Topological aspects of the Bel-Petrov classification.

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Summary. - *The topological aspects of the Bel-Petrov classification of the curvature tensor are examined for compact orientable space-times in which the Einstein equations for the exterior case are satisfied. It is shown that for such space-times of Bel Case III the metric tensor is singularity-free and that the Pontrjagin number identically vanishes. Bel Cases I and II are examined and conditions are given for which the metric is singularity-free and the Pontrjagin number vanishes. Applications to gravitional radiation in general relativity are discussed.*

§ 1. - Introduction.

In a previous paper, [24], the author has given an integral formula for the Pontrjagin number and index of a compact orientable $4k$ dimensional differentiable manifold which has a Riemannian metric of arbitrary signature. In this paper that investigation will be continued for the four dimensional differentiable manifolds of general relativity by considering the Bel–Petrov classification. Some topological results of Avez [2], [3], Chern [8], and Zund [24] are reviewed in § 2. The necessary preliminaries about the Bel–Petrov classification, together with an important lemma are given in § 3. The topological consequences of this classification are presented in § 4.

Throughout this paper, except for minor changes, the notation and terminology of Bel [6], Lichnerowicz [15], and [24] are employed.

§ 2. - Topological preliminaires.

Let V_4 be a four dimensional differentiable manifold which is provided with a Riemannian metric $g_{\alpha\beta}(x^\lambda)$ of hyperbolic normal signature. For brevity such a V_4 will be called a space-time. If V_4 is compact and orientable it is known, Avez [2] and Chern [8], that the Euler–Poincaré characteristic is given by the integral formula

$$\chi(V_4) = -\frac{1}{32\pi^2}\int_{V_4} \Delta \cdot \eta \tag{1}$$

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where

$$\Delta \stackrel{\text{def}}{=} \eta_{\alpha\beta\gamma\delta}\eta_{\lambda\mu\rho\sigma}R^{\alpha\beta,\,\lambda\mu}R^{\gamma\delta,\,\rho\sigma}, \tag{2}$$

$\eta \stackrel{\text{def}}{=} \sqrt{-\det(g_{\alpha\beta})}\, dx^0\Lambda dx^1\Lambda dx^2\Lambda dx^3$, and $\eta_{\alpha\beta\gamma\delta}$ is the LEVI-CIVITA permutation symbol. The results of [3] and [24] show that the PONTRJAGIN number of a compact orientable space-time can be expressed by

$$p[V_4] = -\frac{1}{8\pi^2}\int_{V_4} \widehat{\Delta}\cdot\eta \tag{3}$$

where

$$\widehat{\Delta} \stackrel{\text{def}}{=} \frac{1}{4} R_{\alpha\beta,\,\gamma\delta}R^{\alpha\beta,}{}_{\lambda\mu}\eta^{\gamma\delta\lambda\mu}. \text{ (1)} \tag{4}$$

GÉHÉNIAU and DEBEVER [10] have shown that in V_4 one may construct at most the following six scalars from the curvature tensor and its adjoints:

$$\begin{aligned}
A &\stackrel{\text{def}}{=} \frac{1}{8} R^{\alpha\beta,}{}_{\lambda\mu}R^{\lambda\mu,}{}_{\alpha\beta}\\
B &\stackrel{\text{def}}{=} \frac{1}{8} R^{\alpha\beta,}{}_{\lambda\mu} * R^{\lambda\mu,}{}_{\alpha\beta}\\
C &\stackrel{\text{def}}{=} \frac{1}{8} R^{\alpha\beta,}{}_{\lambda\mu} * R *{}^{\lambda\mu,}{}_{\alpha\beta}\\
D &\stackrel{\text{def}}{=} \frac{1}{16} R^{\alpha\beta,}{}_{\lambda\mu}R^{\lambda\mu,}{}_{\rho\sigma}R^{\rho\sigma,}{}_{\alpha\beta}\\
E &\stackrel{\text{def}}{=} \frac{1}{16} R^{\alpha\beta,}{}_{\lambda\mu}R^{\lambda\mu,}{}_{\rho\sigma} * R^{\rho\sigma,}{}_{\alpha\beta}\\
F &\stackrel{\text{def}}{=} \frac{1}{16} R^{\alpha\beta,}{}_{\lambda\mu}R^{\lambda\mu,}{}_{\rho\sigma} * R *{}^{\rho\sigma,}{}_{\alpha\beta}
\end{aligned} \tag{5}$$

where

$$* R^{\lambda\mu,}{}_{\alpha\beta} \stackrel{\text{def}}{=} \frac{1}{2}\eta^{\lambda\mu\rho\sigma}R_{\rho\sigma,\,\alpha\beta} \tag{6}$$

