness* $\alpha(A)$ of A as the infimum of those $\varepsilon > 0$ such that A can be covered with a finite number of subsets of A having diameter less than or equal to ε. The Kuratowski measure of noncompactness has the following properties.

Let A and B be bounded subsets of a Banach space E. Then

(*a*) $\alpha(\lambda A) = |\lambda|\alpha(A)$, $\lambda \in \boldsymbol{K}$.

(*b*) $\alpha(A) = 0$ if and only if $\overline{A}$ is compact.

(*c*) $|\alpha(A) - \alpha(B)| \leqslant \alpha(A + B) \leqslant \alpha(A) + \alpha(B)$ (see G. DARBO [3]).

(*d*) $\alpha([0, 1]\cdot A) = \alpha(A)$, where $[0, 1]\cdot A = \{tx\colon t \in [0, 1], x \in A\}$.

(*e*) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.

(*f*) If $\dim E = +\infty$, then $\alpha(S_1) = 2$ (see R. D. NUSBAUM [21] and [8]).

(*g*) Let $B(A, \varepsilon) = \cup \{B(x, \varepsilon) : x \in A\}$, where $\varepsilon > 0$ and $B(x, \varepsilon) = \{y \in E : \|x - y\| < \varepsilon\}$, then $\alpha(B(A, \varepsilon)) \leqslant \alpha(A) + 2\varepsilon$.

(*h*) $\alpha(\overline{\mathrm{co}}\, A) = \alpha(A)$, where $\overline{\mathrm{co}}\, A$ stands for the closed convex hull of A (see G. DARDO [3]).

We shall also need the following property of α. Let E and F be Banach spaces and $A \subset E$, $B \subset F$ be bounded subsets. Consider $E \times F$ with the norm

$$\|(x, y)\| = \max \{\|x\|, \|y\|\}.$$

Then

(*i*) $\alpha(A \times B) = \max \{\alpha(A), \alpha(B)\}$.

In fact, assume that $\max \{\alpha(A), \alpha(B)\} = \alpha(A)$ and let $\varepsilon > 0$ be given. Then there exist A_i, $i = 1, \ldots, n_\varepsilon$ and B_j, $j = 1, \ldots, m_\varepsilon$ such that $A = \bigcup_{n=1}^{n_\varepsilon} A_i$, $B = \bigcup_{j=1}^{m_\varepsilon} B_j$ and $\operatorname{diam} A_i < \alpha(A) + \varepsilon$, $i = 1, \ldots, n_\varepsilon$; $\operatorname{diam} B_j < \alpha(A) + \varepsilon$, $j = 1, \ldots, m_\varepsilon$. Moreover, $A \times B \subset \bigcup_{i,j} A_i \times B_j$; $i = 1, \ldots, n_\varepsilon$; $j = 1, \ldots, m_\varepsilon$, and $\operatorname{diam}(A_i \times B_j) \leqslant \max \{\operatorname{diam} A_i, \operatorname{diam} B_j\}$.

Indeed, $\operatorname{diam}(A_i \times B_j) = \sup \{\|(x_1, y_1) - (x_2, y_2)\| : (x_1, y_1), (x_2, y_2) \in A_i \times B_j\} \leqslant \max \{\operatorname{diam} A_i, \operatorname{diam} B_j\}$. Therefore, $\alpha(A \times B) \leqslant \max \{\alpha(A), \alpha(B)\}$. On the other hand the canonical projections $P_E : E \times F \to E$ and $P_F : E \times F \to F$ are such that $\|P_E\| = \|P_F\| = 1$. Thus, $\alpha(A) = \alpha(P_E(A \times B)) \leqslant \|P_E\| \alpha(A \times B) = \alpha(A \times B)$. Hence, $\alpha(A \times B) = \alpha(A)$. This implies that $\alpha(A \times B) = \max \{\alpha(A), \alpha(B)\}$.

We recall the following well-known definitions.

A continuous map $f : E \to E$ is said to be *α-Lipschitz* with constant $K \geqslant 0$ if $\alpha(f(A)) \leqslant K\alpha(A)$ for any bounded subset $A \subset E$. The map f is called *α-contractive* if $0 \leqslant K < 1$ and *α-nonexpansive* if $K = 1$. Clearly, $f : E \to E$ is compact iff it is α-Lipschitz with constant $K = 0$. A continuous map $f : E \to E$ is said to be *condensing* if $\alpha(f(A)) < \alpha(A)$ for any bounded subset $A \subset E$ with $\alpha(A) > 0$.

A useful example of an α-nonexpansive map is represented by the radial retraction $\pi : E \to D$ of a Banach space E onto its unit ball (see R. D. NUSSBAUM [21]).

2.3. *Some facts.*

We list here some well-known results to be used throughout this paper.

THEOREM 2.3.1 (A. M. KRASNOSEL'SKIJ [16]). – *Let $f : E \to E$ be asymptotically linear and compact. Then its asymptotic derivative $f'(\infty) : E \to E$ is compact.*

THEOREM 2.3.2 (see e.g. N. DUNFORD - J. T. SCHWARTZ [5]). – *Let $f : E \to E$ be compact and Fréchet differentiable at zero. Then the Fréchet differential $f'(0)$ of f at zero is compact.*

THEOREM 2.3.3 (see e.g. N. DUNFORD - J. T. SCHWARTZ [5]). – *A bounded linear map $L: E \to F$ is compact if and only if its adjoint $L^*: F^* \to E^*$ is compact.*

THEOREM 2.3.4 (see G. DARBO [3]). – *Let $Q \subset E$ be bounded closed and convex and let $f: Q \to Q$ be α-contractive. Then f has a fixed point $x \in Q$.*

THEOREM 2.3.5 (see B. N. SADOVSKIJ [22]). – *Let $Q \subset E$ be bounded closed and convex and let $f: Q \to Q$ be condensing. Then f has a fixed point $x \in Q$.*

3. – Preliminary results.

The definitions and preliminary results contained in this section are given for continuous maps from a Banach space E into itself even though they could be stated for maps acting between different Banach spaces.

Given a continuous map $f: E \to E$ we introduce three extended real numbers $d(f)$, $\alpha(f)$ and $\beta(f)$ and give some of their properties to be used in the sequel. Our purpose is to avoid cumbersome notations in the nonlinear spectral theory that will be developed in this paper. It is remarkable that all of the above three numbers have a meaning worth of attention in the case when f is a linear operator. In particular, at the end of this section, we will show that a bounded linear operator $L: E \to E$ is Fredholm if and only if $\beta(L)$ and $\beta(L^*)$ are greater than zero, where L^* is the adjoint of L.

3.1. *Definition of $d(f)$, $\alpha(f)$, $\beta(f)$ and their properties.*

Let $f: E \to E$ be a continuous map. Consider the extended real number.

$$d(f) = \liminf_{\|x\| \to +\infty} \frac{\|f(x)\|}{\|x\|}.$$

The main properties of $d(f)$ are contained in the following

PROPOSITION 3.1.1. – *Let f, $g: E \to E$ be continuous maps from a Banach space E (over $\boldsymbol{K}$) into itself. Then (whenever it makes sense)*

(*a*) $0 \leqslant d(f) \leqslant |f|$.

(*b*) $d(\lambda f) = |\lambda| d(f)$, $\lambda \in \boldsymbol{K}$.

(*c*) $d(f) - |g| \leqslant d(f+g) \leqslant d(f) + |g|$.

(*d*) $|d(f) - d(g)| \leqslant |f-g|$. *In particular,* $|f-g| = 0 \Rightarrow d(f) = d(g)$.

(*e*) $d(f)d(g) \leqslant d(f \circ g) \leqslant |f| d(g)$.

(*f*) *If f is a homeomorphism with quasibounded inverse, then* $d(f) = |f^{-1}|^{-1}$.

PROOF. – Due to the fact that f, g need not be quasibounded one has to modify slightly the arguments used in [9], [10] and [12] to prove (a)-(e). We have only to prove (f).

$$|f^{-1}| = \limsup_{\|x\|\to+\infty} \frac{\|f^{-1}(x)\|}{\|x\|} = \limsup_{\|y\|\to+\infty} \frac{\|y\|}{\|f(y)\|} = \left(\liminf_{\|y\|\to+\infty} \frac{\|f(y)\|}{\|y\|}\right)^{-1} = d(f)^{-1}. \qquad \text{Q.E.D.}$$

Given any continuous map $f\colon E \to E$ we consider the following extended real number.

$$\alpha(f) = \inf\{K \geqslant 0\colon \alpha(f(A)) \leqslant K\alpha(A) \text{ for any bounded } A \subset E\},$$

where α is the Kuratowski measure of non compactness.

The following proposition contains the main properties of $\alpha(f)$.

PROPOSITION 3.1.2. – *Let f, $g\colon E \to E$ be continuous maps from a Banach space E (over $\boldsymbol{K}$) into itself. Then (whenever it makes sense)*

(a) $\alpha(\lambda f) = |\lambda|\alpha(f)$, $\lambda \in \boldsymbol{K}$.

(b) $|\alpha(f) - \alpha(g)| \leqslant \alpha(f + g) \leqslant \alpha(f) + \alpha(g)$.

(c) $\alpha(f\circ g) \leqslant \alpha(f)\alpha(g)$.

(d) $\alpha(f) = 0$ *if and only if f is compact.*

(e) *If* $\dim E = +\infty$ *and f is compact, then* $\alpha(\lambda - f) = |\lambda|$.

PROOF. – (a) If $\dim E < +\infty$ then the equality is trivial. In the case when $\dim E = +\infty$ the statement is an easy consequence of the following two facts

i) $\alpha(\lambda A) = |\lambda|\alpha(A)$. (See Notations and definitions 2.2).

ii) $\alpha(f)$ can be equivalently defined as follows

$$\alpha(f) = \sup\{\alpha(f(A))/\alpha(A)\colon \alpha(A) > 0\}.$$

(b) The statement is trivial when $\dim E < +\infty$. If $\dim E = +\infty$ and $\alpha(A) > 0$ then $\alpha((f+g)(A))/\alpha(A) \leqslant [\alpha(f(A)) + \alpha(g(A))]/\alpha(A)$. Taking the sup on both sides of this inequality we get $\alpha(f+g) \leqslant \alpha(f) + \alpha(g)$. The proof of the left hand-side inequality is straight-forward.

(c) It follows immediately from the properties of the Kuratowski measure of noncompactness (see Notations and definitions 2.2) and the definition of $\alpha(f)$.

(d) It follows immediately from the fact that $\alpha(A) = 0 \Leftrightarrow \bar{A}$ is compact.

(e) It follows from (a), (b) and (d) since $\alpha(I) = 1$ ($\dim E = +\infty$, I—the identity on E). Q.E.D.

Let $f\colon E \to E$ be continuous. Consider the extended real number

$$\beta(f) = \sup\{K \geqslant 0 \colon \alpha(f(A)) \geqslant K\alpha(A), \text{ for any bounded } A \subset E\}\,.$$

The following proposition holds.

PROPOSITION 3.1.3. – *Let f, $g\colon E \to E$ be continuous maps from a Banach space E (over $\mathbf{K}$) into itself. Then (whenever it makes sense)*

(*a*) $\beta(\lambda f) = |\lambda|\beta(f)$, $\lambda \in \mathbf{K}$.

(*b*) $\beta(f)\beta(g) \leqslant \beta(f\circ g) \leqslant \alpha(f)\beta(g)$.

(*c*) *If $\beta(f) > 0$, then f is proper on bounded closed sets. If moreover, $d(f) > 0$, then f is proper.*

(*d*) *If* $\dim E = +\infty$, *then* $\beta(f) \leqslant \alpha(f)$, *and* $\beta(f) = +\infty$ *if* $\dim E < +\infty$.

(*e*) $\beta(f) - \alpha(g) \leqslant \beta(f+g) \leqslant \beta(f) + \alpha(g)$.

(*f*) $|\beta(f) - \beta(g)| \leqslant \alpha(f-g)$.

(*g*) *If f is a homeomorphism and $\beta(f) > 0$, then* $\alpha(f^{-1})\beta(f) = 1$.

(*h*) *If* $\dim E = +\infty$ *and f is compact, then* $\beta(\lambda - f) = |\lambda|$.

PROOF. (*a*) If $\dim E < +\infty$ then the equality is trivial. If $\dim E = +\infty$ the statement is a consequence of the following two facts

i) $\alpha(\lambda A) = |\lambda|\alpha(A)$;

ii) $\beta(f)$ can be equivalently defined as follows $\beta(f) = \inf\{\alpha(f(A))/\alpha(A)\colon \alpha(A) > 0\}$.

(*b*) It follows immediately from the properties of the Kuratowski measure of noncompactness and the definition of $\alpha(f)$ and $\beta(f)$.

(*c*) Let us show that $\beta(f) > 0$ and $d(f) > 0$ imply that f is proper. Let $A \subset E$ be compact. Since $d(f) > 0$, then the inverse image $f^{-1}(A)$ of A is bounded. On the other hand if r is any positive real number smaller than $\beta(f)$ we have $\alpha(f(B)) \geqslant r\alpha(B)$ for any bounded set $B \subset E$.

Therefore, setting $B = f^{-1}(A)$ we get

$$0 = \alpha(A) \geqslant r\alpha(f^{-1}(A))\,,$$

i.e. $f^{-1}(A)$ is compact, since it is closed.

In the case when only $\beta(f) > 0$ then the above proof shows that f is proper on bounded closed sets.

(*d*) It follows immediately from the definitions of $\alpha(f)$ and $\beta(f)$.

(*e*) If $\dim E < +\infty$ the inequality is trivially satisfied. If $\dim E = +\infty$, then

$$\begin{aligned}\beta(f+g) = \inf\{\alpha((f+g)(A))/\alpha(A)\colon \alpha(A)>0\} &\leqslant \\ &\leqslant \inf\left\{[\alpha(f(A)) + \alpha(g(A))]/\alpha(A)\colon \alpha(A)>0\right\} \leqslant \\ &\leqslant \inf\left\{\alpha(f(A))/\alpha(A)\colon \alpha(A)>0\right\} + \\ &+ \sup\left\{\alpha(g(A))/\alpha(A)\colon \alpha(A)>0\right\} = \beta(f) + \alpha(g)\,.\end{aligned}$$

Using this inequality we get

$$\beta(f) = \beta(f-g+g) \leqslant \beta(f+g) + \alpha(g)\,.$$

Thus

$$\beta(f) - \alpha(g) \leqslant \beta(f+g)\,.$$

(*f*) It follows from the inequality

$$\beta(f) = \beta(f-g+g) \leqslant \beta(g) + \alpha(f-g)\,.$$

(*g*) If $\dim E < +\infty$ the equality is trivially satisfied (we use the convention $1/\infty = 0$). If $\dim E = +\infty$ we have

$$\begin{aligned}\alpha(f^{-1}) = \sup\left\{\alpha(f^{-1}(A))/\alpha(A)\colon \alpha(A)>0\right\} &= \sup\left\{\alpha(B)/\alpha(f(B))\colon \alpha(B)>0\right\} = \\ &= \left(\inf\left\{\alpha(f(B))/\alpha(B)\colon \alpha(B)>0\right\}\right)^{-1} = (\beta(f))^{-1}\,.\end{aligned}$$

(*h*) It follows from (*a*) and (*e*). The assertion is false if $\dim E < +\infty$ (see (*d*)). Q.E.D.

3.2. *d, α and β in the context of linear operators.*

The following proposition gathers some useful information on d, α and β in the context of linear operators.

Proposition 3.2.1. – *Let $L\colon E \to E$ be a continuous linear operator from a Banach space E (over $\mathbf{K}$) into itself. Then*

(*a*) $\beta(L) \geqslant d(L)$.

(*b*) $\alpha(L) \leqslant \|L\|$.

(*c*) *If L is an isomorphism, then* $d(L) = \|L^{-1}\|^{-1}$.

(*d*) $\beta(L) > 0$ *if and only if L is left semi-Fredholm* (*i.e.* $\operatorname{Im} L$ *is closed and* $\dim \operatorname{Ker} L < +\infty$).

(*e*) $\beta(L^*) > 0$ *if and only if L is right semi-Fredholm* (*i.e.* $\operatorname{Im} L$ *is closed and* $\dim \operatorname{Coker} L < +\infty$). *Thus, L is Fredholm if and only if* $\beta(L) > 0$ *and* $\beta(L^*) > 0$.

PROOF (*a*), (*b*). – If $\dim E < +\infty$ then the inequalities are trivial. Assume that $\dim E = +\infty$. Since

$$d(L) = \inf \{\|L(x)\|/\|x\| : x \neq 0\} ,$$

we have $d(L)\|x\| \leqslant \|L(x)\|$. On the other hand $\|L(x)\| \leqslant \|L\|\,\|x\|$. Thus, for any bounded set $A \subset E$, we have $d(L)\alpha(A) \leqslant \alpha(L(A)) \leqslant \|L\|\alpha(A)$. This inequality ensures that $\beta(L) \geqslant d(L)$ and $\alpha(L) \leqslant \|L\|$.

(*c*) It follows immediately from Proposition 3.1.1-(*f*) and from the equality $|L| = \|L\|$ for any linear bounded operator $L\colon E \to E$.

(*d*) Let us show first that $\beta(L) > 0$ implies that $\dim \operatorname{Ker} L < +\infty$ and $\operatorname{Im} L$ is closed. By Proposition 3.1.3-(*c*) L is proper on D_1—the unit closed ball of E. Thus, $\operatorname{Ker} L \cap D_1$ is compact, i.e. $\dim \operatorname{Ker} L < +\infty$. We prove now that $\operatorname{Im} L$ is closed. Since $\dim \operatorname{Ker} L < +\infty$ there exists a closed subspace $E_0 \subset E$ such that $E = E_0 \oplus \operatorname{Ker} L$. Let $\{y_n\}$ be a sequence in $\operatorname{Im} L$ converging to some y_0 and let $\{x_n\}$ be a sequence in E_0 such that $L(x_n) = y_n$. Since L is proper on bounded closed sets (see Proposition 3.1.3-(*c*)) we get that $y_0 \in \operatorname{Im} L$ in the case when $\{x_n\}$ is bounded. If this is not the case, we may assume $\|x_n\| \to +\infty$. Set $z_n = x_n/\|x_n\|$. Clearly, $L(z_n) = L(x_n/\|x_n\|) = y_n/\|x_n\| \to 0$, as $n \to +\infty$. Put $A = \{z_n : n \in N\}$. Then, $\alpha(L(A)) = 0$. On the other hand $\alpha(L(A)) \geqslant \beta(L)\alpha(A)$. Hence, using the fact that $\beta(L) > 0$ we get $\alpha(A) = 0$, *i.e.* $\{z_n\}$ is compact. Without loss of generality we may assume that $\{z_n\}$ converges to some element $z \in E_0$, $\|z\| = 1$. Thus $L(z) = 0$, contradicting $E_0 \cap \operatorname{Ker} L = \{0\}$.

We have to show now that if $\operatorname{Im} L$ is closed and $\dim \operatorname{Ker} L < +\infty$, then $\beta(L) > 0$. Since $\dim \operatorname{Ker} L < +\infty$, then $E = E_0 \oplus \operatorname{Ker} L$, where E_0 is a closed subspace of E. Let $P_0\colon E \to E_0$ be the canonical projection onto E_0 and set $P_1 = I - P_0$. Put $\hat{L} = L|E_0$. Since $\operatorname{Im} L$ is closed, then $\hat{L}$ is an isomorphism of E_0 onto $\operatorname{Im} L = \operatorname{Im} \hat{L}$ (use the Closed Graph Theorem). Therefore, $d(\hat{L}) > 0$. Thus $\beta(\hat{L}) > 0$ since $\beta(\hat{L}) \geqslant d(\hat{L})$.

On the other hand $L = \hat{L} \circ P_0$. Hence $\beta(L) \geqslant \beta(\hat{L})\beta(P_0)$.

Now, $\beta(P_0) = \beta(I - P_1) = 1$ since P_1 is compact (see Proposition 3.1.3-(*h*)). Therefore, $\beta(L) > 0$.

(*e*) Let us show first that $\beta(L^*) > 0$ implies that $\operatorname{Im} L$ is closed and $\dim \operatorname{Coker} L < +\infty$. On the basis of (*d*) we have that $\operatorname{Im} L^*$ is closed and hence $\operatorname{Im} L$ is closed (see Dunford-Schwartz [5]). Moreover, by (*d*), $\dim \operatorname{Ker} L^* < +\infty$.

Now, let $\{g_1, .., g_n\}$ be a basis for $\operatorname{Ker} L^*$. We have $\operatorname{Im} L = \cap \{\operatorname{Ker} g_i : i = 1, \ldots, n\}$. This implies that $\dim \operatorname{Coker} L < +\infty$.

We shall show now that if $\operatorname{Im} L$ is closed and $\dim \operatorname{Coker} L < +\infty$, then $\beta(L^*) > 0$. It is known that if $\operatorname{Im} L$ is closed then so is $\operatorname{Im} L^*$ (see Dunford-Schwartz [5]). On the other hand there are n continuous linear functionals $f_1, \ldots, f_n$ such that $\operatorname{Im} L = \cap \{\operatorname{Ker} f_i : i = 1, \ldots, n\}$. Clearly, $f_i \in \operatorname{Ker} L^*$, $i = 1, \ldots, n$ and any element of $\operatorname{Ker} L^*$ is a linear combination of $f_1, \ldots, f_n$. Thus $\dim \operatorname{Ker} L^* < +\infty$. The result proved in (*d*) implies that $\beta(L^*) > 0$. Q.E.D.

REMARK 3.2.1. – Proposition 3.2.1 shows that a linear operator $L\colon E \to E$ is Fredholm if and only if $\beta(L)$ and $\beta(L^*)$ are both greater than zero.

The following examples show that the inequalities $\beta(L) \geqslant d(L)$ and $\alpha(L) \leqslant \|L\|$ do not hold in the nonlinear case.

EXAMPLE 3.2.1. – Let $r\colon E \to D_1$ be the radial retraction of an infinite dimensional Banach space E onto its unit closed ball D_1. Clearly $|r| = 0$ and $\alpha(r) \geqslant 1$ since D_1 is not compact (actually, $\alpha(r) = 1$). Thus, $|r| < \alpha(r)$.

EXAMPLE 3.2.2. – Let l^2 be the Hilbert space of square summable sequences of real numbers. Take $f\colon l^2 \to l^2$ defined by $f(x) = (\|x\|, 0, \ldots, 0, \ldots)$. Clearly, f is compact and thus $\beta(f) = 0$. On the other hand $d(f) = 1$. Thus $\beta(f) < d(f)$.

The following proposition contains some equivalent conditions to the property $\beta(L) > 0$ for a bounded linear operator $L\colon E \to E$.

