Cauchy's problem for generalized differential equations.

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Summary. - *We give here the discussion of Cauchy's problem of existence of solution of differential equation for the case of generalized differential equation and initial conditions.*

1. In my paper [1] *I* discuss the foundations of a generalized calculus in abstract spaces and here *I* solve a question raised there about the Cauchy problem for g-differential equations. *I* start from the assumption that the reader is familiar with the main concepts outlined in my paper refered to above. More precisely, *I* intend to study here the equation

$$Df = 0$$

and « initial conditions » for it. So *I* begin by discussing some definitions and concepts needed in what follows. The empty set is indicated by $\emptyset$ and unless stated any set considered is supposed to be non-empty.

2. Given a special g-function f, *I* want to give a meaning to the statement $f = 0$, i.e., « f is equal to zero ».

DEFINITION I. - A special g-function

$$f: (X, V) \longrightarrow [R, V_R)$$

is equal to zero, written $f = 0$, if for any number $\varepsilon > 0$ there is a covering $\sigma \varepsilon V$ such that

$$\forall \tau \varepsilon V, \quad \text{and} \quad \sigma < \tau, \quad A \varepsilon \tau, \quad \Rightarrow f_\tau(A) \subset [0, \varepsilon)$$

where $[0, \varepsilon)$ is the set of all real numbers x, with $0 \leq x < \varepsilon$.

A special g-function such as in Definition I above will be usually called a *zero g-function.* In particular a special g-function which is zero in any open set of any covering of V is a zero g-function. Note also the similarity of this concept with that of differential given in (1; III, B, 6). Their difference lies in the fact that a zero-function is something approaching zero *uniformly* while the differential only does that *locally*.

DEFINITION II. – Let $(X, \mathfrak{F})$ be a Gauss space and M a sub-set of X and given $\alpha \varepsilon \mathfrak{F}$ let us call α_M the collection of all sets of α which intersect M. We say that $\mathfrak{F}$ is of finite type, relative to M, if given any two coverings α, $\beta \varepsilon \mathfrak{F}$, $\alpha < \beta$, there are two integers $k(\alpha, \beta)$ and $\bar{k}(\alpha, \beta)$, such that no set of α_M intersects less than $k(\alpha, \beta)$ sets of β_M and no set of α_M intersects more than $\bar{k}(\alpha, \beta)$ sets of β_M (1).

DEFINITION III. – A Gauss transformation $G : \mathfrak{F} \longrightarrow \mathfrak{F}'$ for the Gauss spaces $(X, \mathfrak{F})$ and $(Y, \mathfrak{F}')$, where $\mathfrak{F}$ and $G(\mathfrak{F}) \subset \mathfrak{F}'$ are of finite type relative to $M \subset X$ and $M' \subset Y$ respectively, is called non-increasing (non-decreasing) if

$$\forall\, \alpha, \quad \beta \varepsilon \mathfrak{F}, \quad \alpha < \beta \Rightarrow k(\alpha, \beta) \geq \bar{k}[G(\alpha), G(\beta)]$$

$$(\bar{k}(\alpha, \beta) \leq k[G(\alpha), G(\beta)]).$$

3. As it is well known in classical analysis, the Cauchy problem for ordinary differential equation of first order is the following: given the equation

$$y' = f(x, y) \tag{1}$$

where $f(x, y)$ is defined in some open set L of the plane, and the point $(x_0, y_0) \varepsilon L$, to find a function $y(x)$ such that

$$\begin{cases} y(x_0) = y_0 \\ y'(x) = f[x, y(x)] \end{cases}$$

for all x in a certain neighborhood $V(x_0)$ of x_0. If such a function $y(x)$ exists it is called a *solution* of (1) satisfying the *initial conditions* $y(x_0) = y_0$. Conditions for the existence and unicity of solutions are well known and *I* do not speak about that here. What *I* have in mind in recalling these classical concepts, is how they can be led to the domain of generalized differential equations in the line of non-determinist mathematics? To be more precise given the g-differential equation

$$Df = \Phi \circ f \tag{2}$$

(1) In this definition, intersection means that they have common *interior points*.

where Φ is a given g-field

$$\Phi\colon (Y,\ V') \longrightarrow [R,\ V_{\Phi}],$$

what is an analogous of Cauchy's problem for the class of continuous g-functions f

$$f\colon (X,\ V) \longrightarrow (Y,\ V');\ (r\colon\ V \longrightarrow V')$$

for given families of coverings V and V'?