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(1) In (24) $\widehat{\Delta}$ was denoted by Δ_4 and $p[V_4]$ was written $p^4[V_4]$.

and

$$(7) \qquad *R*_{\alpha\beta,\lambda\mu} = \frac{1}{4}\eta_{\alpha\beta\gamma\delta}\eta_{\lambda\mu\rho\sigma}R^{\gamma\delta,\rho\sigma}.$$

The six scalars of (5) are called the fundamental scalars.

LEMMA. - In V_4, Δ and $\widehat{\Delta}$ are related to the fundamental scalars by

$$(8) \qquad \Delta = 32C$$

$$(9) \qquad \widehat{\Delta} = 4B,$$

and if V_4 is an EINSTEIN space

$$(10) \qquad \widehat{\Delta} = -32A.$$

PROOF. - Equation (8) is an obvious consequence of (2), (5) and (7). Equation (9) was obtained and discussed by the author in [24] for the case when V_4 is an EINSTEIN space, however it is clearly valid without this restriction. Equation (10) is established by noting that the RUSE-LANCZOS identity [21], [14] reduces to

$$(11) \qquad R_{\alpha\beta,\lambda\mu} + *R*_{\alpha\beta,\lambda\mu} = 0$$

if and only if V_4 is an EINSTEIN space.

Thus the first theree fundamental scalars naturally occur as the integrands of the topological invariants $\chi(V_4)$ and $p[V_4]$ for compact orientable space-times. It is clear from (5) and (11) that $C = -A$ and $F = -D$, hence in an EINSTEIN space there are only four fundamental scalars.

§ 3. - The Bel-Petrov classification.

In this section we present an expose of some of the features of the BEL-PETROV classification as developed by BEL in his thesis [6].

Since $g_{\alpha\beta}(x^\lambda)$ is of hyperbolic normal signature, at each point $x \in V_4$ the line element can be locally reduced to

$$(12) \qquad ds^2 = (\theta^0)^2 - \sum_{j=1}^{3}(\theta^j)^2$$

where the θ^α are a system of linearly independent PFAFFIAN forms. In such a frame the volume element η reduces to

$$(13) \qquad \eta = \theta^0 \Lambda \theta^1 \Lambda \theta^2 \Lambda \theta^3.$$

Throughout the remainder of this paper all tensor components will be considered with respect to the frame θ^α of (12). When Δ or $\widehat{\Delta}$ is evaluated in this frame, (13) may be used, together with the standard partition of unity technique [19], to obtain (1) and (3).

In the BEL–PETROV classification the frame components of $R_{\alpha\beta,\lambda\mu}$ are written in a symmetric 6×6 matrix $\boldsymbol{R} = (R_{IJ})$, where I denotes the row and J denotes the colum j and the $\alpha\beta$ and $\lambda\mu$ indices are relabelled according to the scheme

$\alpha\beta$ or $\lambda\mu$:	23	31	12	10	20	30
I or J:	1	2	3	4	5	6.

If the EINSTEIN equations for the exterior case are satisfied, the matrix can be written in the form

$$\boldsymbol{R} = \begin{pmatrix} -\boldsymbol{Y} & \boldsymbol{Z} \\ \boldsymbol{Z} & \boldsymbol{Y} \end{pmatrix} \tag{14}$$

where $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are real 3×3 traceless matrices, i.e. Tr. $\boldsymbol{Y} =$ Tr. $\boldsymbol{Z} = 0$. For the calculation of the fundamental scalars it is convenient to introduce the matrix of mixed components $\boldsymbol{R}_{\text{mix}} = (R^I{}_J)$,

$$\boldsymbol{R}_{\text{mix}} = \begin{pmatrix} \boldsymbol{Y}_{\text{mix}} & -\boldsymbol{Z}_{\text{mix}} \\ \boldsymbol{Z}_{\text{mix}} & \boldsymbol{Y}_{\text{mix}} \end{pmatrix} \tag{15}$$

and its adjoint components

$$* \boldsymbol{R}_{\text{mix}} = \boldsymbol{R} *_{\text{mix}} = \begin{pmatrix} -\boldsymbol{Z}_{\text{mix}} & -\boldsymbol{Y}_{\text{mix}} \\ \boldsymbol{Y}_{\text{mix}} & \boldsymbol{Z}_{\text{mix}} \end{pmatrix}. \tag{16}$$