PROPOSITION 3.2.2. – *Let $L\colon E \to E$ be a bounded linear operator. Then the following conditions are equivalent*

(1) $\beta(L) > 0$,

(2) *if $A \subset E$ is bounded and $\alpha(A) > 0$ then $\alpha(L(A)) > 0$,*

(3) *the restriction of L to the closed unit ball D_1 is proper,*

(4) *if $\{a_n\}$ is such that $\|a_n\| = 1$ and $La_n \to 0$ then $\{a_n\}$ has a convergent subsequence,*

(5) $\operatorname{Im} L$ *is closed and* $\dim \operatorname{Ker} L < +\infty$.

PROOF. – (1) $\Rightarrow$ (2). It follows immediately from the definition of $\beta(L)$.

(2) $\Rightarrow$ (3). Let $K \subset E$ be compact and put $A = D_1 \cap L^{-1}(K)$. Then $L(A) \subset K$ and $\alpha(L(A)) = 0$. It follows that $\alpha(A) = 0$. Therefore A is compact since it is obviously closed.

(3) $\Rightarrow$ (4). Put $\{a_n\colon n \in \boldsymbol{N}\} = A$ and $K = \{L(A)\} \cup \{0\}$. K is compact. Since $A \subset L^{-1}(K) \cap D_1$ it follows that $\{a_n\}$ is a compact sequence.

(4) $\Rightarrow$ (5). The fact that $\operatorname{Ker} L$ if finite dimensional follows immediately since $\operatorname{Ker} L \cap D_1$ is compact. It remains to show that $\operatorname{Im} L$ is closed.

Since $\dim \operatorname{Ker} L < +\infty$ there exists a closed subspace $E_1 \subset E$ such that $E = E_1 \oplus \operatorname{Ker} L$. We have $L(E) = L(E_1)$ thus it is enough to prove that $L(E_1)$ is closed.

This is the case if $L|E_1\colon E_1 \to L(E_1)$ has a bounded inverse, i.e. there exists $m > 0$ such that $\|L(x)\| \geqslant m\|x\|$ for any $x \in E_1$, $\|x\| = 1$. Assume the contrary. Then there exists a sequence $\{x_n\}$, $\|x_n\| = 1$, $x_n \in E_1$, such that $Lx_n \to 0$. Therefore, by (4), we may assume that $x_n \to x \in E_1$, $\|x\| = 1$. Clearly, $Lx = 0$, and this is a contradiction since $E_1 \cap \operatorname{Ker} L = \{0\}$.

(5) $\Rightarrow$ (1). See Proposition 3.2.1. Q.E.D.

After this paper was written J. MAWHIN brought to our attention that the equivalence « (1) $\Leftrightarrow$ (5) » of Proposition 3.2.2 has already been proved by G. HETZER in [25].

Let $L: E \to E$ be bounded and linear. In Proposition 3.2.1 we proved that $d(L) \leqslant \beta(L)$. Therefore if $d(L) > 0$ then $\beta(L) > 0$. Thus the condition $d(L) > 0$ insures that L is an isomorphism between E and $L(E)$ since it is injective $(d(L) > 0)$ and its image is closed $(\beta(L) > 0)$.

On the other hand there are examples of continuous linear operators L which are injective but $d(L) = 0$. The following proposition shows that this is impossible if $\beta(L) > 0$.

In other words, the condition $d(L) = 0$ implies $\beta(L) = 0$ in the case when the operator is $1 - 1$.

PROPOSITION 3.2.3. – *Let $f: E \to E$ be a continuous positively homogeneous map. Assume that $\beta(f) > 0$. Then the equation $f(x) = 0$ has only the trivial solution if and only if $d(f) > 0$.*

PROOF. – Obviously, the condition $d(f) > 0$ implies that the equation $f(x) = 0$ has only the trivial solution. Let us prove the contrapositive under the assumption $\beta(f) > 0$. In other words, we have to show that the conditions $\beta(f) > 0$ and $d(f) = 0$ imply the existence of an element $x \in E$, $x \neq 0$, such that $f(x) = 0$. Since f is positively homogeneous there exists a sequence $\{x_n\}$, $\|x_n\| = 1$, such that $f(x_n) \to 0$. Put $A = \{x_n : n \in N\}$. Then $0 = \alpha(f(A)) \geqslant \beta(f)\alpha(A)$. It follows that $\alpha(A) = 0$. We may assume, without loss of generality, $x_n \to x$, $\|x\| = 1$. The continuity of f implies that $f(x) = 0$. Q.E.D.

4. – Strong surjections and coincidence theorems.

In the first part of this section we introduce the concept of strong surjective maps and give some of their properties. In the second part we exhibit a coincidence theorem for strong surjections that will play a key role in proving the closedness of spectra for nonlinear operators.

4.1. *Definition of strong surjections and properties.*

A continuous map $f: E \to F$ is called a *strong surjection* if the equation $f(x) = h(x)$ has a solution for any continuous map $h: E \to F$ with $\overline{h(E)}$ compact.

Notice that using Schauder's fixed point theorem is not hard to show that the identity $I: E \to E$ is a strong surjection.

The following propositions give conditions for a continuous map $f: E \to F$ to be a strong surjection.

PROPOSITION 4.1.1. – *Let $f: E \to F$ be a continuous map. Assume that there exist a Banach space G and a continuous map $g: G \to F$ such that $f \circ g$ is a strong surjection. Then f is a strong surjection.*

PROOF. – Let $k: E \to F$ be continuous and such that $\overline{k(E)}$ is compact. We have to show that the equation $f(x) = k(x)$ has a solution $x \in E$. Since $f\circ g$ is a strong surjection and $k\circ g(G)$ is compact, the equation $f\circ g(y) = k\circ g(y)$ has a solution $y_0 \in G$. Hence $g(y_0) = x_0$ is such that $f(x_0) = k(x_0)$. Q.E.D.

Proposition 4.1.1 combined with the fact that the identity is a strong surjection ensures that any continuous map $f: E \to F$ with a continuous right inverse is a strong surjection.

Notice that a suitable definition of strong surjection in the context of topological spaces can be given as follows. Let X, Y be topological spaces. A continuous map $f: X \to Y$ is called a strong surjection if the equation $f(x) = h(x)$ has a solution provided that $h: X \to Y$ is continuous, homotopic to a constant (superfluous on Banach spaces) and $\overline{h(X)}$ is compact. In the case when X is an Absolute Neighbourhood Retract then one can prove (e.g. via the Lefschetz fixed point theorem) that the identity $I: X \to X$ is a strong surjection.

PROPOSITION 4.1.2. – *Let $f: E \to F$ be continuous and such that $\|f(x)\| \to +\infty$ as $\|x\| \to +\infty$. Then the following two conditions are equivalent.*

(*a*) *f is a strong surjection,*

(*b*) *the equation $f(x) = h(x)$ has a solution $x \in E$ for any compact $h: E \to F$ with bounded support.*

PROOF. – Clearly, (*a*) implies (*b*). Therefore we have only to prove that (*b*) implies (*a*). Let $h: E \to F$ be continuous and such that $\overline{h(E)}$ is compact. For any $n \in \mathbf{N}$, let $\sigma_n: E \to [0, 1]$ be continuous and such that

$$\sigma_n(x) = \begin{cases} 1, & \text{if } \|x\| \leqslant n, \\ 0, & \text{if } \|x\| \geqslant 2n. \end{cases}$$

Clearly, the equation $f(x) = \sigma_n(x) h(x)$ has a solution $x_n \in E$. Hence, $\|f(x_n)\| \leqslant \|h(x_n)\|$. This implies that $\|x_{\bar{n}}\| \leqslant \bar{n}$ for some $\bar{n} \in \mathbf{N}$, since $h(E)$ is bounded. Thus, $f(x_{\bar{n}}) = h(x_{\bar{n}})$, since $\sigma_{\bar{n}}(x_{\bar{n}}) = 1$. Q.E.D.

The following result is of independent interest.

PROPOSITION 4.1.3. – (*A Continuation Principle for strong surjections*). *Let $f: E \to F$ be a strong surjection and $h: \mathrm{E}x[0, 1] \to F$ be continuous compact with*

$h(x, 0) = 0$ for any $x \in E$. Assume that

(∗) *there exists a bounded open set $U \subset E$ such that $f^{-1}(0) \subset U$ and $\partial U \cap S = \emptyset$, where $S = \{x \in E: f(x) = h(x, t)$ for some $t \in [0, 1]\}$.*

Then the equation $f(x) = h(x, 1)$ has a solution $x \in U$.

PROOF. – Put $\hat{S} = S \cap U$. Clearly, $\hat{S}$ is not empty. Moreover, $\hat{S}$ is closed since $S \cap \partial U = \emptyset$ and S, by the continuity of f and h, is closed. Let $\varphi: E \to [0, 1]$ be any Uryson's function such that $\varphi(x) = 1$ if $x \in \hat{S}$ and $\varphi(x) = 0$ if $x \notin U$.

Define $k: E \to F$ by $k(x) = h(x, \varphi(x))$ and observe that $\overline{k(E)}$ is compact. Thus, the equation $f(x) = k(x)$ has a solution $x_0 \in E$. Assume $x_0 \notin U$. Then $k(x_0) = h(x_0, \varphi(x_0)) = h(x_0, 0) = 0$. Hence $f(x_0) = 0$, contradicting the assumption $f^{-1}(0) \subset U$. Therefore, $x_0 \in U$. Since $\varphi(x_0) \in [0, 1]$ we get $x_0 \in S$, which implies that $\varphi(x_0) = 1$. Q.E.D.

Notice that if $f: E \to F$ is a strong surjection and $g: E \to F$ is such that $\overline{g(E)}$ is compact, then $f + g$ is a strong surjection. This observation shows that, Proposition 4.1.3 holds true if we replace the conditions « $h(x, 0) = 0$ for all $x \in E$ » and « $f^{-1}(0) \subset U$ » with the weaker assumptions

$$\text{« } \overline{h(E \times \{0\})} \text{ is compact »} \quad \text{and} \quad \text{« } f^{-1}\left(\overline{h(E \times \{0\})}\right) \subset U \text{ ».}$$

We add in passing that condition (∗) is clearly satisfied if S is bounded.

4.2. *Coincidence theorems.*

We shall now give a coincidence theorem for strong surjections that contains as particular cases the well-known fixed point theorems of Schauder, G. Darbo [3] and B. N. Sadovskij [22].

THEOREM 4.2.1. – *Let $f: E \to F$ be a strong surjection. Let $X \subset E$ be closed and $h: X \to F$ be continuous.*

Assume that

(i) $f^{-1}(\overline{\text{co}}\, h(X)) \subset X$,

(ii) *$h(X)$ is bounded,*

(iii) $\alpha(f(A)) = \alpha(h(A)) \Rightarrow \bar{A}$ *compact.*

Then the equation $f(x) = h(x)$ has a solution $x \in X$.

PROOF. – We shall construct a sequence $\{x_n\}$ in E as follows. Take $x_0 \in X$ and choose $x_1 \in f^{-1}(h(x_0))$. Clearly $f(x_1) = h(x_0)$. Now take $x_2 \in f^{-1}(h(x_1))$. Obviously $f(x_2) = h(x_1)$. This procedure yields a sequence $\{x_n\}$ such that $f(x_n) = h(x_{n-1})$. Denote with $A = \{x_n: n \in N\}$. The sequence $\{x_n\}$ was defined in such a way that

$f(A) = f(x_0) \cup h(A)$. Notice that by (ii) $h(A)$ is bounded. Therefore $\alpha(f(A)) = \alpha(h(A))$, and $\bar{A}$ is compact by (iii). Let us denote with A' the set of cluster points of $\{x_n\}$. Since X is closed then $A' \subset X$.

We want to show that $A' \subset f^{-1}(h(A'))$. In other words we have to show that if $x \in A'$ then $f(x) \in h(A')$. Since x is a cluster point of the sequence $\{x_n\}$ then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$. Therefore $h(x_{n_k-1}) = f(x_{n_k}) \to f(x)$. Let $y \in A'$ be a cluster point of $\{x_{n_k-1}\}$. Clearly $h(y) = f(x)$. Denote by $\mathcal{M} = \{M \subset X: A' \subset M,\ M \text{ is closed and } f^{-1}(\overline{\text{co}}\, h(M)) \subset M\}$. The family $\mathcal{M}$ is not empty since $X \in \mathcal{M}$. Put $M_0 = \cap \{M: M \in \mathcal{M}\}$. We have to show now that $f^{-1}(\overline{\text{co}}\, h(M_0)) = M_0$. Since $f^{-1}(\overline{\text{co}}\, h(M_0)) \subset f^{-1}(\overline{\text{co}}\, h(M)) \subset M$ for any $M \in \mathcal{M}$ we have that $f^{-1}(\overline{\text{co}}\, h(M_0)) \subset M_0$. Thus it is enough to show that $f^{-1}(\overline{\text{co}}\, h(M_0)) = M_1 \in \mathcal{M}$. Clearly M_1 is closed. Moreover $f^{-1}(\overline{\text{co}}\, h(M_1)) \subset f^{-1}(\overline{\text{co}}\, h(M_0)) = M_1$ and $A' \subset f^{-1}(h(A')) \subset f^{-1}(\overline{\text{co}}\, h(M_0)) = M_1$. Hence $\overline{\text{co}}\, h(M_0) = f(M_0)$.

Since $h(M_0)$ is bounded we have that $\alpha(\overline{\text{co}}\, h(M_0)) = \alpha(h(M_0)) = \alpha(f(M_0))$, thus M_0 is compact. By Dugundji's extension theorem [4] the map $h: M_0 \to h(M_0)$ can be extended to a continuous map $g: E \to \overline{\text{co}}\, h(M_0)$.

Clearly $\overline{g(E)}$ is compact. This implies that the equation $f(x) = g(x)$ has a solution $x_0 \in E$ since f is a strong surjection. On the other hand $f(x_0) \in \overline{\text{co}}\, h(M_0)$. Thus $x_0 \in f^{-1}(\overline{\text{co}}\, h(M_0)) = M_0$. Therefore, $f(x_0) = g(x_0) = h(x_0)$. Q.E.D.

Notice that condition (iii) is satisfied if $h(X)$ is compact and f is proper.

We show now that some well-known fixed point theorems are direct consequences of Theorem 4.2.1.

COROLLARY 4.2.1 (B. N. SADOVSKIJ [22]). – *Let $C \subset E$ be bounded, closed and convex. Let $h: C \to C$ be condensing. Then h has a fixed point $x \in C$.*

PROOF. – Apply Theorem 4.2.1, taking $f = 1$, the identity on E, and $X = C$. Q.E.D.

Corollary 4.2.1 contains as particular cases Schauder's fixed point theorem and the fixed point theorem of G. Darbo [3].

We give now a coincidence theorem which is of independent interest, representing an extension of the well-known Banach Contraction Principle.

THEOREM 4.2.2. – *Let X and Y be metric spaces, Y being complete. Let $f, h: X \to Y$ be continuous.*

Assume that there exists $K \in \mathbf{R}$, $0 \leqslant K < 1$, such that

$$d(h(x), h(y)) \leqslant K d(f(x), f(y)), \qquad \forall x, y \in X.$$

If f is onto and either f or h is proper then there exists $x \in X$ such that $f(x) = h(x)$.

PROOF. – Construct a sequence $\{x_n\}$ in X as in Theorem 4.2.1. We want to show that $\{h(x_n)\}$ is Cauchy in Y and hence convergent.

Clearly $d(h(x_n), h(x_{n+1})) \leqslant K^n d(h(x_0), h(x_1))$. The fact that $\sum_{n=0}^{+\infty} K^n < +\infty$ implies that $\{h(x_n)\}$ is Cauchy. Let $y_0 \in Y$ be such that $f(x_{n+1}) = h(x_n) \to y_0$. The sequence $\{x_n\}$ is contained in the inverse image under f (or h) of the compact set $A = \{y_0\} \cup \{f(x_n) : n \in \boldsymbol{N}\}$. Hence by the properness of f (or h) we have that $\{x_n\}$ is compact. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ converging to $x_0 \in X$. Clearly $f(x_0)$ and $h(x_0)$ coincide with y_0. Q.E.D.

The Banach Contraction Principle follows immediately by taking $X = Y$ and $f = 1$, the identity on X.

5. – Stably-solvable maps.

In this section we give the definition of stably-solvable maps between Banach spaces. We also give their main properties and show how they are related to strong surjections. Stably-solvable maps will play an important role in the nonlinear spectral theory to be developed in section 7. In particular, using the concept of stably-solvable maps, we will be able to give a decomposition of the spectrum for nonlinear operators in such a way that, when reduced to the linear case it coincides with a well-known decomposition of spectra for linear operators.

We would like to point out that the concept of stably-solvable maps is strictly related to the problem of solvability of nonlinear operator equations. To this purpose see [11] where stably-solvable maps have been introduced and some results (without proofs) are given.

5.1. *Definition of stably-solvable maps and properties.*

Let $f : E \to F$ be a continuous map from a Banach space E into a Banach space F. The map f is said to be *stably-solvable* if the equation

$$f(x) = h(x)$$

has a solution $x \in E$ for any continuous compact map $h : E \to F$ with quasinorm $|h| = 0$.

Obviously, any stably-solvable map $f : E \to F$ is onto. The converse does not hold as the following example shows. Take $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ defined by $f(x) = x/\sqrt{1 + \|x\|}$. Clearly, the map f is continuous, onto and $|f| = 0$. Thus, if we choose $h : \boldsymbol{R}^n \to \boldsymbol{R}^n$, defined by $h(x) = f(x) + x_0$ with $x_0 \neq 0$, then the equation $f(x) = h(x)$ has no solutions.

The following proposition gives a relation between stably-solvable maps and strong surjections.

PROPOSITION 5.1.1. – *Let $f\colon E \to F$ be such that $d(f) > 0$. Then the following conditions are equivalent.*

(*a*) *The equation $f(x) = h(x)$ has a solution $x \in E$ for any compact map $h\colon E \to F$ with bounded support,*

(*b*) *f is strong surjection,*

(*c*) *f is stably-solvable.*

PROOF. – Clearly (*c*) $\Rightarrow$ (*b*) $\Rightarrow$ (*a*). Since $d(f) > 0$ we have that $\lim\limits_{\|x\|\to\infty} \|f(x)\| = +\infty$. Therefore, (*a*) $\Rightarrow$ (*b*) on the basis of Proposition 4.1.2. Hence it is enough to show that (*b*) $\Rightarrow$ (*c*). Let $f\colon E \to F$ be a strong surjection and $h\colon E \to F$ be continuous compact with $|h| = 0$. We have to show that the equation $f(x) = h(x)$ has a solution $x \in E$.

For any $n \in \boldsymbol{N}$ let $\sigma_n\colon E \to [0, 1]$ be a Urysohn's function such that $\sigma_n(x) = 1$ if $\|x\| \leqslant n$ and $\sigma_n(x) = 0$ if $\|x\| \geqslant 2n$. Clearly, the equation $f(x) = \sigma_n(x)h(x)$ has a solution $x_n \in E$ for any $n \in \boldsymbol{N}$. If $\|x_n\| \leqslant n$ for some $n \in \boldsymbol{N}$ we are done, since $\sigma_n(x_n) = 1$. Assume therefore that $\|x_n\| > n$ for any $n \in \boldsymbol{N}$.

Obviously,

$$\frac{f(x_n)}{\|x_n\|} = \frac{\sigma_n(x_n)h(x_n)}{\|x_n\|},$$

for any $n \in \boldsymbol{N}$. Since $h(x_n)/\|x_n\| \to 0$ as $n \to +\infty$ and $\{\sigma_n(x_n)\}$ is bounded we get $f(x_n)/\|x_n\| \to 0$ as $n \to +\infty$, contradicting the inequality $d(f) > 0$. Q.E.D.

Notice that on the basis of Proposition 5.1.1. the identity $I\colon E \to E$ is stably-solvable (I is a strong surjection with $d(I) = 1$).

In the context of bounded linear operators it is possible to characterize the stably-solvable maps. Namely, we have the following.

THEOREM 5.1.1. – *Let $L\colon E \to F$ be linear and bounded. Then L is stably-solvable if and only if it is onto.*

PROOF. – The « only if » part is trivial. Let $L\colon E \to F$ be onto. By Michael's selection theorem [18] there exists a continuous map $g\colon F \to E$ such that $g(y) \in L^{-1}(y)$, where $L^{-1}(y) = \{x \in E\colon L(x) = y\}$, and $\|g(y)\| \leqslant M\|y\|$, $y \in F$, for some $M > 0$.

Let $h\colon E \to F$ be compact with $|h| = 0$. We have to show that the equation $L(x) = h(x)$ has a solution $x \in E$. Now, observe that if $y \in F$ is a solution for the equation $y = h\big(g(y)\big)$, then $g(y)$ is a solution for the equation $Lx = h(x)$, since $L\big(g(y)\big) = y$, $y \in F$. Clearly, the map $h \circ g\colon F \to F$ is compact and $|h \circ g| \leqslant |h||g| = 0$. Since the identity $I\colon F \to F$ is stably-solvable the equation $y = h\big(g(y)\big)$ has a solution $y \in F$. Q.E.D.

Other examples of stably-solvable maps can be given on the basis of the following observations.

1) *Let E, F and G be Banach spaces. Let $f: E \to F$ be stably-solvable and let $r: G \to E$ be a continuous map having a quasibounded right inverse $s: E \to G$. Then the map $g = f \circ r: G \to F$ is stably-solvable.* In fact, let $h: G \to F$ be compact with $|h| = 0$. Since f is stably-solvable and $h \circ s$ is compact with $|h \circ s| \leqslant |h||s| = 0$, the equation $f(x) = h(s(x))$ has a solution $x \in E$. Hence $s(x) \in G$ is a solution of the equation $g(y) = h(y)$.

As an example of the above observation consider the map $f: E \to E$ defined by $f(x) = \|x\|x$. The inverse $f^{-1}(y) = y/\sqrt{\|y\|}$ is quasibounded. Hence f is stably-solvable.

2) *Let E, F and G be Banach spaces. Let $f: E \to F$ be stably-solvable and $g: F \to G$ be right invertible with continuous quasibounded inverse $s: G \to F$. Then $g \circ f: E \to G$ is stably-solvable. Let $h: E \to G$ be compact with $|h| = 0$.* We have to show that the equation $g(f(x)) = h(x)$ has a solution. Clearly, the equation $f(x) = s(h(x))$ has a solution $x \in E$. Now, apply g to both sides of this equation.

3) *Let E, F and G be Banach spaces. Let $f: E \to F$ and $g: G \to E$ be continuous. If g is quasibounded and $f \circ g$ is stably-solvable, then f is stably-solvable.* In fact, take $h: E \to F$ compact with $|h| = 0$. We have to show that the equation $f(x) = h(x)$ has a solution $x \in E$. Since $f \circ g$ is stably-solvable the equation $f(g(y)) = h(g(y))$ has a solution $y \in G$. Clearly, $f(x) = h(x)$ with $x = g(y)$.

5.2. *A Continuation Principle for stably-solvable maps.*

The following result allows us to deal with equations of the form $f(x) = h(x)$, where $f: E \to F$ is stably-solvable and the compact map $h: E \to F$ need not be quasibounded.

THEOREM 5.2.1 (*see* [11]) (*The Continuation Principle for stably-solvable maps*). – *Let $f: E \to F$ be stably-solvable and $h: E \times [0, 1] \to F$ be continuous compact and such that $h(x, 0) = 0$ for any $x \in E$. Let $S = \{x \in E: f(x) = h(x, t)$ for some $t \in [0, 1]\}$. If $f(S)$ is bounded then the equation*

$$f(x) = h(x, 1)$$

has a solution.

PROOF. – There exists an $r > 0$ such that $f(S)$ is contained in the interior of D_r. Let $\varphi: E \to [0, 1]$ be a Urysohn's function such that $\varphi(y) = 1$ if $y \in \overline{f(S)}$ and $\varphi(y) = 0$ if $\|y\| \geqslant r$. Consider the equation $f(x) = \pi \circ h(x, \varphi(f(x)))$, where $\pi: F \to D_r$ is the radial retraction of F onto D_r. Clearly, $\pi \circ h$ is compact and its quasinorm $|\pi \circ h| = 0$. Therefore, there exists a solution $x_0 \in E$ of the above equation. Assume that $\|f(x_0)\| \geqslant r$.

This implies that $\varphi(f(x_0)) = 0$. This contradiction shows that $\|f(x_0)\| < r$. Therefore, $f(x_0) = h(x_0, \varphi(f(x_0)))$. Since $0 \leqslant \varphi(f(x_0)) \leqslant 1$ it follows that $x_0 \in S$. Thus $\varphi(f(x_0)) = 1$ and $f(x_0 = h(x_0, 1)$. Q.E.D.

REMARK 5.2.1. – A condition that ensures the boundedness of $f(S)$ is the following.

$$\limsup_{\substack{\|f(x)\| \to +\infty \\ t \in [0,1]}} \frac{\|h(x, t)\|}{\|f(x)\|} < 1 .$$

In fact, if there would exist a sequence $\{(x_n, t_n)\}$ such that $\{f(x_n)\} \to +\infty$ and $f(x_n) = h(x_n, t_n)$ we would have $\|f(x_n)\| = \|h(x_n, t_n)\|$. Thus

$$\limsup_{\substack{\|f(x)\| \to +\infty \\ t \in [0,1]}} \frac{\|h(x, t)\|}{\|f(x)\|} \geqslant 1 .$$

6. – Regular maps.

In this section we introduce the class of regular maps that will be used later for the definition of the spectrum for nonlinear maps acting on a Banach space E.

6.1. *Definition of regular maps and properties.*

A continuous map $f: E \to F$ from a Banach space E into a Banach space F is said to be *regular* if it is stably-solvable and if $d(f)$ and $\beta(f)$ are both positive.

PROPOSITION 6.1.1. – *The identity $I: E \to E$ acting on a Banach space E is regular.*

PROOF. – The statement is an immediate consequence of the fact that the identity is stably-solvable and $d(I) = \beta(I) = 1$. Q.E.D.

The following proposition characterizes the regular maps among bounded linear operators.

PROPOSITION 6.1.2. – *Let $L: E \to F$ be linear and bounded. Then L is regular if and only if L is an isomorphism.*

PROOF. – Let L be regular. Then L is injective since $d(L) > 0$ and obviously L is surjective. Hence L is an isomorphism. Assume now that L is an isomorphism. Then $d(L) > 0$ (see Proposition 3.2.1-(c)). Furthermore, $\beta(L) \geqslant d(L)$ (see Proposition 3.2.1-(a)), therefore $\beta(L)$ is also positive. Since L is onto, then by Theorem 5.1.1 it is stably-solvable. Q.E.D.

The following proposition shows that the property of being regular is invariant under « small » perturbations. Namely

PROPOSITION 6.1.3. – *Let* $f: E \to F$ *be regular and let* $g: E \to F$ *be such that* $\alpha(g) < \beta(f)$ *and* $|g| < d(f)$. *Then* $f + g$ *is regular.*

PROOF. – We have $d(f+g) \geqslant d(f) - |g| > 0$ and $\beta(f+g) \geqslant \beta(f) - \alpha(g) > 0$. Therefore, it remains to show that the equation $f(x) = -g(x) + k(x)$ has a solution $x \in E$ for any continuous and compact $k: E \to F$ with $|k| = 0$. In this case we have $|-g+k| = |g|$ and $\alpha(-g+k) = \alpha(g)$, therefore, it is enough to show that the equation $f(x) = h(x)$ has a solution $x \in E$, whenever $h: E \to F$ is continuous and such that $|h| < d(f)$ and $\alpha(h) < \beta(f)$. Now, let $b, c \in \boldsymbol{R}$ be such that $|h| < b < c < d(f)$. This implies that there exists $a, r > 0$ such that $\|h(x)\| \leqslant a + b\|x\|$ and $\|f(x)\| \geqslant c(\|x\| - r)$ for all $x \in E$ (to obtain the last inequality choose $r > 0$ such that $\|f(x)\| \geqslant c\|x\|$ for all $\|x\| \geqslant r$). Let $\varrho > 0$ satisfy $a + b\varrho < c(\varrho - r)$. We shall apply Theorem 4.2.1 with $X = D_\varrho = \{x \in E: \|x\| \leqslant \varrho\}$. Clearly, $h(X)$ is contained in the set $C = \{x \in F: \|x\| \leqslant a + b\varrho\}$, hence condition (ii) of Theorem 4.2.1 is satisfied. Now let us take $x \in E$ such that $f(x) \in C$. We have $a + b\varrho \geqslant \|f(x)\| \geqslant c(\|x\| - r)$. This implies $\|x\| \leqslant \varrho$, i.e. $f^{-1}(C) \subset X$. Therefore, condition (i) of Theorem 4.2.1 is satisfied. Moreover, the inequality $\alpha(h) < \beta(f)$ implies $\alpha(h(A)) < \alpha(f(A))$ for any $A \subset X$ with $\alpha(A) > 0$. In fact, if $\alpha(A) > 0$ we have $\alpha(f(A)) \geqslant \beta(f)\alpha(A) > \alpha(h)\alpha(A) \geqslant \alpha(h(A))$. This implies that condition (iii) of Theorem 4.2.1 is also satisfied. Q.E.D.

The following is an interesting property of regular maps.

PROPOSITION 6.1.4. – *Let* $f: E \to F$ *be regular. Then* $f^{-1}: F \multimap E$, *defined by* $f^{-1}(y) = \{x \in E: f(x) = y\}$, *is upper semicontinuous with compact values.*

PROOF. – It is an easy consequence of the following well-known property of upper semicontinuous maps.

Let $g: X \multimap Y$ *be a multivalued map from a metric space* X *into a metric space* Y. *Then* g *is upper semicontinuous if and only if.*

(i) *the graph of* g *is closed* (in $X \times Y$).

(ii) g *sends compact sets into relatively compact sets.*

Now, f^{-1} satisfies condition (i) since f is continuous. Moreover, by Proposition 3.1.3-(c), f is proper. Thus, condition (ii) is also satisfied. Q.E.D.

REMARK 6.1.1. – We recall that a continuous map $\psi: E \to E$ is called a compact vector field if $\psi(x) = x - h(x)$, where $h: E \to E$ is compact.

An important class of regular maps is represented by compact vector fields satisfying the following conditions:

(i) $d(\psi) > 0$,

(ii) $\deg(\psi) \neq 0$, where $\deg(\psi)$ stands for the Leray-Schauder degree $\deg(\psi, B_r, 0)$, which is independent of $r > 0$ for sufficiently large r.

A proof of this fact can be found in [9].

We shall give now the following result for regular maps, which is a consequence of Proposition 6.1.3.

PROPOSITION 6.1.5. – *Let $f: E \to F$ be regular and $h: E \to F$ be α-Lipschitz on bounded sets. Then, there exists $\varepsilon > 0$ such that the equation*

$$f(x) = \lambda h(x)\,, \qquad \lambda \in \boldsymbol{K}\,,$$

is solvable provided that $|\lambda| < \varepsilon$.

PROOF. Since $d(f) > 0$, the inverse image $f^{-1}(X)$ of any bounded set $X \subset F$ is bounded. Then, there exists $r > 0$ such that $\|f(x)\| \leqslant 1$ implies $\|x\| \leqslant r$. Let $\pi: E \to D_r$ be the radial retraction of E onto $D_r = \{x \in E: \|x\| \leqslant r\}$. Since h is α-Lipschitz on D_r and π is α-nonexpansive, the map $h\circ\pi: E \to F$ is α-Lipschitz and its image coincides with $h(D_r)$, which is bounded. Therefore, there exists $\varepsilon > 0$ such that $\varepsilon\alpha(h\circ\pi) < \beta(f)$ and $\varepsilon\|h(\pi(x))\| \leqslant 1$ for all $x \in E$. Take $\lambda \in \boldsymbol{K}$ with $|\lambda| < \varepsilon$.

We have $\alpha(\lambda h\circ\pi) = |\lambda|\alpha(h\circ\pi) < \beta(f)$ and $0 = |\lambda h\circ\pi| < d(f)$. Thus, by Proposition 6.1.3 the equation

$$f(x) = \lambda h(\pi(x))\,,$$

has a solution $x_0 \in E$. It remains to show that $\pi(x_0) = x_0$, *i.e.*, $x_0 \in D_r$. This follows at once from the fact that

$$\|f(x_0)\| = |\lambda|\|(\pi(x_0))\| \leqslant 1\,. \qquad \text{Q.E.D.}$$

The following result shows that starting from regular maps one can produce other regular maps by means of suitable homotopies.

PROPOSITION 6.1.6. – *Let $f: E \to F$ be regular and $h: E \times [0, 1] \to F$ be compact. Assume that*

(i) *$h(x, 0) = 0$ for all $x \in E$,*

(ii) *the set $\{x \in E: f(x) + h(x, t) = 0$ for some $t \in [0, 1]\}$ is bounded,*

(iii) *$d(f(\cdot) + h(\cdot, 1)) > 0$.*

Then the map $f(\cdot) + h(\cdot, 1)$ is regular.

PROOF. – By Proposition 5.1.1 it suffices to to show that the equation $f(x) + h(x, 1) = k(x)$ has a solution provided $k: E \to F$ is compact with bounded support. Now apply Proposition 4.1.3. Q.E.D.

We close this part with the following

EXAMPLE 6.1.1. – Let $f: [0, 1] \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ be continuous and such that $\|f(t, x)\| \leqslant a + b\|x\|$, for some $a, b > 0$. Consider the operator $M: C_n^1([0, 1]) \to C_n^0([0, 1]) \times \boldsymbol{R}^n$, defined by $M(x)(t) = \big(x'(t) - f(t, x(t)),\ x(0)\big)$. Using Proposition 6.1.6 one can show that M is regular (the proof of this fact will be given elsewhere). Therefore if $g: [0, 1] \times \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a C^1 map and $k: C_n^1([0, 1]) \to \boldsymbol{R}^n$ is continuous, then (see Proposition 6.1.5) the problem

$$\begin{cases} x' = f(t, x) + \lambda g(t, x, x') \\ x(0) = x_0 + \lambda k(x) \end{cases}$$

has a solution $x \in C_n^1([0, 1])$, provided that λ is sufficiently small.

6.2. *A characterization of regular maps in finite dimensional spaces.*

Let S_1^{n-1}, S_1^{m-1} be the unit spheres of $\boldsymbol{R}^n$ and $\boldsymbol{R}^m$ respectively and let $f, g: S_1^{n-1} \to S_1^{m-1}$ be continuous. We recall that f and g are said to be *homotopic* ($f \simeq g$) if there exists a continuous map $H: S_1^{n-1} \times [0, 1] \to S_1^{m-1}$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for any $x \in S_1^{n-1}$. This is an equivalence relation and we shall denote by $[f]$ the homotopy class of f. The map $f: S_1^{n-1} \to S_1^{m-1}$ is called *homotopically trivial* if there is a constant map in its equivalence class. For further informations on homotopy classes of maps in the sense mentioned above see S. T. HU [14].

We shall show that to every continuous map $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$ such that $f^{-1}(0)$ is bounded can be associated a unique homotopy class $[f]$. Let $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$ be as above, then there exists $r_0 > 0$ such that $f(x) \neq 0$ for any $x \in \boldsymbol{R}^n$ with $\|x\| \geqslant r_0$: Let $r \geqslant r_0$. Define $f_r: S_1^{n-1} \to S_1^{m-1}$ by $f_r(x) = f(rx)/\|f(rx)\|$, $x \in S_1^{n-1}$. The homotopy class $[f]$ associated to f is the homotopy class of f_r.

We have to show that this definition is independent of $r \geqslant r_0$. Let $s, r \geqslant r_0$. Define the homotpy $H: S_1^{n-1} \times [0, 1] \to S_1^{m-1}$ by

$$H(x, t) = \frac{f(trx + (1-t)sx)}{\|f(trx + (1-t)sx)\|}.$$

Clearly, $H(x, 0) = f_s(x)$ and $H(x, 1) = f_r(x)$. Thus, the homotopy class of f is independent of $r \geqslant r_0$.

PROPOSITION 6.2.1. – *Let $H: \boldsymbol{R}^n \times [0, 1] \to \boldsymbol{R}^m$ be continuous and such that the set $\{x \in \boldsymbol{R}^n: H(x, t) = 0$ for some $t \in [0, 1]\}$ is bounded. Then $[H(\cdot, 0)] = [H(\cdot, 1)]$.*

Proof. – Take $r > 0$ such that $H(x, t) \neq 0$ for any $x \in \boldsymbol{R}^n$ with $\|x\| \geqslant r$ and $t \in [0, 1]$. The result follows immediately by considering the homotopy $\bar{H}: S_1^{m-1} \times [0, 1] \to S_1^{m-1}$ defined by $\bar{H}(x, t) = H(rx, t)/\|H(rx, t)\|$. Q.E.D.

Remark 6.2.1. – The above proposition shows that if $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$ is continuous and such that $d(f) > 0$, then $[f] = [f + w]$, for any continuous $w: \boldsymbol{R}^n \to \boldsymbol{R}^m$ with $|w| = 0$.

The following result characterizes the regular maps on finite dimensional spaces by means of homotopy classes on spheres.

Proposition 6.2.2. – *Let $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$ be continuous with $d(f) > 0$. Then f is regular if and only if the homotopy class $[f]$ associated to f is not trivial.*

Proof. We shall first prove that if f is not regular then $[f]$ is trivial. On the basis of Remark 6.2.1 we may assume that $f(x) \neq 0$ for any $x \in \boldsymbol{R}^n$. The homotopy

$$H(x, t) = \frac{f(tx)}{\|f(tx)\|}, \quad x \in S_1^{n-1}, \ t \in [0, 1],$$

shows that f_1 is homotopic to the constant map $f(0)/\|f(0)\|$. Thus, $[f]$ is trivial. It remains to show that if $[f]$ is trivial then f is not regular. Let $r > 0$ be such that $f(x) \neq 0$ for any $x \in \boldsymbol{R}^n$ with $\|x\| \geqslant r$. Since $[f]$ is trivial then the restriction $f|S_r^{n-1}: S_r^{n-1} \to \boldsymbol{R}^m \setminus \{0\}$ is homotopic to a constant. Therefore, there exists an extension $h: D_r \to \boldsymbol{R}^m \setminus \{0\}$ of $f|S_r^{n-1}$, where $D_r^n = \{x \in \boldsymbol{R}^n: \|x\| \leqslant r\}$. The map $g: \boldsymbol{R}^n \to \boldsymbol{R}^m$ defined by

$$g(x) = \begin{cases} h(x), & \text{if } \|x\| \leqslant r, \\ f(x), & \text{if } \|x\| \geqslant r, \end{cases}$$

is clearly not vanishing. Hence, f is not regular since $|f - g| = 0$. Q.E.D.

6.3. *Regular maps and essential compact vector fields.*

Let E be an infinite dimensional Banach space and E_0 be a finite codimensional closed subspace of E. We recall that a map $f: X \to E_0$, $X \subset E$ is called a compact vector if $(1 - f): X \to E$ is compact (1 being the identity on E). A nonvanishing compact vector field $f: S \to E_0$, $S = \{x \in E: \|x\| = 1\}$, is said to be *essential* if any extension of f to a compact vector field $g: D \to E_0$, $D = \{x \in E: \|x\| \leqslant 1\}$, vanishes at some point $x_0 \in D$ (see A. Granas [13]).

L. Nirenberg [19] gave a characterization of essential compact vector fields using arguments of stable homotopy theory, exhibiting some interesting applications to partial differential equations.

In this part we shall characterize essential compact vector fields in terms of regular maps.

Let $f\colon S \to E_0$ be a compact vector field. Define $\bar{f}\colon E \to E_0$ by

$$\bar{f}(x) = \begin{cases} \|x\| f(x/\|x\|), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Clearly, $\bar{f}$ is a positively homogeneous vector field and coincides with f on S. Moreover $\beta(\bar{f}) = 1$, and $d(\bar{f}) > 0$ if and only if f is nonvanishing.

PROPOSITION 6.3.1. – *Let $f\colon S \to E_0$ be a nonvanishing compact vector field. Then f is essential if and only if $\bar{f}$ is regular.*

PROOF. – (If). Assume that $\bar{f}$ is regular and let $g\colon D \to E_0$ be any compact vector field such that the restriction $g|S = f$. Define $h\colon E \to E_0$ by

$$h(x) = \begin{cases} \bar{f}(x) - g(x), & \text{if } \|x\| \leqslant 1, \\ 0, & \text{if } \|x\| \geqslant 1. \end{cases}$$

Since h is compact with bounded support the equation $\bar{f}(x) = h(x)$ has a solution $x_0 \in D$. Hence, $g(x_0) = 0$. Therefore, f is essential.

(Only if). Assume that f is essential. Since $d(\bar{f}) > 0$ it is enough to show that the equation $\bar{f}(x) = h(x)$ has a solution provided that $h\colon E \to E_0$ is compact with bounded support (see Proposition 5.1.1). Define $g\colon D \to E_0$ by $g(x) = \bar{f}(x) - r^{-1}h(rx)$, where $r > 0$ is such that $h(x) = 0$ for any $\|x\| \geqslant r$.

Clearly, g is a compact vector field and $g|S = f$. Therefore, g vanishes at some point $x_0 \in D$. Hence, $\bar{f}(x_0) = r^{-1}h(rx_0)$, i.e., rx_0 is a solution of the equation $\bar{f}(x) = h(x)$. Q.E.D.

Theorem 6.3.1 below shows that for a compact vector field $f\colon E \to E_0$, with $d(f) > 0$ the property of being regular depends only on its behavior on a sufficiently large sphere $S_r \subset E$.

We need first some notations. Let $f\colon E \to E_0$ be as above. Since $d(f) > 0$, there exists $r_0 > 0$ such that $f(x) \neq 0$ for any $\|x\| \geqslant r_0$. Let $r > r_0$ and denote by f_r the restriction $f|S_r$ and by $\bar{f}_r$ its positive homogeneous extension. Clearly, $\bar{f}_r$ is a compact vector field with $d(\bar{f}_r) > 0$.

Let $r, s > r_0$. The homotopy $H\colon E \times [0, 1] \to E_0$ defined by

$$H(x, t) = x - \|x\|\big(tr^{-1} + (1-t)s^{-1}\big)h\big[\big(tr + (1-t)s\big)x/\|x\|\big],$$

where h is the compact part of f, implies (see Proposition 6.1.6) that $\bar{f}_r$ and $\bar{f}_s$ are either both regular or both non regular. Therefore, by Proposition 6.3.1, f_r and f_s are either both essential or both inessential.

This observation plays a key role in the following

THEOREM 6.3.1. – *Let $f: E \to E_0$ be a compact vector field with $d(f) > 0$. Let $r_0 > 0$ be such that $f(x) \neq 0$ for $\|x\| \geq r_0$. Then f is regular if and only if f_s is essential for some $s \geq r_0$ (hence f_r is essential for all $r \geq r_0$).*

PROOF. – (If). We have to show that the equation $f(x) = h(x)$ has a solution provided that $h: E \to E_0$ is compact with bounded support. Let $r > r_0$ be such that $h(x) = 0$ whenever $\|x\| \geq r$. Clearly, f_r is essential. Since $f - h$ is an extension of f_r to D_r, there exists $x_0 \in D_r$ such that $f(x_0) = h(x_0)$. Thus, f is regular.

(Only if). Assume that f is regular and take $s > r_0$: Let $g: D_s \to E_0$ be a compact field extending f_s. Set $h: E \to E_0$, as follows

$$h(x) = \begin{cases} f(x) - g(x), & \text{if } \|x\| \geq s, \\ 0, & \text{if } \|x\| \leq s. \end{cases}$$

Since f is regular the equation $f(x) = h(x)$ has a solution $x_0 \in D_s$. Hence $g(x_0) = 0$. Q.E.D.

7. – On the structure of the space $C(E. F)$.

This section is of fundamental importance in developping the spectral theory for nonlinear operators.

In the first part of this section we study the space $C(E, F)$ of all continuous maps from a Banach space E into a Banach space F. We endow $C(E, F)$ with a suitable (for our purposes) topology and examine the subset $\sigma(E, F)$ of $C(E, F)$ consisting of all non regular maps from E into F.

The second part is devoted to the analysis of a decomposition of $\sigma(E, F)$.

In the third part we give a Continuation Principle for regular maps that will be used in the study of bifurcation points (see Section 11).

7.1. *The subset $\sigma(E, F)$ of $C(E, F)$.*

We shall denote by $C(E, F)$ the (vector) space of all continuous maps from E into F, by $Q(E, F)$ the subspace of $C(E, F)$ consisting of all quasibounded, α-Lipschitz maps and by $L(E, F)$ the subspace of $Q(E, F)$ of all bounded linear operators from E into F. We introduce a topology on $C(E, F)$ in such a way that it induces a Banach structure on $Q(E, F)$ and reduces to the standard topology of uniform convergence on bounded sets in $L(E, F)$.

Given $\varepsilon > 0$ we denote by

$$U(\varepsilon) = \{f \in C(E, F): \|f(x)\| < \varepsilon(1 + \|x\|),\ \alpha(f) < \varepsilon\}.$$

We endow $C(E, F)$ with the topology induced by taking the family $\{U(\varepsilon): \varepsilon > 0\}$ as a fundamental system of neighbourhoods of the origin. By translation this yields a fundamental system of neighbourhoods of any point of $C(E, F)$.

This topology will be called the *strong* topology of $C(E, F)$ which is stronger than the q-topology defined below (see Remark 7.1.1).