It is easy to show by using these expressions that

$$\Delta = -\frac{1}{64} \text{Tr.} (\boldsymbol{R}_{\text{mix}} \boldsymbol{R}_{\text{mix}}) \tag{17}$$

$$\widehat{\Delta} = \frac{1}{8} \text{Tr.} (\boldsymbol{R}_{\text{mix}} * \boldsymbol{R}_{\text{mix}}). \tag{18}$$

According to the BEL–PETROV classification, if $R_{\alpha\beta} = 0$, one has six canonical forms for the curvature tensor depending on the degeneracy of the eigenvalues of $\boldsymbol{R}_{\text{mix}}$:

CASE I

$$
R_{\text{mix}} = \left[\begin{array}{ccc|ccc}
\alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\
0 & \alpha_2 & 0 & 0 & -\beta_2 & 0 \\
0 & 0 & \alpha_3 & 0 & 0 & -\beta_3 \\
\hline
\beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\
0 & \beta_2 & 0 & 0 & \alpha_3 & 0 \\
0 & 0 & \beta_3 & 0 & 0 & \alpha_3
\end{array}\right] \tag{19}
$$

where

$$
\sum_{j=1}^{3} \alpha_j = 0 \quad \text{and} \quad \sum_{j=1}^{3} \beta_j = 0. \tag{20}
$$

CASE II$_a$

$$
R_{\text{mix}} = \left[\begin{array}{ccc|ccc}
2\alpha & 0 & 0 & 2\beta & 0 & 0 \\
0 & -\alpha & 0 & 0 & -\beta & 0 \\
0 & 0 & -\alpha & 0 & 0 & -\beta \\
\hline
-2\beta & 0 & 0 & 2\alpha & 0 & 0 \\
0 & \beta & 0 & 0 & -\alpha & 0 \\
0 & 0 & \beta & 0 & 0 & -\alpha
\end{array}\right] \tag{21}
$$

CASE II$_b$

$$
R_{\text{mix}} = \left[\begin{array}{ccc|ccc}
2\alpha & 0 & 0 & 2\beta & 0 & 0 \\
0 & \sigma - \alpha & \tau & 0 & -(\beta + \tau) & -\sigma \\
0 & -\tau & -(\alpha + \sigma) & 0 & -\sigma & \tau - \beta \\
\hline
-2\beta & 0 & 0 & 2\alpha & 0 & 0 \\
0 & \beta + \tau & \sigma & 0 & \sigma - \alpha & -\tau \\
0 & \sigma & \beta - \tau & 0 & -\tau & -(\alpha + \sigma)
\end{array}\right] \tag{22}
$$

(Either τ or σ can be made to vanish by a suitable rotation in the 23 2-plane).

CASE III$_a$

$$
R_{\text{mix}} = \left[\begin{array}{ccc|ccc}
0 & \mu & -\nu & 0 & -\nu & -\mu \\
\mu & 0 & 0 & -\nu & 0 & 0 \\
-\nu & 0 & 0 & -\mu & 0 & 0 \\
\hline
0 & \nu & \mu & 0 & \mu & -\nu \\
\nu & 0 & 0 & \mu & 0 & 0 \\
\mu & 0 & 0 & -\nu & 0 & 0
\end{array}\right] \tag{23}
$$

(Either μ or ν can be made to vanish by a suitable rotation in the 23 2-plane).

CASE $\mathrm{III_b}$

$$\left[\begin{array}{ccc:ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma & -\tau & 0 & -\tau & -\sigma \\ 0 & -\tau & -\sigma & & -\sigma & \tau \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tau & \sigma & 0 & \sigma & -\tau \\ 0 & \sigma & -\tau & 0 & -\tau & -\sigma \end{array}\right]$$

(Same indetermination of frame as in CASE $\mathrm{II_b}$).

CASE 0

$$\boldsymbol{R}_{\text{mix}} = (0) \tag{25}$$

which is merely the MINKOWSKI space-time of special relativity.

In our opinion one of the advantages of this method of classification lies in the natural manner in which the fundamental scalars appear and play a basic role in the determination of the BEL–PETROV cases:

THEOREM 1 [2]. - If the EINSTEIN equations for the exterior case are satisfied then the exterior case are satisfied then the space-time V_4 is of

1°) CASE III. - If and only if $A = B = D = E = 0$

2°) CASE II. - If and only if $(A + iB)^3 = 6(D + iE)^2$

3°) CASE. - If and only if neither 1° nor 2° is satisfied.