It should be remarked that $C(E, F)$ with the strong topology is not a topological vector space. This is due to the fact that the $U(\varepsilon)$'s are not absorbing. However, it can be shown that the subspace $Q(E, F)$ of $C(E, F)$ is actually a Banach space.

Denote by $\sigma(E, F)$ the subset of $C(E, F)$ of all maps which are not regular. We have the following

THEOREM 7.1.1. – *$\sigma(E, F)$ is a closed subset of $C(E, F)$.*

PROOF. – We shall prove that $C(E, F)\setminus\sigma(E, F)$ is open. Let $f \notin \sigma(E, F)$ and take $\varepsilon > 0$ such that $\varepsilon < \min\{d(f), \beta(f)\}$. It suffices to show that $f + g$ is regular for all $g \in U(\varepsilon)$. This follows immediately from Proposition 6.1.3. In fact, $g \in U(\varepsilon)$ implies that $|g| < \varepsilon$ and $\alpha(g) < \varepsilon$. Q.E.D.

Notice that, on the basis of Proposition 6.1.2, the above theorem represents an extension of the well-known fact that the set of all linear isomorphisms from a Banach space E into a Banach space F form an open subset of the Banach space $L(E, F)$ of all bounded linear operators from E into F.

REMARK 7.1.1. – By choosing in $C(E, F)$ the following fundamental system of neighbourhoods of the origin

$$V(\varepsilon) = \{f \in C(E, F): q(f) < \varepsilon\},$$

where $q(f) = \max\{\alpha(f), |f|\}$, we get a topology weaker than the strong topology. An inspection of the proof of Theorem 7.1.1 shows that $\sigma(E, F)$ is closed in this topology, which will be called in the sequel the *q-topology* of $C(E, F)$.

7.2. *A decomposition of $\sigma(E, F)$.*

In what follows, unless otherwise stated, we shall take $C(E, F)$ endowed with the strong topology.

Define the following subsets of $\sigma(E, F)$.

$$\sigma_\delta(E, F) = \{f \in \sigma(E, F): f \text{ is not stably-solvable}\},$$

$$\sigma_\pi(E, F) = \{f \in \sigma(E, F): d(f) = 0 \quad \text{or } \beta(f) = 0\}.$$

We have the following obvious property.

PROPOSITION 7.2.1. $\sigma(E, F) = \sigma_\pi(E, F) \cup \sigma_\delta(E, F)$.

We want to show that the boundary $\partial\sigma(E, F)$ is contained in $\sigma_\pi(E, F)$. To this aim we need the following two lemmata.

LEMMA 7.2.1. – *$\sigma_\pi(E, F)$ is closed in $C(E, F)$.*

PROOF. – From the inequalities $|d(f) - d(g)| \leqslant |f - g|$ and $|\beta(f) - \beta(g)| \leqslant \alpha(f - g)$; $f, g \in C(E, F)$ it follows that β and d are continuous maps from $C(E, F)$ into the compact interval $[0, +\infty]$. Thus $\sigma_\pi(E, F)$ is closed. Q.E.D.

LEMMA 7.2.2. – *$\sigma(E, F) \setminus \sigma_\pi(E, F)$ is open in $C(E, F)$.*

PROOF. – Let $f \in \sigma(E, F) \setminus \sigma_\pi(E, F)$. Since $\sigma_\pi(E, F)$ is closed it is enough to find a neighbourhood $U(\varepsilon)$ of the origin such that $f + g$ is not stably-solvable for any $g \in U(\varepsilon)$. Assume the contrary. Then there exists a sequence $\{g_n\}$ in $Q(E, F)$ such that $\|g_n(x)\| < (1/n)(1 + \|x\|)$, $\alpha(g_n) < 1/n$ and $f + g_n$ is stably-solvable. Since f is not stably-solvable there exists a compact map $h: E \to F$ with $|h| = 0$ such that, $f(x) \neq h(x)$ for any $x \in E$. On the other hand (since $f + g_n$ is stably-solvable) there exists a sequence $\{x_n\}$ in E such that $f(x_n) + g_n(x_n) = h(x_n)$. The sequence $\{x_n\}$ is bounded. In fact, if this is not the case then, by taking a suitable subsequence, we may assume that $\|x_n\| \to +\infty$ We have.

$$\|f(x_n)\| \leqslant \|g_n(x_n)\| + \|h(x_n)\| < (1/n)(1 + \|x_n\|) + \|h(x_n)\| .$$

This implies $\|f(x_n)\|/\|x_n\| \to 0$. Thus $d(f) = 0$, contradicting $f \notin \sigma_\pi(E, F)$. Therefore, there exists $M > 0$ such that $\|x_n\| \leqslant M$ for any $n \in \mathbf{N}$. We have

$$\|f(x_n) - h(x_n)\| \leqslant \|g_n(x_n)\| < (1/n)(1 + M) .$$

This shows that $f(x_n) - h(x_n) \to 0$. Since $\beta(f - h) = \beta(f) > 0$ and $\{x_n\}$ is bounded then $\{x_n\}$ is a compact sequence. By taking a convergent subsequence of $\{x_n\}$ we get an element $\bar{x} \in E$ such that $f(\bar{x}) = h(\bar{x})$. A contradiction. Q.E.D.

We are now in a position of proving the following

THEOREM 7.2.1. – *The boundary $\partial\sigma(E, F)$ of $\sigma(E, F)$ is contained in $\sigma_\pi(E, F)$.*

PROOF. – Let $f \in \partial\sigma(E, F)$. Clearly, $f \in \sigma(E, F)$ since $\sigma(E, F)$ is closed. Assume that $f \notin \sigma_\pi(E, F)$. Then $f \in \sigma(E, F) \setminus \sigma_\pi(E, F)$ which is open, contradicting $f \in \partial\sigma(E, F)$. Q.E.D.

We have now the following consequence of Theorem 7.2.1.

COROLLARY 7.2.1. – *Let Ω be a connected component of $C(E, F) \setminus \sigma_\pi(E, F)$. If $f \in \Omega$ is regular, then any other map $g \in \Omega$ is regular.*

PROOF. – We recall first that a topological space X is connected if and only if any nonempty subset A of X with empty boundary coincides with X. Now, Ω is a connected topological space. Define $A = \{g \in \Omega\colon g \text{ is regular}\}$. Clearly, $A \neq \emptyset$ and its boundary, relative to Ω, is empty in view of Theorem 7.2.1. Thus, $A = \Omega$. Q.E.D.

Let E, F be Banach spaces and let Λ—the « parameter space »—be a topological space. Consider a continuous map $\varphi\colon E\times\Lambda \to F$. Assume that the map $\lambda \mapsto \varphi(\cdot, \lambda)$ from Λ into $C(E, F)$ is continuous. We introduce the following definition. An element $\lambda \in \Lambda$ is said to be a *spectral value* for φ if $\varphi(\cdot, \lambda) \in \sigma(E, F)$. Denote by $\sigma(\varphi)$ the set of all spectral values of φ. Analogously, one can define $\sigma_\pi(\varphi)$ and $\sigma_\delta(\varphi)$. Clearly, $\sigma(\varphi) = \sigma_\pi(\varphi) \cup \sigma_\delta(\varphi)$. Moreover $\sigma(\varphi)$ is closed and $\partial\sigma(\varphi) \subset \sigma_\pi(\varphi)$. Other properties of $\sigma(\varphi)$ and $\sigma_\pi(\varphi)$ can be obtained from the results contained in this section.

We close with the following observation. It is well-known that the set of all linear bounded operators from E into F, which are not onto is closed in $L(E, F)$. It is, therefore, natural to ask (see Proposition 5.1.1) whether the set $\sigma_\delta(E, F)$ of all non stably-solvable maps from E into F is closed or not in $C(E, F)$. We do not know the answer to this question.

7.3. *A Continuation Principle for regular maps.*

We point out that in the proof of Proposition 4.1.3 the compactness of the perturbing homotopy plays an essential role. The results contained in this section allow us to prove a Continuation Principle for regular maps where the compactness assumption on the perturbation is dropped (see Theorem 7.3.1 below).

We introduce the following notation. Let $H\colon E\times[0, 1] \to F$ and $\sigma\colon E \to [0, 1]$ be continuous maps. Define $H_\sigma\colon [0, 1] \to C(E, F)$ by $H_\sigma(\lambda)(x) = H\big(x, \lambda\sigma(x)\big)$. We need the following

LEMMA 7.3.1. – *Let $X \subset E$ be bounded and $\sigma\colon E \to [0, 1]$ be such that $\sigma(x) = 1$, for any $x \in X$. Assume that $H\colon E\times[0, 1] \to F$ is such that $H_\sigma\colon [0, 1] \to C(E, F)$ is continuous (in the strong topology of $C(E, F)$). Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\|H(x, t_1) - H(x, t_2)\| < \varepsilon$$

whenever $|t_1 - t_2| < \delta$ and $x \in X$.

PROOF. – The continuity of H_σ implies that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|H_\sigma(\lambda)(x) - H_\sigma(\lambda_1)(x)\| \leqslant \varepsilon(1 + \|x\|)$, whenever $|\lambda - \lambda_1| < \delta$. Therefore, for any $x \in X$ we have

$$\|H(x, \lambda) - H(x, \lambda_1)\| = \|H_\sigma(\lambda)(x) - H_\sigma(\lambda_1)(x)\| \leqslant (1 + M)\,,$$

where $M = \sup\{\|x\|\colon x \in X\} < +\infty$. Q.E.D.

We have the following Continuation Principle.

Theorem 7.3.1. – *Let $H: E\times[0,1]\to F$ be continuous and such that*

(*a*) *$H(\cdot, 0)$ is regular,*

(*b*) *there exists $a>0$ such that $\beta(H(\cdot, t))\geqslant a$ for any $t\in[0,1]$,*

(*c*) *any compact set $K\subset E$ has an open neighbourhood U such that the map H_σ is continuous, whenever $\sigma: E\to[0,1]$ is a continuous map whose support* $\operatorname{supp}\sigma\subset U$.

Then, the equation $H(x,1)=0$ has a solution $x\in E$, provided that the set $H^{-1}(0)$ is compact.

Proof. – By assumption the set $S=\{x\in E: H(x,t)=0 \text{ for some } t\in[0,1]\}$ is compact. Let U be a neighbourhood of S satisfying condition (*c*). Due to the fact that E is normal, there exists an open set V such that $S\subset V\subset \overline{V}\subset U$. By taking a continuous function $\varphi: E\to[0,1]$ such that $\operatorname{supp}\varphi\subset U$ and $\varphi(x)=1$ for any $x\in\overline{V}$, we obtain, on the basis of Lemma 7.3.1, that for any $\varepsilon>0$ there exists $\delta>0$ such that $\|H(x,t_1)-H(x,t_2)\|<\varepsilon$, whenever $|t_1-t_2|<\delta$ and $x\in V$. Now, let $\sigma: E\to[0,1]$ be continuous and such that $\operatorname{supp}\sigma\subset V$ and $\sigma(x)=1$ for any $x\in S$. It suffices to show that the equation $H(x,\sigma(x))=0$ has a solution $x_0\in E$. In fact, since $0\leqslant\sigma(x_0)\leqslant 1$, we have that $x_0\in S$ and $\sigma(x_0)=1$.

Observe that $H_\sigma(0)=H(\cdot,0)$ is regular, i.e., $H_\sigma(0)\notin\sigma(E,F)$. Since the map H_σ is continuous it is enough to show that $H_\sigma(\lambda)\notin\sigma_\pi(E,F)$ for any $\lambda\in[0,1]$. In fact, if this is the case, we obtain that $H_\sigma(1)$ is regular, since $\partial\sigma(E,F)\subset\sigma_\pi(E,F)$.

Let us prove that $d(H_\sigma(\lambda))>0$ and $\beta(H_\sigma(\lambda))>0$ for any $\lambda\in[0,1]$. The first inequality is easily verified since $H_\sigma(\lambda)(x)=H(x,0)$ for any $\lambda\in[0,1]$ and any $x\in E$ of sufficiently large norm. It remains to show that $\beta(H_\sigma(\lambda))>0$ for any $\lambda\in[0,1]$.

Let $A\subset E$ be bounded with $\alpha(A)>0$ and let $\varepsilon>0$ be given. By our choice of V there exists $n\in\boldsymbol{N}$ such that $|t_1-t_2|<1/n$ implies that $\|H(x,t_1)-H(x,t_2)\|<\varepsilon\alpha(A)$ for any $x\in V$. Define $A_0=\{x\in A:\sigma(x)=0\}$, $A_K=\{x\in A: (K-1)/n<\sigma(x)\leqslant K/n\}$, $K=1,2,\dots,n$. Observe that $A=\cup\{A_K: K=0,1,\dots,n\}$, $A_K\subset V$ for any $K\geqslant 1$ and $\alpha(A)=\alpha(A_i)$ for some i (recall $\alpha(X\cup Y)=\max\{\alpha(X),\alpha(Y)\}$).

Therefore,

$$\frac{\alpha(H_\sigma(\lambda)(A))}{\alpha(A)}=\frac{\alpha(H_\sigma(\lambda)(A))}{\alpha(A_i)}\geqslant\frac{\alpha(H_\sigma(\lambda)(A_i))}{\alpha(A_i)}\,.$$

Since, $A\subset E$ and $\varepsilon>0$ are arbitrary, we obtain $\beta(H_\sigma(\lambda))\geqslant a$ for any $\lambda\in[0,1]$, if we show that $\alpha(H_\sigma(\lambda)(A))/\alpha(A)\geqslant a-2\varepsilon$. This is clearly satisfied if the above inequality holds for $i=0$. In fact,

$$a\leqslant\beta(H(\cdot,0))\leqslant\frac{\alpha(H(A_0,0))}{\alpha(A_0)}=\frac{\alpha(H_\sigma(\lambda)(A_0))}{\alpha(A_0)}\,,$$

since $\sigma(x) = 0$ for any $x \in A_0$. Assume, therefore, that $i \geqslant 1$. Clearly, for any $x \in A_i$ and any $\lambda \in [0, 1]$ we have $|\lambda\sigma(x) - \lambda i/n| \leqslant 1/n$. Thus, $\|H_\sigma(\lambda)(x) - H(x, \lambda i/n)\| = \|H(x, \lambda\sigma(x)) - H(x, \lambda i/n)\| < \varepsilon\alpha(A)$, for any $x \in A_i$, since $A_i \subset V$. This implies that

$$\alpha(H_\sigma(\lambda)(A_i)) \geqslant \alpha(A_i, \lambda i/n)) - 2\varepsilon\alpha(A)\,,$$

$\big($recall $\alpha(B(X, \varepsilon)) \leqslant \alpha(X) + 2\varepsilon\big)$. Therefore,

$$\frac{\alpha(H_\sigma(\lambda)(A_i))}{\alpha(A_i)} \geqslant \frac{\alpha(H(A_i, \lambda i/n)) - 2\varepsilon\alpha(A)}{\alpha(A_i)} \geqslant a - 2\varepsilon\,,$$

since $\beta\, H(\cdot, \lambda i/n)) \geqslant a$ and $\alpha(A) = \alpha(A_i)$. Q.E.D.

We have the following

COROLLARY 7.3.1. – *Let $H: E \times [0, 1] \to F$ be continuous and satisfy conditions (a), (b) and (c) of Theorem 7.3.1. Assume that*

(d) for any $t \in [0, 1]$ we have

$$\lim_{t \to t_0} H(x, t) = H(x, t_0)$$

uniformly on any bounded subset of E.

Then, the equation $H(x, 1) = h(x)$ has a solution for any compact map $h: E \to F$ with bounded support provided that the set $S = \{x \in E: H(x, t) = 0$ for some $t \in [0, 1]\}$ is bounded. If, moreover, $d(H(\cdot, 1)) > 0$, then $H(\cdot, 1)$ is regular.

PROOF. – Let $h: E \to F$ be compact with bounded support. Set

$$\hat{H}(x, t) = H(x, t) - h(x)$$

and

$$\hat{S} = \{x \in E: \hat{H}(x, t) = 0 \text{ for some } t \in [0, 1]\}\,.$$

On the basis of Theorem 7.3.1 $H(x, 1) = 0$ has a solution if we show that $\hat{S}$ is compact. Take a sequence $\{x_n\}$ in $\hat{S}$. There exists a sequence $\{t_n\}$ in $[0, 1]$ such that $\hat{H}(x_n, t_n) = 0$. We may assume that $t_n \to t \in [0, 1]$. Now, $\hat{S}$ is bounded since $\hat{H}(x, t) = H(x, t)$ for $\|x\|$ sufficiently large. Hence, by condition (d), we get

$$\hat{H}(x_n, t) = \hat{H}(x_n, t) - \hat{H}(x_n, t_n) \to 0$$

as $n \to +\infty$. Now, $\beta(\hat{H}(\cdot, t)) = \beta(H(\cdot, t)) > 0$.

Therefore, $\{x_n\}$ has a convergent subsequence. This implies that $\hat{S}$ is compact. Therefore, the equation $H(x, 1) = h(x)$ has a solution $x \in E$.

Now, if $d(H(\cdot, 1)) > 0$, then on the basis of Proposition 5.1.1 we get that $H(\cdot, 1)$ is regular. Q.E.D.

The proof of Corollary 7.3.1 shows that if we replace the assumption $d(H(\cdot, 1)) > 0$ with the weaker assumption

$$\lim_{\|x\|\to+\infty} \|H(x, 1)\| = +\infty,$$

we obtain that $H(\cdot, 1)$ is a proper strong surjection (use Proposition 4.1.2).

Notice that conditions (*c*) and (*d*) of Corollary 7.3.1 are verified if $H: E\times[0, 1] \to F$ is such that

(C_1) *for any continuous* $\sigma: E \to [0, 1]$ *with bounded support the map* $H_\sigma: [0, 1] \to C(E, F)$ *is continuous.*

We shall give now some other sufficient conditions ensuring that assumption (*c*) of Theorem 7.3.1 is satisfied.

(C_2) *The continuous map* $H: E\times[0, 1] \to F$ *is such that* $t \mapsto H(\cdot, t)$ *is continuous from* $[0, 1]$ *to* $C(E, F)$.

Actually, (C_2) implies (C_1). In fact, if $\sigma: E \to [0, 1]$ has bounded support and $\varepsilon > 0$, then there exists $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies

$$\|H(x, t_1) - H(x, t_2)\| \leqslant \varepsilon(1 + \|x\|) .$$

Thus

$$\|H_\sigma(\lambda_1)(x) - H_\sigma(\lambda_2)(x)\| \leqslant \varepsilon(1 + \|x\|) ,$$

whenever $|\lambda_1 - \lambda_2| < \delta$. On the other hand, there exists $\delta_1 > 0$ such that $|t_1 - t_2| < \delta_1$ implies $\alpha(H(\cdot, t_1) - H(\cdot, t_2)) < \varepsilon$. With a technique similar to that used in Theorem 7.3.1 one obtains $\alpha(H_\sigma(\lambda_1) - H_\sigma(\lambda_2)) < \varepsilon$, provided that $|\lambda_1 - \lambda_2| < \delta_1$. Thus, $H_\sigma: [0, 1] \to C(E, F)$ is continuous.

(C_3) $H(x, t) = f(x) + \lambda(t)g(x)$, *where* $f, g: E \to F$, $\lambda: [0, 1] \to \mathbf{K}$ *are continuous and, moreover,* g *is locally* α*-Lipschitz (in particular this holds if* $g = g_1 + h$, *where* g_1 *is a* C^1 *map and* h *is compact).*

The implication (C_3) $\Rightarrow$ (*c*) is based on the following three facts.

(1) for any compact set $K \subset E$ there exists a bounded open set $U \supset K$ such that the restriction $g|U$ is α-Lipschitz;

(2) for any continuous $\sigma: E \to [0,1]$ with $\operatorname{supp} \sigma \subset U$ we have

$$H_\sigma(t_1)(x) - H_\sigma(t_2)(x) = \lambda(t_1\sigma(x)) - \lambda(t_2\,\sigma(x))\,g(x)$$

(3) $\lambda(t_1\sigma(x)) - \lambda(t_2\sigma(x)) = 0$ if $x \notin U$ and $\lambda(t\sigma(x)) \to \lambda(t_0\sigma(x))$ uniformly in E, for any $t_0 \in [0,1]$, as $t \to t_0$.

REMARK 7.3.1. – Notice that condition (C_3) with the further assumption *g sends bounded sets into bounded sets*, implies conditions (c) and (d) of Corollary 7.3.1.

The following results on monotone operators are consequences of Corollary 7.3.1.

We recall first that a continuous map $f: E \to E$ is called *strongly monotone* if there exists $a > 0$ such that

$$\operatorname{Re}\langle f(x) - f(y), z'\rangle \geqslant a\|x - y\|^2$$

for all $x, y \in E$ and some $z' \in J(x-y)$, where $J: E \multimap E^*$ is the duality map defined by $J(x) = \{x' \in E^*: x'(x) = \|x\|^2 \text{ and } \|x'\| = \|x\|\}$. The symbol $\langle \cdot, \cdot \rangle$ stands for the pairing between E and its dual E^*.

PROPOSITION 7.3.1. – *Let $f: E \to E$ be a strongly monotone locally α-Lipschitz map, sending bounded sets into bounded sets. Then f is regular.*

PROOF. – Define a homotopy joining the identity with f by $H(x, t) = x + t(f(x) - x)$. We want to show that H satisfies all of the conditions of Corollary 7.3.1. Condition (a) is clearly satisfied since $H(x, 0) = x$. Conditions (c) and (d) are also satisfied since f is locally α-Lipschitz and sends bounded sets into bounded sets (see Remark 7.3.1). Moreover, $\operatorname{Re}\langle H(x,t) - H(y,t), z'\rangle \geqslant b\|x-y\|^2$, where $z' \in J(x-y)$ and $b = \min\{1, a\}$. Thus $\|H(x,t) - H(y,t)\| \geqslant b\|x-y\|$. This inequality gives the following. First, the set $S = \{x \in E: H(x,t) = 0 \text{ for some } t \in [0,1]\}$ is bounded. Second, $\beta(H(\cdot, t)) \geqslant b$ for any $t \in [0,1]$. Third, $d(H(\cdot, 1)) \geqslant b$. Now, apply Corollary 7.3.1. Q.E.D.

From Proposition 7.3.1 we get a more general result. We recall first two definitions. A continuous map $f: E \to E$ is said to be *monotone* if $\operatorname{Re}\langle f(x) - f(y), z'\rangle \geqslant 0$ for all $x, y \in E$ and some $z' \in J(x-y)$. The map f is called *coercive* if

$$\operatorname{Re}\langle f(x), y'\rangle / \|x\| \to +\infty \qquad \text{as } \|x\| \to +\infty,$$

where $y' \in J(x)$.

PROPOSITION 7.3.2. – *Let $f: E \to E$ be monotone, coercive and proper. Assume, moreover, that f is locally α-Lipschitz, sending bounded sets into bounded sets. Then f is a strong surjection.*

PROOF. – Clearly, by Proposition 7.3.1, the maps $f_n(x) = (1/n)x + f(x)$, $n \in \mathbf{N}$, are regular. Therefore, if $h: E \to E$ is continuous and such that $\overline{h(E)}$ is compact, there exists a sequence $\{x_n\}$ in E satisfying $(1/n)x_n + f(x_n) = h(x_n)$. This equality implies

$$(1/n)\|x_n\| + \frac{\mathrm{Re}\,\langle f(x_n), y_n'\rangle}{\|x_n\|} = \frac{\mathrm{Re}\,\langle h(x_n), y_n'\rangle}{\|x_n\|},$$

where $y_n' \in J(x_n)$. Thus, $\mathrm{Re}\,\langle f(x_n), y_n'\rangle/\|x_n\| \leqslant \|h(x_n)\|$. Since f is coercive and $h(E)$ is bounded, the sequence $\{x_n\}$ must be bounded. Therefore, $f(x_n) - h(x_n) \to 0$ as $n \to +\infty$. Since $\{h(x_n)\}$ is a compact sequence, so is $\{f(x_n)\}$. Now, the properness of f implies the existence of a cluster point $\bar{x}$ of $\{x_n\}$. Clearly, $f(\bar{x}) = h(\bar{x})$. Therefore, f is a strong surjection. Q.E.D.

In Section 10 we shall give a result generalizing Proposition 7.3.2 (see Theorem 10.2.1).

We would like to remark that Corollary 7.3.1 contains the following well-known Continuation Principle for linear operators which is a classical tool in proving the existence of solutions of partial differential equations.

PROPOSITION 7.3.3 (see J. SCHAUDER [23]). – *Let $L_0, L_1: E \to E$ be linear and bounded operators. Assume*

(*a*) *L_0 is an isomorphism,*

(*b*) *there exists a real number $k > 0$ such that $\|x\| \leqslant k\|H(x, t)\|$ for any $x \in E$ and $t \in [0, 1]$, where $H(x, t) = L_0(x) + t\big(L_1(x) - L_0(x)\big)$. Then L_1 is an isomorphism.*

PROOF. – We have that $H(\cdot, 0) = L_0(\cdot)$ is regular since L_0 is an isomorphism (see Proposition 6.1.2). Furthermore, condition (*b*) ensures that the remaining assumptions of Corollary 7.3.1 are satisfied. Hence, $H(\cdot, 1) = L_1(\cdot)$ is regular and, being linear and bounded, is an isomorphism. Q.E.D.

We close this section with the observation that Corollary 7.3.1 also contains the criterion for existence of solutions for compact vector fields based upon the homotopy invariance property of the Leray-Schauder topological degree (see [9] and Remark 6.1.1).

8. – The spectrum for nonlinear maps.

In the first part of this section we introduce a notion of spectrum $\sigma(f)$ for nonlinear maps $f: E \to E$, where E is a Banach space over $\boldsymbol{K}$. We show that this definition of spectrum coincides with the usual concept of spectrum for linear operators in the case when f is linear (this motivates the notation $\sigma(f)$).

We also consider a decomposition of the spectrum $\sigma(f)$. Namely, $\sigma(f)$ is expressed as the union of the approximate point spectrum $\sigma_\pi(f)$ of f and the approximate defect spectrum $\sigma_\delta(f)$ of f.

In the case when f is linear $\sigma_\pi(f)$ and $\sigma_\delta(f)$ are exactly the approximate point spectrum and the approximate defect spectrum for linear operators respectively. For these definitions see e.g. S. K. BERBERIAN [1].

The main result contained in the first part of this section is represented by Theorem 8.1.2 where we show that $\sigma(f)$ is a closed subset of $\boldsymbol{K}$ and that the boundary $\partial\sigma(f)$ of $\sigma(f)$ is contained in $\sigma_\pi(f)$. This extends well-known results of the linear spectral theory.

After giving a finer decomposition of $\sigma(f)$ we collect in Proposition 8.1.2 other properties of $\sigma(f)$.

Theorem 8.1.3 gives sufficient conditions for the surjectivity of the map $\lambda - f$, where f is a continuous map acting on a Banach space E and $\lambda \in \boldsymbol{K}$.

We close this part with Proposition 8.1.3 devoted to the study of the spectrum of a compact map defined on an infinite dimensional Banach space.

The second part of this section deals with the problem of nonemptiness of $\sigma(f)$. At the end of this part we give an example of a continuous map defined on $\boldsymbol{C}^2$ with empty spectrum.

In the third of this section we show that the multivalued map $f \mapsto\!\circ\, \sigma(f)$, that associates to f its spectrum, is upper semicontinuous. This is a generalization of a well-known result of the linear theory.

8.1. *Spectrum for nonlinear maps: definition and properties.*

Let $f: E \to E$ be a continuous map acting on a Banach space E over the field $\boldsymbol{K}$. Define the *spectrum* $\sigma(f)$ of f as follows

$$\sigma(f) = \{\lambda \in \boldsymbol{K}: \lambda - f \text{ is not regular}\},$$

or, equivalently,

$$\sigma(f) = \{\lambda \in \boldsymbol{K}: \lambda - f \in \sigma(E, E)\}.$$

Define also

$$\sigma_\pi(f) = \{\lambda \in \boldsymbol{K}: \lambda - f \in \sigma_\pi(E, E)\}.$$

and

$$\sigma_\delta(f) = \{\lambda \in \boldsymbol{K}: \lambda - f \in \sigma_\delta(E, E)\}.$$

We have the following

THEOREM 8.1.1. – *Let $L: E \to E$ be bounded and linear. Then*

(*a*) $\sigma(L)$ *is the usual spectrum of* L,

(*b*) $\sigma_\delta(L)$ *is the approximate defect spectrum of* L,

(*c*) $\sigma_\pi(L)$ *is the approximate point spectrum of* L.

PROOF. – (*a*) Follows directly from Proposition 6.1.2.

(*b*) We recall that the approximate defect spectrum of L is the set of all $\lambda \in \boldsymbol{K}$ such that $\lambda - L$ is not onto (see S. K. BERBERIAN [1]). On the basis of Theorem 5.1.1 we get

$$\sigma_\delta(L) = \{\lambda \in \boldsymbol{K}\colon \lambda - L \text{ is not onto}\}.$$

(*c*) We recall (see [1]) that the approximate point spectrum of L is the set of all $\lambda \in \boldsymbol{K}$ such that

$$\inf\{\|\lambda x - Lx\|\colon \|x\| = 1\} = 0 .$$

Let $\lambda \in \boldsymbol{K}$ be such that the above equality holds. Since

$$d(\lambda - L) = \liminf_{\|x\|\to+\infty} \frac{\|\lambda x - Lx\|}{\|x\|} = \inf_{\|x\|=1} \|\lambda x - Lx\| .$$

It follows that $d(\lambda - L) = 0$. Thus $\lambda \in \sigma_\pi(L)$.

Conversely, assume that $\lambda - L \in \sigma_\pi(E, E)$. In view of the fact that $\beta(\lambda - L) \geqslant d(\lambda - L)$ (see Proposition 3.2.1-(*a*)) we may assume that $d(\lambda - L) = 0$. This implies that

$$\inf\{\|\lambda x - Lx\|\colon \|x\| = 1\} = 0 . \qquad \text{Q.E.D}$$

On the basis of Theorem 8.1.1 $\sigma_\delta(f)$ and $\sigma_\pi(f)$ will be called *the approximate defect spectrum of* f and *the approximate point spectrum* of f respectively, also in the case when f is not linear.

The following results are well-known in the linear spectral theory. Let $L \in L(E, E)$, then

(*a*) $\sigma(L)$ is closed. More precisely, $\sigma_\pi(L)$ and $\sigma_\delta(L)$ are closed;

(*b*) the boundary $\partial\sigma(L)$ of $\sigma(L)$ is contained in $\sigma_\pi(L)$.

The next theorem shows that these properties hold true in the context of nonlinear maps (except for the fact that $\sigma_\delta(f)$ is closed).

THEOREM 8.1.2. – *Let* $f \in C(E, E)$. *Then*

(*a*) $\sigma(f) = \sigma_\pi(f) \cup \sigma_\delta(f)$, *is closed*;

(*b*) $\sigma_\pi(f)$ *is closed*;

(*c*) $\partial\sigma(f) \subset \sigma_\pi(f)$.

PROOF. – Consider the continuous map $\psi\colon \boldsymbol{K} \to C(E, E)$ defined by $\psi(\lambda) = \lambda - f$. We have $\sigma(f) = \psi^{-1}(\sigma(E, E))$, $\sigma_\pi(f) = \psi^{-1}(\sigma_\pi(E, E))$ and, by the continuity of ψ,

$\partial\sigma(f) = \partial\psi^{-1}(\sigma(E, E)) \subset \psi^{-1}(\partial\sigma(E, E))$. Now, since $\sigma(E, E)$ and $\sigma_\pi(E, E)$ are closed (see Theorem 7.1.1 and Lemma 7.2.1), then (*a*) and (*b*) hold true from the continuity of ψ.

Moreover, since $\partial\sigma(E, E) \subset \sigma_\pi(E, E)$ (see Theorem 7.2.1) we have

$$\partial\sigma(f) \subset \psi^{-1}(\partial\sigma(E, E)) \subset \psi^{-1}(\sigma_\pi(E, E)) = \sigma_\pi(f) . \qquad \text{Q.E.D.}$$

The following theorem gives sufficient conditions for the equation $\lambda x - f(x) = y$, $\lambda \in \boldsymbol{K}$, $f: E \to E$ continuous, to be solvable for any $y \in E$.

THEOREM 8.1.3. – *Let $f: E \to E$ be continuous. Then*

(*a*) *If λ_1, λ_2 belong to the same component of $\boldsymbol{K} \setminus \sigma_\pi(f)$, then $\lambda_1 - f$ and $\lambda_2 - f$ are either both regular or both not regular.*

(*b*) *If $\boldsymbol{K} = \boldsymbol{C}$, $\sigma(f)$ is bounded (this is true in particular if f is quasibounded and α-Lipschitz) and λ belongs to the unbounded component of $\boldsymbol{C} \setminus \sigma_\pi(f)$, then $\lambda - f$ is regular.*

(*c*) *If $\boldsymbol{K} = \boldsymbol{R}$, $\sigma(f)$ is bounded above (below) and λ belongs to the right (left) unbounded component of $\boldsymbol{R} \setminus \sigma_\pi(f)$, then $\lambda - f$ is regular.*

PROOF. – (*b*) and (*c*) follow from (*a*) which is a consequence of Corollary 7.2.1. Q.E.D.

Theorem 8.1.3 represents an extension of a result previously proved in [8] for the particular case when $f: E \to E$ is quasibounded and compact.

We recall that a continuous map f belongs to $\sigma_\pi(E, E)$ if either $d(f) = 0$ or $\beta(f) = 0$. On the basis of this fact one can consider a finer decomposition of $\sigma(f)$. More precisely, $\sigma_\pi(f)$ can be regarded as the union of the following two sets

$$\sigma_\beta(f) = \{\lambda \in \boldsymbol{K}: \beta(\lambda - f) = 0\}$$

and

$$\Sigma(f) = \{\lambda \in \boldsymbol{K}: d(\lambda - f) = 0\} .$$

The definition and properties of $\Sigma(f)$ where given in [9], [10] and [12].

Moreover we have the following

PROPOSITION 8.1.1. – *Let $f: E \to E$ be continuous. Then $\sigma_\beta(f)$ and $\Sigma(f)$ are closed.*

PROOF. – The result follows immediately from the proof of Lemma 7.2.1 and the continuity of the function $\psi: \boldsymbol{K} \to C(E, E)$ defined by $\psi(\lambda) = \lambda - f$. Q.E.D.

In what follows the notation $\sigma(f) \equiv \sigma(g)$ stands for $\sigma_\delta(f) = \sigma_\delta(g)$, $\Sigma(f) = \Sigma(g)$ and $\sigma_\beta(f) = \sigma_\beta(g)$ simultaneously.

PROPOSITION 8.1.2. – *Let $f, g: E \to E$ be continuous. Then*

(a) $\sigma(\lambda f) \equiv \lambda\sigma(f)$, $\lambda \in \boldsymbol{K}$;

(b) $\sigma(\lambda + f) \equiv \lambda + \sigma(f)$, $\lambda \in \boldsymbol{K}$.

(c) $\sigma(A \circ f \circ A^{-1}) \equiv \sigma(f)$, *for any linear isomorphism* $A : E \to E$.

(d) $r(f) \leqslant q(f)$, *where* $r(f) = \sup\{|\lambda|: \lambda \in \sigma(f)\}$ *and* $q(f) = \max\{\alpha(f)), |f|\}$.

(e) $q(f - g) = 0$ *implies* $\sigma(f) \equiv \sigma(g)$.

(f) *Let* $f_i: E_i \to E_i$, $(i = 1, 2)$, *be continuous.*

Then $\sigma(f_1 \times f_2) \supset \sigma(f\,) \cup \sigma(f_2)$. *More precisely,* $\Sigma(f_1 \times f_2) = \Sigma(f_1) \cup \Sigma(f_2)$; $\sigma_\beta(f_1 \times f_2) = \sigma_\beta(f_1) \cup \sigma_\beta(f_2)$ *and* $\sigma_\delta(f_1 \times f_2) \supset \sigma_\delta(f_1) \cup \sigma_\delta(f_2)$.

PROOF. – (a) The equality is obvious if $\lambda = 0$. Let $\lambda \neq 0$. Clearly, $\sigma_\beta(\lambda f) = \lambda\sigma_\beta(f)$ and $\Sigma(\lambda f) = \lambda\Sigma(f)$. Moreover, $\sigma_\delta(\lambda f) = \lambda\sigma_\delta(f)$ since f is stably-solvable if and only if so is λf.

(b) The proof runs as in (a).

(c) The equality $\Sigma(A \circ f \circ A^{-1}) = \Sigma(f)$ has been established in [9]. To prove $\sigma_\beta(f) = \sigma_\beta(A \circ f \circ A^{-1})$ it suffices to show that $\beta(f) = 0$ if and only if $\beta(A \circ f \circ A^{-1}) = 0$. Clearly, $\beta(A \circ f \circ A^{-1}) = 0$ implies $\beta(f) = 0$, since $\beta(A \circ f \circ A^{-1}) \geqslant \beta(A)\beta(f)\beta(A^{-1})$ and $\beta(A)$, $\beta(A^{-1})$ are different from zero (see Proposition 3.1.3-(g)).
The converse implication is obtained by setting $g = A \circ f \circ A^{-1}$ and by considering $\beta(A^{-1} \circ g \circ A)$.

(d) The fact that $|\lambda| > q(f)$ implies $d(\lambda - f) > 0$ has been shown in [8]. It also implies $\beta(\lambda - f) > 0$ since $\beta(\lambda - f) \geqslant |\lambda| - \alpha(f) \geqslant |\lambda| - q(f)$ (see Proposition 3.1.3-(e)). It remains only to show that if $|\lambda| > q(f)$ then $\lambda - f$ is stably-solvable. Since $q(f + h) = q(f)$ for any compact map h with $|h| = 0$, it is enough to show that the equation $\lambda x - f(x) = 0$ has a solution for any $|\lambda| > q(f)$. Now the map $\lambda^{-1} f$ is an α-contraction and the inequality $|f| < |\lambda|$ ensures that there exists $\Gamma > 0$ such that $\lambda^{-1} f$ maps the closed ball D_r into itself. By Darbo's fixed point theorem [3], the equation $x = \lambda^{-1} f(x)$ has a solution.

(e) Clearly, $|f - g| = 0$ implies $\Sigma(f) = \Sigma(g)$ (see [9]). Moreover, $\alpha(f - g) = 0$ implies $\beta(f) = \beta(g)$ (see Proposition 3.1.3-(f)). Finally, $q(f - g) = 0$ implies that $f = g + h$, where h is compact with $|h| = 0$. Thus f is stably-solvable if and only if so is g.

(f) For the sake of simplicity we take $E_1 \times E_2$ with norm $\|(x, y)\| = \max\{\|x\|, \|y\|\}$. The inclusion $\sigma_\beta(f_1) \cup \sigma_\beta(f_2) \subset \sigma_\beta(f_1 \times f_2)$ follows from the fact that if $\beta(f_1) = 0$ then $\beta(f_1 \times f_2) = 0$.
In fact, let $A_n \subset E$, $n \in \boldsymbol{N}$ be such that $\alpha(A_n) \neq 0$, $n \in \boldsymbol{N}$ and $\alpha(f_1(A_n))/\alpha(A_n)$ goes to zero as $n \to +\infty$. It suffices to consider the sequence of subsets of $E_1 \times E_2$

of the form $Q_n = A_n \times \{0\}$, $n \in \boldsymbol{N}$. Clearly, $\alpha(Q_n) = \alpha(A_n)$ and $\alpha(f_1 \times f_2(Q_n)) = \alpha(f_1(A_n))$. The opposite inclusion is a consequence of the following facts.

For any $Q \subset E_1 \times E_2$ we have $\alpha(Q) = \max\{\alpha(P(Q)), \alpha(P_2(Q))\}$, where $P_i: E_1 \times E_2 \to E_i$, $i = 1, 2$, are the canonical projections of $E_1 \times E_2$ onto E_i, $i = 1, 2$. Let $Q_n \subset E_1 \times E_2$, $n \in \boldsymbol{N}$ be such that $\alpha(f_1 \times f_2(Q_n))/\alpha(Q_n)$ tends to zero as $n \to +\infty$. Without loss of generality we may assume that for infinitely many indices $\alpha(Q_n) = \alpha(P_1(Q_n))$. On the other hand $P_1(f_1 \times f_2(Q_n)) = f_1(P_1(Q_n))$, $n \in \boldsymbol{N}$, and $\alpha(P_1(f_1 \times f_2(Q_n))) \leqslant \alpha(f_1 \times f_2(Q_n))$. Hence

$$\frac{\alpha(f_1(P_1(Q_n)))}{\alpha(P_1(Q_n))} = \frac{\alpha(P_1(f_1 \times f_2)(Q_n)))}{\alpha(P_1(Q_n))} \leqslant \frac{\alpha(f_1 \times f_2(Q_n))}{\alpha(Q_n)} \to 0 .$$

The equality $\Sigma(f_1 \times f_2) = \Sigma(f_1) \cup \Sigma(f_2)$ has been established in [10].

To show that $\sigma_\delta(f_1 \times f_2) \supset \sigma_\delta(f_1) \cup \sigma_\delta(f_2)$ it sufficies to prove that if $f_1 \times f_2: E_1 \times E_2 \to E_1 \times E_2$ is stably-solvable then so are $f_1: E_1 \to E_1$ and $f_2: E_2 \to E_2$: Let $h_1: E_1 \to E_1$ be compact with $|h_1| = 0$. Consider the map $h: E_1 \times E_2 \to E_1 \times E_2$ defined by $h(x, y) = (h_1(x), 0)$. Clearly, h is compact and $|h| = 0$. Hence the system

$$\begin{cases} f_1(x) + h_1(x) = 0 \\ f_2(y) = 0 \end{cases}$$

has a solution $(x, y) \in E_1 \times E_2$. This implies that f_1 is stably-solvable. The proof that f_2 is stably-solvable runs as above. Q.E.D.

REMARK 8.1.1. – We do not know whether the inclusion $\sigma_\delta(f_1 \times f_2) \subset \sigma_\delta(f_1) \cup \sigma_\delta(f_2)$ holds true. This would obviously imply $\sigma(f_1 \times f_2) \equiv \sigma(f_1) \cup \sigma(f_2)$.

REMARK 8.1.2. – From the proof of Theorem 8.1.1-(*c*) it follows that if $L: E \to E$ is bounded and linear, then $\sigma_\pi(L) = \Sigma(L)$. Thus, in this case, we have that $\sigma(L) = \sigma_\delta(L) \cup \Sigma(L)$ and $\partial\sigma(L) \subset \Sigma(L)$. It is, therefore, natural to ask whether the inclusion $\partial\sigma(f) \subset \Sigma(f)$ remains valid in the context of nonlinear maps.

The following example shows that the inclusion $\partial\sigma(f) \subset \Sigma(f)$ does not hold in general, and justifies the introduction of $\sigma_\beta(f)$.

In [7] an example of an α-Lipschitz retraction r of the unit closed ball D of a Banach space E onto its boundary was given. Define $f: E \to E$ by

$$f(x) = \begin{cases} r(x), & \text{if } x \in D \\ x, & \text{if } x \notin D . \end{cases}$$

Clearly $|f| = 1$ and f is α-Lipschitz. Hence $\sigma(f)$ is bounded. On the other hand $\Sigma(f) = \{1\}$. Therefore the inclusion $\partial\sigma(f) \subset \Sigma(f)$ would imply that $\sigma(f) = \{1\}$, contradicting the fact that $0 \in \sigma(f)$, being f not onto.

Theorem 8.1.2 shows that it is of interest to give estimates of $\sigma_\pi(f)$. In this direction we have the following

Proposition 8.1.3. – *Let $f\colon E \to E$ be continuous. Then*

(*a*) $\lambda \in \Sigma(f) \Rightarrow d(f) \leqslant |\lambda| \leqslant |f|$,

(*b*) $\lambda \in \sigma_\beta(f) \Rightarrow \beta(f) \leqslant |\lambda| \leqslant \alpha(f)$.

Proof. – Since $d(\lambda) = \alpha(\lambda) = \beta(\lambda) = |\lambda|$, then (*a*) and (*b*) follow from the inequalities $d(f-g) \geqslant d(f) - |g|$ and $\beta(f-g) \geqslant \beta(f) - \alpha(g)$. Q.E.D.

We shall restrict now our attention to the case when $f\colon E \to E$ is compact. As we will see in this case one can obtain a more precise information about the structure of the spectrum. Namely.

Proposition 8.1.4. – *Let $f\colon E \to E$ be a compact map defined on an infinite dimensional Banach space E. Then*

(*a*) $\sigma_\beta(f) = \{0\}$, *therefore* $\sigma_\pi(f) = \{0\} \cup \Sigma(f)$,

(*b*) $f(E) \neq E$. *In particular,* $0 \in \sigma_\delta(f)$,

(*c*) $0 \notin \Sigma(f)$ *implies that the connected component of* $\mathbf{K} \setminus \Sigma(f)$ *containing zero lies entirely in* $\sigma_\delta(f)$. *In particular,* 0 *is an interior point of* $\sigma_\delta(f)$.

(*d*) *If moreover f is positively homogeneous, then* $\Sigma(f) \setminus \{0\} = \{\lambda \in \mathbf{K}\colon \lambda x = f(x)$, for some $x \neq 0\}$.