The BEL cases are related to the PETROV types as used by SACHS, [22], by the scheme

BEL CASE	I	$\mathrm{II_a}$	$\mathrm{II_b}$	$\mathrm{III_a}$	$\mathrm{III_b}$	0
PETROV TYPE	I	D	II	III	N	0.

Further details about the BEL–PETROV classification can be found in [4] and [5].

§ 4. - Topological consequences in General Relativity.

The results of § 2 and § 3 have shown that the BEL–PETROV classification is intimately related to two fundamental scalars and that in a compact

(2) This theorem was proven by L BEL in his thesis (6).

orientable space-time for which $R_{\alpha\beta} = 0$ two of these scalars, A and B, occur in the integral formulae for the EULER–POINCARÉ characteristic and PONTRJAGIN number. By the first part of Theorem 1, one obtains.

THEOREM 2. - Let V_4 be a compact orientable space-time in which $R_{\alpha\beta}=0$. If V_4 is of BEL CASE III then $\chi(V_4) = 0$ and $p[V_4] = 0$.

This result can also be immediately seen by using (23) and (24) to verify that the expressions for Δ and $\widehat{\Delta}$ given in (17) and (18) vanish identically. Part of this theorem, $p[V_4] = 0$, was given by AVEZ [3].

It is well known, [1], that $\chi(V_4) = 0$ is the necessary and sufficient condition for the existence of a continuous non-zero tangent vector field on V_4. In particular, [17], [23], this is equivalent to the existence of a singularity-free metric tensor field $g_{\alpha\beta}(x^\lambda)$ on V_4. Although EINSTEIN never formally required that the physically meaningful solutions of his equations be singularity-free, he often expressed the desirability of such solutions [9]. Thus part of Theorem 2 states that a compact orientable space-time of BEL Case III, in which $R_{\alpha\beta} = 0$, always admits topologically a singularity-free metric. The physical interest in this result is related to the fact that space-time of BEL Case III are frequently [7], [17] identified as representing the most idealized form of gravitational radiation. In fact BEL Case III_b exhibits the same type of algebraic structure

$$
(26) \qquad \begin{aligned} l^\alpha R_{\alpha\beta,\,\lambda\mu} &= 0 \\ l^\alpha * R_{\alpha\beta,\,\lambda\mu} &= 0 \end{aligned}
$$

possessed by the tensor $F_{\alpha\beta}$ in singular electromagnetic fields [16]. A number of exact solutions are known for BEL Case III: e.g. the plane and plane-fronted gravitational waves of Case III_b, [7], [26]; and the Case III_a solution of KERR and GOLDBERG [13]. Unfortunately it is not known whether any of these solutions are compact.

The BEL–PETROV classification makes no assumption about the real scalars appearing in the canonical matrices (19) - (25) other than requiring that $\sum_{j=1}^{3} \alpha_j = 0$ and $\sum_{j=1}^{3} \beta_j = 0$ in Case I. Hence by direct calculation of Δ and $\widehat{\Delta}$ one obtains the following:

THEOREM 3. - Let V_4 be a compact orientable space-time on which $R_{\alpha\beta} = 0$. Then one has

BEL Case I:

$$
(27) \qquad \chi(V_4) = \frac{1}{\pi^2} \int_{V_4} \Big(\sum_{j=1}^{3} \alpha_j^2 - \sum_{j=1}^{3} \beta_j^2 \Big) \cdot \eta
$$

(28)
$$p[V_4] = \frac{1}{\pi^2} \int_{V_4} \sum_{j=1}^{3} \alpha_j \beta_j \cdot \eta$$

BEL Case II:

(29)
$$\chi(V_4) = \frac{6}{\pi^2} \int_{V_4} (\alpha^2 - \beta^2) \cdot \eta$$

BEL Case II$_a$:

(30)
$$p[V_4] = -\frac{2}{3\pi^2} \int_{V_4} \alpha\beta \cdot \eta$$

BEL Case II$_b$:

(31)
$$p[V_4] = -\frac{1}{\pi^2} \int_{V_4} \{5\alpha\beta + (\tau\sigma - \alpha\tau + \sigma\beta)\} \cdot \eta.\ (^3)$$

It is clear that in general $\chi(V_4)$ and $p[V_4]$ need not vanish for space-times of BEL Case I and II. Using (27) and (29) one may select special space-times of Case I and II for which $\chi(V_4) = 0$:

THEOREM 4. - Let V_4 be a compact orientable space-time in which $R_{\alpha\beta}=0$. Then $g_{\alpha\beta}$ is singularity-free in each of the following special space-times:

BEL Case I:

(32) $\quad$ 1° $\quad \alpha_j = \varepsilon\beta_j, \quad \varepsilon = \pm 1, \quad j = 1, 2, 3$

or

(33) $\quad$ 2° $\quad$ The α_j are a permutation of the β_j.