Proof. – (*a*) follows immediately from the equality $\beta(\lambda - f) = |\lambda|$. (*b*) We have $f(E) = \cup \{f(D_n)\colon n \in \boldsymbol{N}\}$. Now, $f(D_n)$, $n \in \boldsymbol{N}$, are nowhere dense subsets of E since f is compact and $\dim E = +\infty$. Thus, $f(E) \neq E$ since E is of second category. (*c*) From (*a*) and the condition $0 \notin \Sigma(f)$ it follows that 0 is an isolated point of $\sigma_\pi(f)$. Therefore, on the basis of Theorem 8.1.3-(*a*), it suffices to show that for λ small enough $\lambda - f$ is not onto. Assume that this is not the case. Then there exists a sequence $\{\lambda_n\}$ in $\boldsymbol{K}$ such that $\lambda_n \to 0$ and $\lambda_n - f$ is onto for any $n \in \boldsymbol{N}$.

Take $a > 0$ such that $2a < d(f)$. There exists $b > 0$ such that $\|f(x)\| \geqslant 2a\|x\| - b$. In fact, by the definition of $d(f)$, there exists $r > 0$ such that $\|f(x)\| \geqslant 2a\|x\|$, when $\|x\| \geqslant r$. Hence, if we take $b = 2ar$ the inequality $\|f(x)\| \geqslant 2a\|x\| - b$ holds for all $x \in E$.

Therefore, if $|\lambda| < a$ we have $\|\lambda x - f(x)\| \geqslant a\|x\| - b$. Let $y \in E$ with $\|y\| \leqslant 1$. There exists a sequence $\{x_n\}$ in E such that $\lambda_n x_n - f(x_n) = y$. Without loss of generality we may assume that $|\lambda_n| < a$ for any $n \in \boldsymbol{N}$. Thus, $1 \geqslant \|\lambda_n x_n - f(x_n)\| \geqslant$ $= a\|x_n\| - b$. Hence, $\|x_n\| \leqslant (1 + b)/a = c$. Therefore, $f(x_n) \to -y$, since $\lambda_n x_n \to 0$. This shows that $\overline{f(D_c)} \supset D_1$, which is impossible since $\overline{f(D_c)}$ is compact. (*d*) follows from

$$\liminf_{\|x\| \to +\infty} \frac{\|\lambda x - f(x)\|}{\|x\|} = \inf_{\|x\| = 1} \|\lambda x - f(x)\|$$

and the compactness of f. Q.E.D.

8.2. *Nonemptiness of the spectrum for nonlinear maps.*

Let E be a Banach space over the field $\boldsymbol{C}$ of complex numbers. It is well-knwon that the spectrum $\sigma(L)$ of a linear bounded operator $L: E \to E$ is not empty (it may be empty if E is a Banach space over the reals $\boldsymbol{R}$).

We shall now investigate the problem of nonemptiness of the spectrum of a nonlinear map $f: E \to E$, where E is over $\boldsymbol{K}$ ($\boldsymbol{C}$ or $\boldsymbol{R}$).

The first result in this direction regards compact maps defined on infinite dimensional Banach spaces.

THEOREM 8.2.1. – *Let* $f: E \to E$ *be continuous compact and* $\dim E = +\infty$. *We have*

(*a*) $\sigma_\beta(f) = \{0\}$,

(*b*) $0 \in \sigma_\delta(f)$,

(*c*) *if* $\sigma(f) \neq \boldsymbol{K}$, *then* $\Sigma(f) \neq \emptyset$,

(*d*) *if* $0 \notin \Sigma(f)$ *and* $\sigma(f)$ *is bounded, then the connected component of* $\boldsymbol{K}\setminus\Sigma(f)$ *containing* 0 *is bounded. Thus,* $\Sigma(f)$ *contains a positive and a negative value.*

PROOF. – (*a*) and (*b*) have been proved in Proposition 8.1.4. (*c*) If $0 \in \Sigma(f)$ we are done. Assume $0 \notin \Sigma(f)$, then Proposition 8.1.4-(*e*) ensures that $0 \notin \partial\sigma(f)$. Now. $\partial\sigma(f) \subset \sigma_\pi(f)$ (see Theorem 8.1.2-(*c*)) and $\sigma_\pi(f) = \{0\} \cup \Sigma(f)$, (see Proposition 8.1.4-(*a*)). Thus, $\partial\sigma(f) \subset \Sigma(f)$. The result now follows from the fact that $\partial\sigma(f) \neq \emptyset$ since $0 \in \sigma(f)$ and $\sigma(f) \neq \boldsymbol{K}$.

(*d*) Follows directly from Proposition 8.1.4-(*e*). Q.E.D.

Theorem 8.2.1-(*d*) was already proved in [12] for the case when f is quasibounded.

The following proposition is a further result about the nonemptiness of the spectrum for possibly noncompact maps.

PROPOSITION 8.2.1. – *Let* $f: E \to E$ *be quasibounded and* α*-Lipschitz. Assume that* $\dim E = +\infty$ *and* $\alpha(f) < d(f)$. *Then* $\Sigma(f) \neq \emptyset$.

Proposition 8.2.1 is a consequence of a more general result that will be proved later (see Theorem 11.1.1).

The following example shows that Proposition 8.2.1 is false without quasiboundedness assumptions. Let E be an infinite dimensional Banach space and $e \in E$ be such that $\|e\| = 1$. Define $f: E \to E$ by $f(x) = \|x\|^2 e$. Clearly, $0 = \alpha(f) < d(f) = +\infty$. But, $\Sigma(f) = \emptyset$.

Nevertheless, we have the following

PROPOSITION 8.2.2. – *Let* E *be an infinite dimensional Banach space and* $f: E \to E$ *be such that* $\alpha(f) < d(f)$. *Then* 0 *is an interior point of* $\sigma_\delta(f)$.

Proof. – We will show first that if $f\colon E\to E$ is onto then $\alpha(f)\geqslant d(f)$. If $d(f)=0$ we are done. Assume $d(f)>0$ and take $0<a<d(f)$. There exists $b>0$ such that $\|f(x)\|\geqslant a\|x\|-b$ for any $x\in E$. This implies that if $f(x)\in D_n=\{x\in E\colon \|x\|\leqslant n\}$ then $\|x\|\leqslant(n+b)/a=r_n$. This inequality and the surjectivity of f gives $f(D_{r_n})\subset D_n$. Therefore,

$$\alpha(f)\geqslant\frac{\alpha(f(D_{r_n}))}{\alpha(D_{r_n})}\geqslant\frac{2n}{2r_n}=\frac{n}{n+b}\,a\,,$$

(we recall that if $\dim E=+\infty$ then $\alpha(D_r)=2r$, see R. D. Nussbaum [21] and [8]). Therefore, $\alpha(f)\geqslant\lim na/(n+b)=a$. Thus, $\alpha(f)\geqslant d(f)$.

The continuity of α and d and the assumption $\alpha(f)<d(f)$ imply that there exists $\varepsilon>0$ such that $\alpha(\lambda-f)<d(\lambda-f)$ for any $|\lambda|<\varepsilon$. Hence, $\lambda-f$ is not onto if $|\lambda|<\varepsilon$. Q.E.D.

Observe that the above proposition implies $\sigma_\pi(f)\neq\emptyset$ in the case when $\sigma(f)\neq\boldsymbol{K}$ (recall that $\partial\sigma(f)\subset\sigma_\pi(f)$).

In the context of finite dimensional real spaces we have the following.

Proposition 8.2.3. – *Let $f\colon\boldsymbol{R}^{2n+1}\to\boldsymbol{R}^{2n+1}$ ($n\in\boldsymbol{N}$) be quasibounded. Then $\Sigma(f)\neq\emptyset$.*

This proposition will be proved later in a more general form (see Theorem 11.1.2).

Notice that Proposition 8.2.3 is false without quasiboundedness assumption (even if $\Sigma(f)$ is replaced by $\sigma(f)$). In fact, consider $f\colon\boldsymbol{R}\to\boldsymbol{R}$ defined by $f(x)=x^3$. We have $d(f)=\beta(f)=+\infty$. This implies that $\sigma_\pi(f)=\emptyset$. Moreover, $\lambda-f$ is clearly stably-solvable for any $\lambda\in\boldsymbol{R}$. Thus, $\sigma(f)=\emptyset$.

We give now an example of a continuous map with empty spectrum.

Define $f\colon\boldsymbol{C}^2\to\boldsymbol{C}^2$ by $f(x,y)=(\bar{y},i\bar{x})$, $(x,y)\in\boldsymbol{C}^2$. The fact that $\Sigma(f)=\emptyset$ has been established in [12]. Moreover, $\sigma_\beta(f)=\emptyset$ since $\beta(\lambda-f)=+\infty$ for any $\lambda\in\boldsymbol{C}$. It remains to show that $\sigma_\delta(f)=\emptyset$. This is a consequence of the following two facts:

(i) $\partial\sigma(f)\subset\sigma_\pi(f)$,

(ii) $|\lambda|>q(f)$, $\lambda\in\boldsymbol{K}$, implies that $\lambda-f$ is regular.

Since, in our case, $\sigma_\pi(f)=\Sigma(f)\cup\sigma_\beta(f)=\emptyset$, then either $\sigma(f)=\boldsymbol{C}$ or $\sigma(f)=\emptyset$. But $q(f)=1$ implies that $\sigma(f)\neq\boldsymbol{C}$. Hence $\sigma(f)=\emptyset$.

8.3. *Upper semicontinuity of the spectrum.*

It is well-known (see e.g. T. Kato [15]) that the multivalued map that associates to every $A\in L(E,E)$ its spectrum $\sigma(A)$ is upper semicontinuous.

In this section we will extend this result to the context of nonlinear maps. To this aim we show first that the multivalued map that to every $f\in C(E,E)$ associates its spectrum $\sigma(f)$ is upper semicontinuous in a sense to be specified below. From this fact will follow that when we restrict our attention to maps $f\in Q(E,E)$ then we get the usual upper semicontinuity, thus generalizing the result of the linear case. Throughout this section the topology considered in $C(E, E)$ and $Q(E, E)$ is the q-topology.

THEOREM 8.3.1. – *Let $f: E \to E$ be continuous and let $M \subset \mathbf{K}$ be compact. Assume that $M \cap \sigma(f) = \emptyset$. Then there exist $\varepsilon > 0$ such that $\mu \notin \sigma(g)$, whenever $\mu \in M$ and $q(f - g) < \varepsilon$.*

PROOF. – Assume the contrary. Then there exist a sequence $\{f_n\}$ in $C(E, E)$ and a sequence $\{\mu_n\}$ in M such that $q(f_n - f) \to 0$ and $\mu_n \in \sigma(f_n)$. Without loss of generality we may assume that $\mu_n \to \mu \in M$. We have $q\big((\mu_n - f_n) - (\mu - f)\big) \leqslant |\mu_n - \mu| + q(f_n - f)$. Hence $\mu_n - f_n$ converges to $\mu - f$ in the q-topology of $C(E, E)$. The closedness of $\sigma(E, E)$ implies that $\mu - f \in \sigma(E, E)$, i.e. $\mu \in \sigma(f)$. A contradiction. Q.E.D.

Analogous results hold true if we replace in Theorem 8.3.1 $\sigma(f)$ by $\sigma_\pi(f)$, $\sigma_\beta(f)$ or $\Sigma(f)$ respectively.

REMARK 8.3.1. – The following existence result is a direct consequence of Theorem 8.3.1.

Let $f: E \to E$ be continuous and let $M \subset \mathbf{K}$ be compact and such that $M \cap \sigma(f) = \emptyset$. Then there exists $\varepsilon > 0$ such that the equation

$$\mu x - f(x) = g(x)$$

has a solution $x \in E$ for any $\mu \in M$ and $g \in C(E, E)$ with $q(g) < \varepsilon$.

Theorem 8.3.1 says that the multivalued map $f \mapsto\!\circ\, \sigma(f)$ is upper semicontinuous in the following sense.

For any $f \in C(E, E)$ and any open neighbourhood U of $\sigma(f)$ whose complement in $\mathbf{K}$ is bounded there exists a neighbourhood V of f (in the q-topology, see Remark 7.1.1) such that $\sigma(g) \subset U$ for any $g \in V$.

In the subspace $Q(E, E)$ of $C(E, E)$ we have the following stronger result.

THEOREM 8.3.2. – *The multivalued map $\sigma: Q(E, E) -\!\circ\, \mathbf{K}$, which assigns to each $f \in Q(E, E)$ its spectrum $\sigma(f)$, is upper semicontinuous (with compact values).*

PROOF. – Let $U \supset \sigma(f)$ be open. Take $r > q(f) + 1$ and set $M = D_r \setminus U$, where $D_r = \{\mu \in \mathbf{K}: |\mu| \leqslant r\}$. Clearly, M is compact and $M \cap \sigma(f) = \emptyset$. Choose $\varepsilon > 0$ as in Theorem 8.3.1 and such that $\varepsilon < 1$. For any g such that $q(f - g) < \varepsilon$ we have $\sigma(g) \cap M = \emptyset$. Moreover,

$$q(g) \leqslant q(f) + q(f - g) < q(f) + \varepsilon < q(f) + 1 < r\,.$$

Therefore, $\sigma(g) \subset D_r$ since the spectral radius $r(g)$ of g is less than or equal to $q(g) < r$ (see Proposition 8.1.2-(d)). Thus $\sigma(g) \subset D_r \setminus M \subset U$. Q.E.D.

Theorem 8.3.2 remains valid if $\sigma(f)$ is replaced by $\Sigma(f)$, $\sigma_\beta(f)$ and $\sigma_\pi(f)$ respectively.

The upper semicontinuity of the map $f \mapsto \Sigma(f)$, for f quasibounded, was already proved in [12].

As a consequence of Theorem 8.3.2 we have the following.

COROLLARY 8.3.1. – *The set*

$$\Omega(E) = \{f \in Q(E, E) : \sigma(f) \neq \emptyset\},$$

is closed.

PROOF. – We shall show that the complement of $\Omega(E)$ is open. Let $f \notin \Omega(E)$. By Theorem 8.3.2 given any open set $U \supset \sigma(f)$ there exists a neighbourhood V of f such that $\sigma(g) \subset U$ for any $g \in V$. Now, take $U = \emptyset$. Q.E.D.

9. – Fredholm alternative for nonlinear maps.

In the first part of this section we observe that a well-known alternative for linear operators can be interpreted in the setting of the linear spectral theory. This interpretation leads naturally to the definition of alternative maps. We give also an extension of the Fredholm alternative to the context of nonlinear compact maps.

The second part is devoted to the study of (not necessarily compact) alternative maps.

9.1. *Fredholm alternative for nonlinear compact maps.*

Let $K: E \to E$ be a compact linear operator. We recall the following well-known Fredholm alternative:

Let $\lambda \in \boldsymbol{K}$, assume moreover $\lambda \neq 0$ if $\dim E = +\infty$. Then the equation $\lambda x - Kx = y$ is solvable for any $y \in E$ if and only if the equation $\lambda x - Kx = 0$ has only the trivial solution (i.e. λ is not an eigenvalue for K).

Notice that if $\dim E = +\infty$, then $\beta(\lambda - K) = |\lambda|$ and $\beta(\lambda - K) = +\infty$ if $\dim E < +\infty$. Therefore, $\lambda \in \boldsymbol{K} (\lambda \neq 0$ if $\dim E = +\infty)$ is an eigenvalue for K if and only if $\lambda \in \sigma_\pi(K)$ (see Proposition 3.2.3). This shows that the above alternative is equivalent to the equality $\sigma(K) = \sigma_\pi(K)$.

This fact motivates the following definition. A continuous map $f: E \to E$ is said to be *alternative* if $\sigma(f) = \sigma_\pi(f)$.

In order to extend the above Fredholm alternative to the nonlinear case we recall the definition of asymptotically odd map introduced in [9].

A map $f: E \to E$ is said to be *asymptotically odd* if there exists an odd map $g: E \to E$ such that $|f - g| = 0$. If f is also compact then there exists a compact odd map $k: E \to E$ such that $|f - k| = 0$. In fact, define the odd compact map k by $k(x) = \big(f(x) - f(-x)\big)/2$. We have

$$\|f(x) - k(x)\| \leq \tfrac{1}{2}\|f(x) - g(x)\| + \tfrac{1}{2}\|g(-x) - f(-x)\|,$$

hence $|f - k| = 0$.

LEMMA 9.1.1. – *Let $f: E \to E$ be odd and compact. Then f is alternative.*

PROOF. – Let $\lambda \in \boldsymbol{K}$ be such that $d(\lambda - f) > 0$. It is enough to show that $\lambda - f$ satisfieds property (a) of Proposition 5.1.1. Let $h: E \to E$ be compact with bouded support. Clearly, there exists $r > 0$ such that $h(x) = 0$ and $\lambda x - f(x) \neq 0$ for any x with $\|x\| \geq r$. Since $\lambda \neq 0$ $\big($recall that $\beta(f) = 0\big)$ the vector field $G: D_r \to E$ defined by $G(x) = x - \lambda^{-1}f(x) + \lambda^{-1}h(x)$ is odd and singularity free on $S_r = \{x \in E: \|x\| = r\}$. By Borsuk's theorem (see e.g. A. GRANAS [13]) it vanishes at some point $x_0 \in D_r$. Thus $\lambda x_0 - f(x_0) + h(x_0) = 0$ and hence $\lambda - f$ is stably-solvable. Q.E.D.

Theorem 9.1.1 below can be regarded as an extension of the Fredholm alternative for linear compact operators and (by Proposition 3.2.3) it contains as well the non-linear version given in [6] (Theorem 3.2). Actually in [6] the Authors study an operator of the form $\lambda T - S$, where T, $S: E \to F$ and $\lambda \in \boldsymbol{K}$. However, since T is a suitable homeomorphism, this operator can be reduced into one of the form $\lambda - f$ (f compact and homogeneous), by considering the composite map

$$F \xrightarrow{T^{-1}} E \xrightarrow{\lambda T - S} F\,.$$

THEOREM 9.1.1. – *Let $f: E \to E$ be asymptotically odd and compact. Then f is alternative.*

PROOF. – There exists a compact odd map $g: E \to E$ such that $|f - g| = 0$. Since $f - g$ is compact we get $q(f - g) = 0$. By Proposition 8.1.2-(e) we have $\sigma(f) \equiv \sigma(g)$. By Lemma 9.1.1 $\sigma(g) = \sigma_\pi(g)$. Hence $\sigma(f) = \sigma_\pi(g) = \sigma_\pi(f)$. Q.E.D.

9.2. *Further examples of alternative maps.*

Notice that the equalities $\sigma(\mu + f) = \mu + \sigma(f)$, $\sigma_\pi(\mu + f) = \mu + \sigma_\pi(f)$, imply that $\mu + f$ is alternative whenever so is f. This furnishes examples of non compact alternative maps. Other examples are given below.

There are several definitions of *essential spectrum* $\sigma_e(A)$ for a bounded linear operator $A: E \to E$. We shall use the following

$$\sigma_e(A) = \cap \{\sigma(A + K): K \text{ compact and linear}\}.$$

It is well-known that this definition coincides with the following

$$\sigma_e(A) = \{\lambda \in \boldsymbol{K} \colon \lambda - A \text{ is not Fredholm of index zero}\}.$$

We say that a bounded linear operator $A\colon E \to E$ is *balanced* if $\sigma_\beta(A) = \sigma_e(A)$. Notice, that in general $\sigma_\beta(A) \subset \sigma_e(A)$, since $\beta(\lambda - A) = 0$ means that $\lambda - A$ is not left semi-Fredholm (see Proposition 3.2.1-(d)).

We shall show (see Theorem 9.2.1 below) that any balanced operator is alternative. The following are examples of balanced operators.

(a) The linear operator $x \to \lambda x$ is balanced for any $\lambda \in \boldsymbol{K}$. Clearly, $\sigma_e(\lambda) = \sigma_\beta(\lambda) = \{\lambda\}$ if $\dim E = +\infty$ and $\sigma_e(\lambda) = \sigma_\beta(\lambda) = \emptyset$ if $\dim E < +\infty$.

(b) Any linear operator $A\colon E \to E$, $\dim E < +\infty$, is balanced. In this case $\sigma_e(A) = \sigma_\beta(A) = \emptyset$.

(c) Any normal operator A acting on a Hilbert space H is balanced (recall that A is normal if $A \circ A^* = A^* \circ A$, where A^* is the adjoint of A, thus any self-adjoint operator is normal). In fact, if $A\colon H \to H$ is normal; then $H = \operatorname{Ker} A \oplus \overline{\operatorname{Im} A}$ (see e.g. A. Taylor [24]). Therefore, if $\lambda \notin \sigma_\beta(A)$ we have that $\lambda - A$ is normal, $\dim \operatorname{Ker}(\lambda - A) < +\infty$ and $\operatorname{Im}(\lambda - A)$ is closed. Thus, $H = \operatorname{Ker}(\lambda - A) \oplus \operatorname{Im}(\lambda - A)$. This shows that $\lambda - A$ is Fredholm of index 0, i.e., $\lambda \notin \sigma_e(A)$.

(d) Let $A\colon E \to E$ be bounded, linear and such that the iterate A^n, for some $n \in \boldsymbol{N}$, is compact. Then A is balanced. To prove this we may assume that $\dim E = +\infty$. We have $0 \in \sigma_\beta(A)$, since $\beta(A)^n \leqslant \beta(A^n) \leqslant \alpha(V^n) = 0$ (see Proposition 3.1.3-(b)). It remains to show that $\lambda \neq 0$ implies that $\lambda \notin \sigma_e(A)$ (recall $\sigma_\beta(A) \subset \sigma_e(A)$). We have $\lambda^n - A^n = T \circ (\lambda - A)$, where

$$T = \lambda^{n-1} + \lambda^{n-2} A + \ldots + A^{n-1}.$$

Therefore, if $\lambda \neq 0$ we get $0 < |\lambda|^n = \beta(\lambda^n - A^n) \leqslant \alpha(T)\beta(\lambda - A)$. Thus, $\beta(\lambda - A) > 0$. Since A^{*n} is also compact we obtain that $\beta(\bar{\lambda} - A^*) > 0$. This shows that $\lambda - A$ is Fredholm (see Proposition 3.2.1-(e)). Since $(tA)^n$ is compact for any $t \in \boldsymbol{R}$ we get that $\lambda - tA$ is Fredholm for any $t \in \boldsymbol{R}$ and $\lambda \neq 0$. The continuity of the index yields $\operatorname{ind}(\lambda - A) = \operatorname{ind}(\lambda) = 0$.

(e) Let $A\colon E \to E$ be a bounded linear operator whose spectral radious is zero (i.e., $\sqrt[n]{\|A^n\|} \to 0$ as $n \to +\infty$). We have (if $\dim E = +\infty$)

$$\beta(A) = \sqrt[n]{\beta(A)^n} \leqslant \sqrt[n]{\beta(A^n)} \leqslant \sqrt[n]{\alpha(A^n)} \leqslant \sqrt[n]{\|A^n\|}.$$

Thus, $0 \in \sigma_\beta(A)$. On the other hand $\sigma_\beta(A) \subset \sigma_e(A) \subset \sigma(A) = \{0\}$. Hence, $\sigma_\beta(A) = \sigma_e(A)$.

(f) Clearly, if A is balanced, then so is $\mu + A$ for any $\mu \in \boldsymbol{K}$, since $\sigma_\beta(\mu + A) = \mu + \sigma_\beta(A)$ and $\sigma_e(\mu + A) = \mu + \sigma_e(A)$.

The following result can be regarded as an extension of Theorem 9.1.1.

THEOREM 9.2.1. – *Let $A: E \to E$ be balanced and $h: E \to E$ be compact and asymptotically odd. Then $A + h$ is alternative.*

PROOF. – We have to show that $\sigma(A + h) = \sigma_\pi(A + h)$, i.e., if $\lambda \notin \sigma_\pi(A + h)$ then $\lambda - (A + h)$ is regular. Since $\beta(\lambda - A) = \beta(\lambda - (A + h)) > 0$ we get $\lambda \notin \sigma_\beta(A)$. Hence, $\lambda - A$ is Fredholm of index 0, since A is balanced. Therefore, there exists a compact linear operator $K: E \to E$ such that $\lambda - (A + K) = L$ is an isomorphism. Thus, it is enough to show that $L^{-1}\circ(\lambda - (A + h)) = 1 - L^{-1}\circ(h + K)$ is regular.

Obviously,

$$\beta(1 - L^{-1}\circ(h + K)) = 1$$

and

$$d(1 - L^{-1}\circ(h + K)) = d(L^{-1}\circ(\lambda - (A + h))) \geqslant d(L^{-1})\cdot d(\lambda - (A + h)) > 0\,.$$

Hence $1 \notin \sigma_\pi(L^{-1}\circ(h + K))$. Now, $L^{-1}\circ(h + K)$ is asymptotically odd and compact. Thus, by Theorem 9.1.1 we obtain $1 - L^{-1}\circ(h + K)$ is regular. Q.E.D.

Another class of alternative maps is given in Section 12 (see Theorem 12.2.1).

10. – Topological consequences of the nonlinear spectral theory.

In the first part of this section we show that some well-know results of topological character in Banach spaces can be derived from the spectral theory for nonlinear maps.

In the second part the notion of hypocompact maps is given, generalizing the notion of compact (α-contractive or condensing) vector fields as well as the notion of monotone operators. We also give some results regarding hypocompact maps.

10.1. *Existence theorems and retractions.*

Let S be the unit sphere of a Banach space E and $f: S \to E$ be continuous with bounded image. In this section we shall denote by $\bar{f}$ $E \to E$ the following extension of f.

$$\bar{f}(x) = \begin{cases} \|x\| f(x/\|x\|)\,, & \text{if } \|x\| \neq 0 \\ 0\,, & \text{if } x = 0\,. \end{cases}$$

The following facts are easy to check.

(*a*) $\bar{f}$ is continuous and positively homogeneous,

(*b*) $d(\bar{f}) = \inf\{\|f(x)\| : x \in S\}$,
$|\bar{f}| \;= \sup\{\|f(x)\| : x \in S\}$.

(*c*) If $\beta(\lambda - \bar{f}) > 0$, then $\lambda \in \Sigma(\bar{f})$ if and only if $\lambda x = f(x)$ for some $x \in S$. (see Proposition 3.2.3).

THEOREM 10.1.1. – *Let $f\colon S^n \to S^n$ be continuous, where $S^n = \{x \in R^{n+1}\colon \|x\| = 1\}$. Assume that f is not onto. Then f has a fixed point* (*and an antipodal point*).

PROOF. – Clearly, $d(\bar{f}) = |\bar{f}| = 1$. Thus $\lambda - \bar{f}$ is regular for any $\lambda \in \boldsymbol{K}$ with $|\lambda| > 1$ and $\sigma_\pi(\bar{f}) = \Sigma(\bar{f}) \subset S^0 = \{-1, 1\}$. Since $\bar{f}$ is not onto 0 belongs to $\sigma(\bar{f})$. It follows that $\sigma(\bar{f}) = [-1, 1]$ and $\Sigma(\bar{f}) = S^0$ (recall that $\partial\sigma(\bar{f}) \subset \sigma_\pi(\bar{f})$). Therefore, $1 \in \Sigma(\bar{f})$. Thus f has a fixed point. Q.E.D.

THEOREM 10.1.2. – *Let $S = \{x \in E\colon \|x\| = 1\}$ be the unit sphere in an infinite dimensional Banach space E and let $f\colon S \to S$ be continuous and compact. Then f has a fixed point* (*and an antipodal point*).

PROOF. – Since $\bar{f}\colon E \to E$ is compact, we have $\beta(1 - \bar{f}) = 1 > 0$. Thus, it suffices to show that $1 \in \Sigma(\bar{f})$. Clearly, $\Sigma(\bar{f}) \subset S^1 \cap \boldsymbol{K}$ and $\sigma(\bar{f})$ is bounded. Therefore, on the basis of Theorem 8.2.1-(*d*), the component of $\boldsymbol{K} \setminus \Sigma(\bar{f})$ containing 0 must be bounded. This implies that $\Sigma(\bar{f}) = S^1 \cap \boldsymbol{K}$. Q.E.D.

THEOREM 10.1.3. – *Let $f\colon S^n \to S^n$ be continuous and odd. Then f is onto.*

PROOF. – Clearly, $\sigma_\pi(\bar{f}) = \Sigma(\bar{f}) \subset S^0$. Thus $0 \notin \sigma_\pi(\bar{f})$. Since $\bar{f}$ is alternative $0 \notin \sigma(\bar{f})$ and hence $\bar{f}$ is onto. Therefore, f is onto. Q.E.D.

THEOREM 10.1.4. – *Let E be an infinite dimensional Banach space and $f\colon S \to S$ be compact. Then f cannot be odd.*

PROOF. – Assume that f is odd. Clearly, $\Sigma(\bar{f}) \subset S^1 \cap \boldsymbol{K}$ and $\sigma_\beta(\bar{f}) = \{0\}$. Since $\bar{f}$ is alternative $\sigma(\bar{f}) = \sigma_\pi(\bar{f})$. This is impossible since 0 is an interior point of $\sigma_\delta(\bar{f})$ (see Proposition 8.1.4-(*e*)). Q.E.D.

THEOREM 10.1.5 (Birkoff-Kellog theorem). – *Let E be an infinite dimensional Banach space and let $f\colon S \to E$ be continuous and compact such that $f(S)$ is bounded away from zero. Then f has a positive eigenvalue.*

PROOF. – Clearly, $d(\bar{f}) > 0$, so $0 \notin \Sigma(\bar{f})$. On the other hand $|\bar{f}| < +\infty$, since $f(S)$ is bounded. Thus, by Theorem 8.2.1-(*d*), the connected component of $\boldsymbol{K} \setminus \Sigma(\bar{f})$ containing 0 is bounded. Therefore, there exists $r > 0$ such that $r \in \Sigma(\bar{f})$. Hence, r is an eigenvalue since $\beta(r - \bar{f}) = r > 0$. Q.E.D.

THEOREM 10.1.6 (Hopf theorem on spheres). – *Let $f\colon S^{2n} \to \boldsymbol{R}^{2n+1}$ be continuous. Assume $(f(x), x) = 0$ for all $x \in S^{2n}$, where $(\cdot,\cdot)$ is the euclidean inner product on $\boldsymbol{R}^{2n+1}$. Then, f vanishes at some point $x \in S^{2n}$.*

PROOF. – Clearly, $\bar{f}: \boldsymbol{R}^{2n+1} \to \boldsymbol{R}^{2n+1}$ is quasibounded. Thus, by Proposition 8.2.3, $\Sigma(\bar{f}) \neq \emptyset$. Let $\lambda \in \Sigma(\bar{f})$. Then there exists $x \in S^{2n}$ such that $\lambda x = f(x)$. This implies $\lambda = (f(x), x) = 0$. Q.E.D.

THEOREM 10.1.7 (Borsuk-Ulam Theorem). – *Let $\varphi: S \to E$ be a compact vector field such that $\varphi(S)$ is contained in a proper closed subspace F of E. Then there exists $x \in S$ such that $\varphi(x) = \varphi(-x)$.*

PROOF. – Define $\psi: S \to F$ by $\psi(x) = \varphi(x) - \varphi(-x)$. Clearly, $\beta(\bar{\psi}) > 0$ (actually, $\beta(\bar{\psi}) = +\infty$ if $\dim E < +\infty$ and $\beta(\bar{\psi}) = 2$ if $\dim E = +\infty$). Thus, $0 \notin \sigma_\beta(\bar{\psi})$. On the other hand $\bar{\psi}$, being not onto, is not regular. Hence, $0 \in \Sigma(\bar{\psi})$, since $\bar{\psi}$ is alternative (see Section 9). By Proposition 3.2.3, there exists $x \in S$ such that $\psi(x) = 0$. Q.E.D.

THEOREM 10.1.8 – *Let U be an open and bounded subset of a Banach space E. Then the boundary ∂U of U is not a retract of $\bar{U}$ under a compact vector field.*

PROOF. – Assume that there exists a compact vector field $\varphi: \bar{U} \to \partial U$ which is a retraction of $\bar{U}$ onto its boundary. Define

$$f(x) = \begin{cases} \varphi(x)\,, & \text{if } x \in \bar{U} \\ x\,, & \text{if } x \notin U\,. \end{cases}$$

Clearly, $q(1-f) = 0$. Thus, on the basis of Proposition 8.1.2-(*e*), we get $\sigma(f) = \sigma(1) = \{1\}$. Therefore, $0 \notin \sigma(f)$ and f must be onto which is obviously impossible. Q.E.D.

10.2. *Hypocompact maps.*

Let $\varphi: X \to E$, $X \subset E$, be continuous and of the form $\varphi(x) = x - f(x)$. We recall that φ is called a *compact*, *α-contractive*, *condensing* or *α-nonexpansive vector field* if f is compact, α-contractive, condensing or α-nonexpansive respectively. It is easy to see that these vector fields are included in the following more general class of maps.

A continuous map $f: X \to E$ is called *hypocompact* if

$$\beta(\lambda + f) > 0 \qquad \text{for any } \lambda > 0\,.$$

Any monotone operator $f: E \to E$ is a hypocompact map since $\beta(\lambda + f) \geqslant \lambda$ for any $\lambda > 0$ (see the proof of Proposition 7.3.1). Moreover, if f is hypocompact and h is compact, then $\lambda + \mu f + h$ is hypocompact provided that $\lambda, \mu \geqslant 0$. In particular, any compact map is hypocompact.

The following result is a generalization of Proposition 7.3.2.

THEOREM 10.2.1. – *Let $f: E \to E$ be a coercive, proper hypocompact map. Assume that*

(*a*) *f is locally α-Lipschitz, sending bounded sets into bounded sets,*

(*b*) $\liminf \beta(1 + tf) > 0$ *as* $t \to 0^+$.

Then f is a strong surjection.

PROOF. – Let us prove first that if $\lambda < 0$, then $\lambda \notin \sigma(f)$. This will be accomplished in three steps. First, since f is hypocompact then $\lambda \notin \sigma_\beta(f)$ whenever $\lambda < 0$. Second, take $\lambda > 0$, then

$$\lambda + \frac{\operatorname{Re}\langle f(x), x'\rangle}{\|x\|^2} = \frac{\operatorname{Re}\langle \lambda x + f(x), x'\rangle}{\|x\|^2} \leqslant \frac{\|\lambda x + f(x)\|}{\|x\|},$$

where $x' \in J(x)$. The coerciveness of f implies that

$$\liminf \operatorname{Re}\langle f(x), x'\rangle / \|x\|^2 \geqslant 0 \qquad \text{as } \|x\| \to +\infty .$$

Hence, $d(\lambda + f) > 0$ and this implies that if $\lambda < 0$ then $\lambda \notin \Sigma(f)$. Third, $-1 \notin \sigma(f)$, i.e. $1 + f$ is regular. Consider the homotopy $H(x, t) = x + tf(x)$. Clearly, H satisfies conditions (*a*), (*c*) and (*d*) of Corollary 7.3.1. Moreover, $\beta\big(H(\cdot, t)\big) = t\beta(1/t + f)$ if $t > 0$. Hence, the function $t \mapsto \beta\big(H(\cdot, t)\big)$ is continuous for $t > 0$. This fact, the assumption $\liminf \beta(1 + tf) > 0$ as $t \to 0^+$ and $\beta\big(H(\cdot, 0)\big) = 1$ imply $\beta\,\big(H(\cdot, t)\big) \geqslant a$ for some $a > 0$ and $t \in [0, 1]$, i.e. condition (*b*) of Corollary 7.3.1 is also satisfied.

To apply Corollary 7.3.1, in order to show that $1 + f$ is regular, it remains only to show that the set $S = \{x \in E: H(x, t) = 0 \text{ for some } t \in [0, 1]\}$ is bounded. Let $x \in S$, then $1 + t\big(f(x), x\big)/\|x\|^2 = 0$ for some $t \in [0, 1]$. This equality and the coerciveness of f imply that S cannot be unbounded.

Since $\partial\sigma(f) \subset \sigma_\pi(f)$, $-1 \notin \sigma(f)$ and any $\lambda < 0$ is such that $\lambda \notin \sigma_\pi(f)$, then we have $\lambda \notin \sigma(f)$ whenever $\lambda < 0$, i.e., $\lambda + f$ is regular for any $\lambda > 0$.

The same argument used in the proof of Proposition 7.3.2 yields that f is a strong surjection. Q.E.D.

Notice that condition $\liminf \beta(1 + tf) > 0$ as $t \to 0^+$ is satisfied in the following two cases.

(1) f is α-Lipschitz. In fact, in this case $\lim \beta(1 + tf) = 1$ as $t \to 0$.

(2) f is monotone. In fact, $\beta(1 + tf) > 1$ for any $t > 0$.

The following consequence of Theorem 10.2.1 extends Theorem 10.1.9 as well its generalizations to α-contractive and condensing vector fields.

COROLLARY 10.2.1. – *Let U be a bounded open subset of a Banach space E. Then the boundary ∂U of U is not a retract of $\overline{U}$ under a locally α-Lipschitz hypocompact map.*

PROOF. – Assume that there exists a locally α-Lipschitz hypocompact retraction $r: \overline{U} \to \partial U$. Define

$$f(x) = \begin{cases} r(x), & \text{if } x \in \overline{U} \\ x, & x \notin U. \end{cases}$$

All the assumptions of Theorem 10.2.1 are verified by the map $\lambda + f$, wherever $\lambda > 0$. Hence, $\lambda + f$ is a strong surjection (actually $\lambda + f$ is regular if $\lambda > 0$, since $d(\lambda + f) = \lambda + 1 > 0$). This implies that $\operatorname{Im} f$ is dense in E. In fact, take $p \in E$. Then there exists a sequence $\{x_n\}$ in E such that $(1/n)x_n + f(x_n) = p$. Since, f is the identity outside U the sequence x_n must be bounded. Thus, $f(x_n) \to p$, so $\operatorname{Im} f$ is dense in E. This contradicts the fact that $U \cap \operatorname{Im} f = \emptyset$. Q.E.D.

11. – Bifurcation points for possibly non differentiable maps.

In the first part of this section we examine some questions related to asymptotic bifurcation points for equations of the form

$$\lambda x - g(x) = 0, \quad \lambda \in \boldsymbol{K},$$

where $g: E \to E$ is continuous. We show that the set $B(g)$ of all asymptotic bifurcation points of this equation is a (closed) subset of $\Sigma(g)$. We also give some results regarding the nonemptiness of $B(g)$.

In the second part we study the more general problem of asymptotic bifurcation points for equations of the following type.

$$\varphi(x, \lambda) = 0, \quad \lambda \in \boldsymbol{K},$$

where $\varphi: E \times \boldsymbol{K} \to F$ is a continuous map. We give extensions to this context of some of the results obtained in the first part of this section.

11.1. *Bifurcation points for maps from a Banach space into itself.*

Let $g: E \to E$ be a continuous map. A point $\lambda \in \boldsymbol{K}$ is said to be an *asymptotic bifurcation point* for g (see A. M. KRASNOSEL'SKIJ [16]) if there exists a sequence $\{(\lambda_n, x_n)\}$ in $\boldsymbol{K} \times E$ such that $\|x_n\| \to +\infty$ $\lambda_n \to \lambda$ and $\lambda_n x_n = g(x_n)$. The set of all asymptotic bifurcation points for g will be denoted by $B(g)$.

PROPOSITION 11.1.1. – *Let $g: E \to E$ be continuous. Then $B(g)$ is a closed subset of $\Sigma(g)$.*

PROOF. – Obviously, $B(g)$ is closed. Moreover, if $\lambda \in B(g)$, then there exists a sequence $\{(\lambda_n, x_n)\}$ in $\boldsymbol{K}\times E$ such that $\|x_n\| \to +\infty$, $\lambda_n \to \lambda$ and $\lambda_n x_n = g(x_n)$. Thus,

$$\frac{\|\lambda x_n - g(x_n)\|}{\|x_n\|} = |\lambda_n - \lambda| \to 0 \qquad \text{as } n \to +\infty .$$

Hence, $\lambda \in \Sigma(g)$. Q.E.D.

THEOREM 11.1.1. – *Let E be an infinite dimensional Banach space and $g\colon E \to E$ be quasibounded and α-Lipschitz. Assume that $d(g) > \alpha(g)$. Then $B(g) \neq \emptyset$.*

PROOF. – Take $\varepsilon > 0$ such that $d(g) - \varepsilon > \alpha(g)$. Then the map $g_\varepsilon\colon E \to E$ defined by

$$g_\varepsilon(x) = \frac{g(x)}{d(g) - \varepsilon}$$

is an α-contraction with $d(g_\varepsilon) > 1$. We can therefore assume, without loss of generality, that we are dealing with a map $g\colon E \to E$ which is α-contractive quasibounded and $d(g) > 1$. Choose $r > 0$ such that $\|g(x)\| > \|x\|$ for any $x \in E$ with $\|x\| \geqslant r$ and let $\pi_n\colon E \to D_{r+n}$ be the radial retraction of E onto D_{r+n}. The map $g_{r+n}\colon S_{r+n} \to S_{r+n}$ defined by $g_{r+n}(x) = \pi_n \circ g(x)$ is α-contractive. Thus, by R. D. Nussbaum's fixed point theorem [20], there exists $x_y \in S_{r+n}$, and $\lambda_n > 1$ such that $\lambda_n x_n = g(x_n)$. Let us show that the sequence $\{\lambda_n\}$ is bounded. In fact,

$$|\lambda_n| = \|\lambda_n x_n\| / \|x_n\| = \|g(x_n)\| / x_n\| .$$

Thus,

$$\limsup_{n\to+\infty} |\lambda_n| \leqslant \limsup_{n\to+\infty} \frac{\|g(x_n)\|}{\|x_n\|} \leqslant |g| < +\infty .$$

Hence, the sequence $\{\lambda_n\}$ is compact and any cluster point of $\{\lambda_n\}$ is a bifurcation point for g. Q.E.D.

The following is an existence theorem for bifurcation points of continuous maps. It also gives a more detailed information about the spectrum of nonlinear maps.

THEOREM 11.1.2. – *Let $f\colon E \to E$ be continuous. Let $\lambda_0, \lambda_1 \in \boldsymbol{K}\setminus\sigma_\pi(f)$ be such that $\lambda_0 \notin \sigma(f)$ and $\lambda_1 \in \sigma(f)$. Then $B(f) \cup \sigma_\beta(f)$ separates λ_0 from λ_1, i.e., λ_0 and λ_1 belong to different components of $\boldsymbol{K}\setminus(B(f) \cup \sigma_\beta(f))$.*

PROOF. – We have to show that given a continuous path $\lambda\colon [0, 1] \to \boldsymbol{K}$, with $\lambda(0) = \lambda_0$ and $\lambda(1) = \lambda_1$, then $\lambda(t)$ must belong either to $B(f)$ or to $\sigma_\beta(f)$ for some $t \in [0, 1]$. Assume this is not the case. Consider the homotopy $H\colon E\times[0, 1] \to E$ defined by $H(x, t) = \lambda(t) - f(x)$. This homotopy satisfies conditions (a)-(d) of Corollary 7.3.1. In fact, (a) is satisfied since $H(\cdot, 0) = \lambda_0 - f$ is regular. Condition (b)

is satisfied since the function $t \mapsto \beta(H(\cdot, t))$ is continuous and $\lambda(t) \notin \sigma_\beta(f)$ for all $t \in [0, 1]$. Thus $\inf \{\beta(H(\cdot, t)) : t \in [0, 1]\} > 0$. Conditions (*c*) and (*d*) are satisfied (see the examples following Corollary 7.3.1 and recall that the identity is α-Lipschitz). Clearly, the condition $\lambda(t) \notin B(f)$ for any $t \in [0, 1]$ implies that the set $S = \{x \in E: \lambda(t)x = f(x)$, for some $t \in [0, 1]\}$ is bounded. Moreover, $d(H(\cdot, 1)) > 0$ since $\lambda_1 \notin \sigma_\pi(f)$. Thus, on the basis of Corollary 7.3.1, $\lambda_1 - f$ must be regular, contradicting the assumption $\lambda_1 \in \sigma(f)$. Q.E.D.

We shall now investigate the set of asymptotic bifurcation points $B(g)$ for a map g defined on a finite dimensional space.

The example following Proposition 8.2.3 shows that $B(g)$ may be empty in the case when $\dim E$ is even.

We shall prove that if $\dim E$ is odd and g is quasibounded, then $B(g)$ is not empty. This extends Proposition 8.2.3.

THEOREM 11.1.3. – *Let* $g: \boldsymbol{R}^{2n+1} \to \boldsymbol{R}^{2n+1}$ *be quasibounded. Then* $B(g) \neq \emptyset$.

PROOF. – Let us prove first that if $|\lambda| > |g|$ then $[\lambda - g] = [\lambda]$. Clearly, the homotopy $H: \boldsymbol{R}^{2n+1} \times [0, 1] \to \boldsymbol{R}^{2n+1}$, defined by $H(x, t) = \lambda x - tg(x)$, satisfies Proposition 6.2.1, since the set of solutions of the equation $\lambda x - tg(x) = 0$, $t \in [0, 1]$, is bounded. This follows at once from the fact that there exists $r > 0$ such that $\|x\| \geqslant r$ implies

$$\|x\| |\lambda| > \|g(x)\| .$$

Hence, if $x \in E$ is a solution of the above equation then $\|x\| < r$.

On the other hand, $[\lambda] = [1]$ if $\lambda > 0$ and $[\lambda] = [-1]$ if $\lambda < 0$. Therefore, if $\lambda_0, \lambda_1 \in \boldsymbol{R}$ are such that $\lambda_0 < -|g|$ and $\lambda_1 > |g|$, then $[\lambda_0 - g] = [-1]$ and $[\lambda_1 - g] = [1]$.