BEL Case II:

(34)
$$\alpha = \varepsilon\beta, \quad \varepsilon = \pm 1, \quad \alpha, \ \beta \neq 0.$$

Similarly one can select special space-times for which $\widehat{\Delta} = 0$:

THEOREM 5. - Let V_4 be a compact orientable space-time in which $R_{\alpha\beta}=0$. Then $p[V_4] = 0$ in each of the following special space-times:

BEL Case I:

(35)
$$\sum_{j=1}^{3} \alpha_j \beta_j = 0 \quad \text{where} \quad \sum_{j=1}^{\alpha} \alpha_j = \sum_{j=1}^{3} \beta_j = 0\ (^4)$$

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(3) The integrand may be simplified. See the remark following (22).

(4) For example $\alpha_j = 0$ or $\beta_j = 0$ for $j = 1, 2, 3$.

BEL Case $\mathrm{II_a}$:

(36) $$1^\circ \quad \alpha = 0, \quad \beta \neq 0$$

or

(37) $$2^\circ \quad \beta = 0, \quad \alpha \neq 0$$

BEL Case $\mathrm{II_b}$:

(38) $$1^\circ \quad \tau = 0, \quad \sigma = -5\alpha$$

or

(39) $$2^\circ \quad \sigma = 0, \quad \tau = 5\beta.$$

The proof is immediate by using (28), (30) and (31). It is interesting to note that (37) is similar to the situation for the SCHWARZSCHILD solution where $\alpha = -\frac{km}{r^3}$ and $\beta = 0$. The special space-times of Theorems 4 and 5 are not the same except in BEL Case $\mathrm{II_b}$ for the obvious choices of σ or τ.

If V_4 is compact and orientable the PONTRJAGIN number $p[V_4]$ is related to the index of V_4, $\tau(V_4)$, by HIRZEBRUCH'S index theorem, [11],

(40) $$\tau(V_4) = \frac{1}{3}\, p[V_4].$$

By definition this index is equal to the difference between the number of positive and negative signs in the quadratic form associated to the cohomology product $f^2 \cup g^2$, $f^2, g^2 \in H^2(V_4;\, \mathbb{R})$. The coefficients of this quadratic form are written as the intersection matrix a_{ij}, [12], and given by

(41) $$J(\overset{i}{z}_2,\, \overset{j}{z}_2) = a_{ij}$$

where the indices i, j range from 1, 2,... up to the second BETTI number $b_2(V_4)$ of V_4, and $\{\overset{i}{z}_2\}$, $i = 1, \ldots, b_2(V_4)$ is a basis for $H_2(V_4;\, \mathbb{R})$. The intersection matrix is then related to the cohomology product by the formula

(42) $$J(\mathfrak{D}f^2,\, \mathfrak{D}g^2) = (f^2 \cup g^2)(z_4)$$

where $\mathfrak{D}: H^q(V_4;\, \mathbb{R}) \longrightarrow H_{4-q}(V_4;\, \mathbb{R})$, $q = 1, \ldots, 4$, is the isomorphism of the POINCARÉ duality theorem; f^2, $g^2 \in H^2(V_4;\, \mathbb{R})$, and z_4 is the generator of $H_4(V_4;\, \mathbb{R})$. By the familiar properties of the cohomology product, the POINCARÉ duality and (46) it follows that the intersection matrix is symmetric and non-singular.

In [24] the author prematurely assumed that $\tau(V_4) = -2$, i.e. that the signatures of a_{ij} and $g_{\alpha\beta}$ were identical. This is not necessarily true. Theorem 2, by (40), asserts that for compact orientable space-times with $R_{\alpha\beta} = 0$ that $\tau(V_4) = 0$ for BEL–PETROV Case III. The general determination for $\tau(V_4)$ for connected four dimensional differentiable manifolds has recently been investigated by MILNOR, [18].

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