If we show that $[1] \neq [-1]$ then, on the basis of Proposition 6.2.1 the set $S = \{x \in \boldsymbol{R}^n: t\lambda_0 x + (1-t)\lambda_1 x = f(x)$, for some $t \in [0, 1]\}$ is unbounded. Thus, there exists a sequence $\{x_n, \lambda_n)\}$ in $S \times [\lambda_0, \lambda_1]$ such that $\|x_n\| \to +\infty$ and $\{\lambda_n\}$ is convergent to some $\lambda \in [\lambda_0, \lambda_1]$. Hence, $\lambda \in B(g)$. Therefore, it remains to show that $[1] \neq [-1]$. The Leftschetz number of the identity coincides with the Euler characteristic of S^{2n} and equals 2. On the other hand, the antipodal map on S^{2n} is fixed point free and thus its Lefschetz number is zero. Hence, the identity and the antipodal map are not homotopic. Q.E.D.

11.2. *Bifurcation points for maps between different Banach spaces.*

Let $\varphi: E \times \boldsymbol{K} \to F$ be continuous. Consider the equation

$$(*) \qquad \varphi(x, \lambda) = 0 , \qquad \lambda \in \boldsymbol{K} .$$

A point $\lambda \in \boldsymbol{K}$ is called an *asymptotic bifurcation point* for the equation $(*)$ (see A. M. KRASNOSEL'SKIJ [16]) if there exists a sequence $\{(x_n, \lambda_n)\}$ in $E \times \boldsymbol{K}$ such that, $\|x_n\| \to +\infty$, $\lambda_n \to \lambda$ and $\varphi(x_n, \lambda_n) = 0$.

The set of all asymptotic bifurcation points for the equation $(*)$ will be denoted by $B(\varphi)$.

We have the following.

PROPOSITION 11.2.1. – *The set $B(\varphi)$ is closed.*

PROOF. – Let $\{\lambda_n\}$ be a sequence of points of $B(\varphi)$ converging to λ. For any $n \in \boldsymbol{N}$ we can choose $\mu_n \in \boldsymbol{K}$, $x_n \in E$ such that $|\mu_n - \lambda_n| < 1/n$, $\|x_n\| \geqslant n$ and $\varphi(x_n, \mu_n) = 0$. Clearly $\mu_n \to \lambda$ and $\lambda \in B(\varphi)$. Q.E.D.

We have shown that if $f\colon E \to E$ is continuous then $B(f) \subset \Sigma(f)$ (see Proposition 11.1.1). Analogously, one could show that

$$B(\varphi) \subset \Sigma(\varphi) = \{\lambda \in \boldsymbol{K} \colon d(\varphi(\cdot, \lambda)) = 0\},$$

provided that the map $\lambda \mapsto \varphi(\cdot, \lambda)$ is continuous from $\boldsymbol{K}$ into $C(E, F)$ with the strong topology.

By slightly modifying the proof of Theorem 11.1.3 and taking into account condition (C_2) following Corollary 7.3.1 we get a more general result.

THEOREM 11.2.1. – *Let $\varphi\colon E \times \boldsymbol{K} \to F$ be as above. Let $\lambda_0, \lambda_1 \in \boldsymbol{K} \setminus \sigma_\pi(\varphi)$ be such that $\lambda_0 \notin \sigma(\varphi)$ and $\lambda_1 \in \sigma(\varphi)$. Then $B(\varphi) \cup \sigma_\beta(\varphi)$ separates λ_0 from λ_1.*

Here, $\sigma_\beta(\varphi) = \{\lambda \in \boldsymbol{K} \colon \beta(\varphi(\cdot, \lambda)) = 0\}$.

For the definitions of $\sigma(\varphi)$ and $\sigma_\pi(\varphi)$ see Section 7.

We add in passing that $\boldsymbol{K}$ in Theorem 11.2.1 can be replaced by any path connected topological space Λ (the « parameter space »).

12. – The numerical range for nonlinear maps.

In this section H will stand for a complex Hilbert space and $(\cdot, \cdot)$ will denote the inner product on H which is linear in the first variable and conjugate linear in the second.

In the first part we consider a notion of numerical range for continuous maps acting on H. This notion turns out to be equivalent to that given previously in [10]. By means of the numerical range for nonlinear maps we are able to extend to the nonlinear context the notion of self-adjoint map. We obtain that the sum of a bounded linear self-adjoint operator with a (possibly nonlinear) compact self-adjoint map is alternative (see Corollary 12.1.1). This is done in the second part.

In the third part of this section we show that the concept of numerical range for nonlinear maps is strictly related to the asymptotic behaviour of the solutions of nonlinear ordinary differential equations.

12.1. *Definition of the numerical range and properties.*

Let H be a complex Hilbert space and $f\colon H \to H$ be continuous. Define $f_N\colon H\setminus\{0\} \to H$, *the normal component* of f, by

$$f_N(x) = \frac{(f(x), x)}{\|x\|^2}\, x\,.$$

The map f_N is clearly continuous (and can be continuously extended to H by putting $f_N(0) = 0$ in the case when $f(0) = 0$). The map $f_T = f - f_N$ is called *the tangent component* of f.

Observe that if f_N and f_T are compact then so is f. The opposite implication is not true as the following example shows.

EXAMPLE 12.1.1. – Let l^2 be the Hilbert space of square summable sequences of complex numbers and let $L\colon l^2 \to l^2$ be defined by

$$L(x) = (e_1, x)\, e_1\,,$$

where $\{e_n\}$ is any orthonormal basis.

We have

$$L_N(x) = \frac{(e_1, x)^2}{\|x\|^2}\, x\,.$$

L_N is not compact since it maps the bounded sequence $\{e_1 + e_n\}$ into the sequence $\{(e_1 + e_n)/2\}$ which does not have cluster points.

The following properties are easy to verify.

PROPOSITION 12.1.1. – *Let $f, g\colon H \to H$ be continuous. Then*

(*a*) $(f_N)_N = f_N$;

(*b*) $|f_N| \leqslant |f|$;

(*c*) $(f + g)_N = f_N + g_N$;

(*d*) $(\lambda f)_N = \lambda f_N$;

(*e*) $f = f_N$ *if and only if* $f(x) = \varphi(x)x$ *for all* $x \neq 0$ *and some continuous function* $\varphi\colon H\setminus\{0\} \to \mathbf{C}$.

Let $L: H \to H$ be linear and bounded. We recall that the numerical range, $w(L)$, of L is defined as the set

$$w(L) = \{(Lx, x): \|x\| = 1\}.$$

It is known that $w(L)$ is convex and its closure contains the spectrum of L. Moreover, $\overline{w(L)} = \overline{\text{co}}\, \sigma(L)$ in the case when L is normal (i.e. $L \circ L^* = L^* \circ L$).

The following theorem holds.

THEOREM 12.1.1. – *Let $L: H \to H$ be linear and bounded. Then the closure, $\overline{w(L)}$, of the numerical range of L coincides with $\Sigma(L_N)$.*

PROOF. – Since L_N is homogeneous we have that

$$\Sigma(L_N) = \Big\{\lambda \in \boldsymbol{C}: \inf_{\|x\|=1} \|\lambda x - L_N(x)\| = 0\Big\}.$$

On the other hand if $\|x\| = 1$ then

$$\lambda x - L_N(x) = \big(\lambda - (Lx, x)\big)\, x.$$

Hence

$$\|\lambda x - L_N(x)\| = |\lambda - (Lx, x)|.$$

This implies that

$$\Sigma(L_N) = \overline{\{(Lx, x): \|x\| = 1\}} = \overline{w(L)}. \qquad \text{Q.E.D.}$$

REMARK 12.1.1. – The above theorem shows that $\Sigma(L_N)$ is convex and therefore implies that L_N is not linear in the case when $\dim H < +\infty$ and $\sigma(L)$ has more than one point.

Theorem 12.1.1 justifies the following definition of numerical range for (possibly nonlinear) continuous maps of a Hilbert space H into itself.

Let $f: H \to H$ be continuous and let $f_N: H \setminus \{0\} \to H$ be the normal component of f. We define the *numerical range*, $n(f)$, of f as the set $n(f) = \Sigma(f_N)$. This clearly makes sense even if f_N is not defined at 0.

The following theorem holds.

THEOREM 12.1.2. – *Let $f: H \to H$ be continuous. Then $\Sigma(f) \subset n(f)$.*

PROOF. – It is enough to show that

$$d(\lambda - f_N) \leqslant d(\lambda - f)$$

for all $\lambda \in \boldsymbol{C}$. Since $(\lambda - f)_N = \lambda - f_N$ we have only to prove that $d(f_N) \leqslant d(f)$. This follows at once from the fect that

$$\|f_N(x)\| = \frac{\|(f(x), x)\, x\|}{\|x\|^2} \leqslant \frac{\|f(x)\|\,\|x\|^2}{\|x\|^2} = \|f(x)\| . \qquad \text{Q.E.D.}$$

The numerical range has the following properties.

PROPOSITION 12.1.2. – *Let $f, g\colon H \to H$ be continuous. Then*

(*a*) $n(\lambda f) = \lambda n(f)$, $n(\lambda + f) = \lambda + n(f)$, $\lambda \in \boldsymbol{C}$.

(*b*) *If f_N is quasibounded and $\lambda \in n(f)$ then $|\lambda| \leqslant |f_N|$.*

(*c*) *If $|f_N - g_N| = 0$ then $n(f) = n(g)$.*

(*d*) *If f_N is quasibounded then $n(f + g) \subset n(f) + n(g)$.*

PROOF. – We shall prove only (*d*). The other properties are easy to verify. Assume that $\lambda \in n(f + g)$. Then there exists a sequence $\{x_n\}$, $\|x_n\| \to +\infty$ such that

$$\frac{\big(f(x_n) + g(x_n), x_n\big)}{\|x_n\|^2} \to \lambda .$$

On the other hand

$$\frac{\big(f(x_n), x_n\big)}{\|x_n\|^2}$$

is bounded and we may therefore assume, without loss of generality, that

$$\frac{\big(f(x_n), x_n\big)}{\|x_n\|^2} \to \mu .$$

Thus, $\mu \in n(f)$ and $\big(g(x_n), x_n\big)/\|x_n\|^2 \to \lambda - \mu$, which is clearly a point in $n(g)$. Q.E.D.

We give now a result concerning the structure of $n(f)$.

THEOREM 12.1.3. – *Let $f\colon H \to H$ be such that f_N is quasibounded. Then $n(f)$ is nonempty, connected and compact.*

PROOF. – Let $\varphi\colon H \setminus \{0\} \to \boldsymbol{C}$ be defined by $\varphi(x) = \big(f(x), x\big)/\|x\|^2$. Since f_N is quasibounded the set $\varphi(H \setminus D_n)$ is bounded for sufficiently large n. On the other and $H \setminus D_n$ is connected and so is $\varphi(H \setminus D_n)$. Therefore, it is enough to show that

$$n(f) = \Sigma(f_N) = \cap \{\overline{\varphi(H \setminus D_n)} : n \in \boldsymbol{N}\} ,$$

since (as it is well known) the intersection of a decreasing sequence of nonempty connected and compact sets is nonempty, connected and compact.

Let $\lambda \in \cap \{\varphi(H \setminus D_n) : n \in N\}$. Then there exists a sequence $\{x_n\}$, $x_n \in H \setminus D_n$ for any $n \in N$, such that $\lambda - \varphi(x_n) \to 0$ and this implies that $\lambda \in \Sigma(f_N) = n(f)$.

Assume now that $\lambda \in \Sigma(f_N)$. Then there exists a sequence $\{x_n\}$, $\|x_n\| > n$ for any $n \in N$, such that $\lambda - \varphi(x_n) \to 0$.

We want to prove that $\lambda \in \overline{\varphi(H \setminus D_n)}$ for any n.

If $\lambda \notin \overline{\varphi(H \setminus D_{\bar{n}})}$ for some $\bar{n} \in N$ then there exists a neighborhood V of λ such that V has empty intersection with $\overline{\varphi(H \setminus D_n)}$ for all $n > \bar{n}$.

On the other hand $\varphi(x_n) \in \varphi(H \setminus D_n)$ since $\|x_n\| > n$. Hence, $\varphi(x_n)$ cannot be convergent to λ, which is a contradiction. Q.E.D.

In [10] the numerical range of a quasibounded map $f : H \to H$ was defined in a different way, namely

$$n(f) = \cap \{\overline{\varphi(H \setminus D_n)} : n \in N\},$$

where $\varphi(x) = (f(x), x)/\|x\|^2$. The proof of Theorem 12.1.3 shows that the two definitions are equivalent.

Further results on the numerical range and extensions to the context of Banach spaces are obtained by J. CANAVATI in [2].

12.2. *Nonlinear self-adjoint maps.*

Recall that a bounded linear operator $A : H \to H$ is called self-adjoint if $(Ax, y) = (x, Ay)$ for all $x, y \in H$. Let $f : H \to H$ be continuous and assume that $(f(x), y) = (x, f(y))$ for all $x, y \in H$. Then it is easy to check that f is linear. Therefore, the above definition cannot be extended to the nonlinear case.

On the other hand it is known that a bounded linear operator $A : H \to H$ is self-adjoint if and only if (Ax, x) is real for every $x \in H$. This allows us to extend the definition of self-adjoint operator to the nonlinear case.

A continuous α-lipschitz map $f : H \to H$ is said to be *self-adjoint* if f_N is quasibounded and $n(f) \subset \boldsymbol{R}$.

Proposition 12.1.2 implies that the set of all self-adjoint maps from H into H is a closed real subspace of $C(H, H)$.

It is known that a bounded linear self-adjoint operator $A : H \to H$ is alternative, i.e. $\sigma(A) = \sigma_\pi(A)$. We want to show now that this result can be extended to a more general class of nonlinear self-adjoint maps.

We need first the following lemma, which is also of independen interest.

LEMMA 12.2.1. - *Let $f : H \to H$ be continuous. Then $r(f) \leqslant \max \{\alpha(f), |f_N|\}$, where $r(f)$ in the spectral radius of f.*

PROOF. - Let $|\lambda| > \max \{\alpha(f), |f_N|\}$. Then $\beta(\lambda - f) \geqslant |\lambda| - \alpha(f) > 0$ and $\lambda \notin \Sigma(f)$, since $\Sigma(f) \subset n(f)$. Therefore, $\lambda \notin \sigma_\pi(f)$. We have only to show that if $h : H \to H$ is continuous and such that $h(E)$ is relatively compact, then the equation $\lambda x = f(x) +$

$+ h(x)$ is solvable. This is equivalent to require that the map $g(x) = \lambda^{-1}(f(x) + h(x))$ has a fixed point.

Given $n \in \boldsymbol{N}$, let $\pi_n : H \to H$ be the radial retraction of H onto the closed ball $D_n = \{x \in H : \|x\| \leqslant n\}$. Since π_n is α-nonexpansive (actually, in this case, it is nonexpansive) and g is α-contractive, by Darbo's Fixed Point Theorem, the map $g_n(x) = \pi_n(g(x))$ has a fixed point $x_n \in D_n$.

If $\|\pi_n(g(x_n))\| < n$ for some $n \in \boldsymbol{N}$ we are done. In fact, in this case, $\pi_n(g(x_n)) = g(x_n)$. Assume therefore $\|\pi_n(g(x_n))\| \geqslant n$ for all $n \in \boldsymbol{N}$. This implies $\|x_n\| \to +\infty$ as $n \to +\infty$. Clearly, there exists a sequence $\{t_n\}$, $t_n \leqslant 1$ for all $n \in \boldsymbol{N}$, such that $x_n = t_n g(x_n)$, i.e. $\lambda x_n = t_n((x_n) + h(x_n))$. Hence,

$$|\lambda| > |f_N| = |f_N + h_N| \geqslant \limsup_{n\to+\infty} \frac{|(f(x_n) + h(x_n), x_n)|}{\|x_n\|^2} \geqslant$$

$$\geqslant \limsup_{n\to+\infty} \frac{|(\lambda x_n, x_n)|}{\|x_n\|^2} = |\lambda| . \quad \text{A contradiction.} \qquad \text{Q.E.D.}$$

We are now in a position of proving the following result.

THEOREM 12.2.1. – *Let $f: H \to H$ be self-adjoint and such that $\sigma_\beta(f) \subset \boldsymbol{R}$. Then*

(*a*) *$\sigma(f)$ is a compact subset of $\boldsymbol{R}$,*

(*b*) *f is alternative.*

PROOF. – (*a*) We have $\sigma_\pi(f) = \sigma_\beta(f) \cup \Sigma(f) \subset \sigma_\beta(f) \cup n(f) \subset \boldsymbol{R}$. On the other hand $\partial\sigma(f) \subset \sigma_\pi(f)$ and, by Lemma 12.2.1, $\sigma(f)$ is bounded.

So $\sigma(f) \subset \boldsymbol{R}$.

(*b*) Let $\lambda \notin \sigma_\pi(f)$. We have to show that $\lambda \notin \sigma(f)$. Now, by (*a*), λ belongs to the unbounded component of $\boldsymbol{C} \setminus \sigma_\pi(f)$. Thus, by Theorem 8.1.3-(*b*), $\lambda \notin \sigma(f)$. Q.E.D.

COROLLARY 12.2.1. – *Let $A: H \to H$ be bounded, linear and $h: H \to H$ be compact. If A and h are both self-adjoint, then properties (a) and (b) of Theorem 12.2.1 hold.*

PROOF. – Since $A + h$ is self-adjoint, it is enough to show that $\sigma_\beta(A + h) \subset \boldsymbol{R}$ We have $\sigma_\beta(A + h) = \sigma_\beta(A)$, since h is compact, and $\sigma_\beta(A) \subset \Sigma(A)$, since A is linear. Therefore, $\sigma_\beta(A) \subset \Sigma(A) \subset n(A) \subset \boldsymbol{R}$. Q.E.D.

12.3. *Boundedness of solutions for nonlinear ordinary differential equations.*

Let $f: \boldsymbol{C}^n \to \boldsymbol{C}^n$ be continuous, with quasibounded normal part. Consider the following ordinary differential equation

$$\dot{z} = f(z) .$$

Theorem 12.3.1 below gives a sufficient condition for the solutions of this equation to be bounded as $t \to +\infty$.

THEOREM 12.3.1. – *Let $f: \boldsymbol{C}^n \to \boldsymbol{C}^n$ be continuous with $|f_N| < +\infty$: Assume that there exists a linear isomorphism $A: \boldsymbol{C}^n \to \boldsymbol{C}^n$ such that $n(A \circ f \circ A^{-1}) \subset \boldsymbol{C}_-$, where $\boldsymbol{C}_- = \{\lambda \in \boldsymbol{C}: \operatorname{Re} \lambda < 0\}$. Then all the solutions of the differential equation,*

$$\dot{z} = f(z)$$

are bounded as $t \to +\infty$.

PROOF. – Put $w = Az$. Then the differential equation $\dot{z} = f(z)$ becomes $\dot{w} = A \circ f \circ A^{-1}(w) = g(w)$. Therefore, it is enough to show that all the solutions of the differential equation $\dot{w} = g(w)$ are bounded (as $t \to +\infty$). Since the compact set $n(g)$ is contained in $\boldsymbol{C}_-$, then there exists $\varepsilon > 0$ such that $\operatorname{Re} \lambda < -\varepsilon$ for all $\lambda \in n(g)$. From the definition of numerical range it follows that there exists $r > 0$ such that $\operatorname{Re}\big(g(w), w\big) < -\varepsilon \|w\|^2$, whenever $\|w\| > r$. Let $w(\cdot)$ be any solution of $\dot{w} = g(w)$.

Then $(d/dt)\|w(t)\|^2 = \big(\dot{w}(t), w(t)\big) + \big(w(t), \dot{w}(t)\big) = 2 \operatorname{Re}\big(g(w(t)), w(t)\big)$. Now, the last term is negative if

$$\|w(t)\| > r\,, \quad i.e., \quad \frac{d}{dt}\|w(t)\|^2 < 0 \quad \text{if } \|w(t)\| > r\,.$$

This implies that $w(\cdot)$ is bounded. Q.E.D.

Let $L: \boldsymbol{C}^n \to \boldsymbol{C}^n$ be linear. It is well known that the following result holds. *The solutions of the differential equation*

$$\dot{z} = Lz$$

are bounded as $t \to +\infty$, provided that $\sigma(L) \subset \boldsymbol{C}_-$.

On the other hand if $\sigma(L) \subset \boldsymbol{C}_-$, the numerical range $n(L)$ need not be contained in $\boldsymbol{C}_-$. However, the following proposition shows that there exists a linear isomorphism $A: \boldsymbol{C}^n \to \boldsymbol{C}^n$ such that $n(A \circ L \circ A^{-1}) \subset \boldsymbol{C}_-$. Therefore Theorem 12.3.1 can be regarded as an extension of the result mentioned above.

PROPOSITION 12.3.1. – *Let $A: \boldsymbol{C}^n \to \boldsymbol{C}^n$ be linear. Then for any open set U containing $\overline{\mathrm{co}}\, \sigma(A)$ there exists a linear isomorphism $L: \boldsymbol{C}^n \to \boldsymbol{C}^n$ such that*

$$n(L^{-1} \circ A \circ L) \subset U\,.$$

PROOF. – By choosing a suitable basis on $\boldsymbol{C}^n$ the linear operator A can be represented with an upper triangular matrix M (*e.g.* its Jordan canonical form). Write M in the form

$$M = D + N$$

where D is the diagonal matrix which is obtained by taking the diagonal of M. Hence, $\sigma(M) = \sigma(D)$. Given $t > 0$ we consider as $L(t)$ the diagonal matrix $L_t = \operatorname{diag}(t, t^2, \dots, t^n)$. Clearly, $L_t^{-1} = \operatorname{diag}(t^{-1}, t^{-2}, \dots, t^{-n})$. By standard computations one can show that

$$\|L_t^{-1} N L_t\| \to 0\,, \quad \text{as } t \to 0\,,$$

and

$$L_t^{-1} D L_t = D\,.$$

On the other hand

$$n(L_t^{-1} M L_t) \subset n(L_t^{-1} D L_t) + n(L_t^{-1} N L_t) = n(D) + n(L_t^{-1} N L_t)\,.$$

Since D is obviously normal we have $n(D) = \overline{\text{co}}\, \sigma(D) = \overline{\text{co}}\, \sigma(M)$. Hence,

$$n(L_t^{-1} M L_t) \subset \overline{\text{co}}\, \sigma(M) + n(L_t^{-1} N L_t)\,.$$

The result now is a consequence of the fact that if

$$\lambda \in n(L_t^{-1} N L_t)\,, \quad \text{then } |\lambda| \leqslant \|L_t^{-1} N L_t\|\,. \qquad \text{Q.E.D.}$$

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