We have to clarify two things: the meaning of solution» of (2) and the meaning of «initial conditions» in non-determinist terminology.

Let us begin with the concept of solution of (2). Suppose M is some given sub-set of X. We say that f is a *solution of* (2) *in* M if

$$\forall\ \sigma \varepsilon V, \qquad A \varepsilon \sigma_M \Rightarrow D_\sigma f(A) = \Phi_{\sigma'}[f_\sigma(A)], \qquad (\sigma' = r(\sigma)).$$

Let us give now a set $M \subset X$ and a set $M' \subset Y$. We say that a solution of (2) *satisfies initial conditions relative to the pair* $(M,\ M')$ if

$$\forall\ \sigma \varepsilon V; \qquad \forall\ A \varepsilon \sigma_M \Rightarrow A' = f_\sigma(A) \subset M'.$$

Now, the Cauchy problem for g-differential equations can be easily stated, namely, given the g-differential equation

$$Df = \Phi \circ f \tag{2}$$

for a given g-field Φ and given the pair $(M,\ M')$ as before, to find a continuous g-function

$$f\colon (X,\ V) \longrightarrow (Y,\ V')$$

which is a solution of equation (2), satisfying initial conditions relative to the pair $(M,\ M')$.

In this general form *I* do not know how to solve this problem. So *I* study the particular case where the g-field Φ is zero, that is

$$Df = 0,$$

and the pair $(M,\ M')$ is such that M is arbitrary and M' is open. We make this more precise in the existence theorem which follows. I hope that the method used in this particular case can be improved in such a way that we can solve the general case also.

4. EXISTENCE THEOREM. – Suppose $(X,\ \mathfrak{F})$ and $(Y,\ \mathfrak{F}')$ are Gauss spaces and $(M,\ M')$ is a pair with $M \subset X$ arbitrary and $M' \subset Y$ open. Let $G\colon \mathfrak{F} \longrightarrow \mathfrak{F}'$

be a non-increasing Gauss transformation with $\mathfrak{F}$ and $G(\mathfrak{F}) \subset \mathfrak{F}'$ of finite type relatively to A and M' respectively [1]. Let V and V' be two families of open coverings of X and Y respectively and suppose the following conditions are satisfied:

a) V is a sequence $\{\sigma_i\}$, $(i = 1, 2, ...)$ of open coverings such that $i < j \Rightarrow \sigma_i < \sigma_j$ and any $\sigma_i \varepsilon V$ has only a finite number of sets intersecting M;

b) V' is cofinal in the set of all coverings of Y;

c) given any open set $H \subset X$, with $H \cap M \neq \emptyset$ and any integer n, there is a covering $\alpha \varepsilon \mathfrak{F}$ having more than n sets intersecting $H \cap M$.

Then given the g-differential equation

$$Df = 0 \tag{3}$$

there is a continuous g-function f

$$f\colon (X, V) \to (Y, V')$$

which is a solution of (3) in M and satisfies initial conditions relative to the pair (M, M').

PROOF: Take $\sigma_1 \varepsilon V$ and consider any $A_1 \varepsilon \sigma_1$ with $A_1 \cap M \neq \emptyset$. Take $\alpha_1 \varepsilon \mathfrak{F}$ arbitrary and $F'_1 \varepsilon \alpha'_1 = G(\alpha_1)$ such that $F'_1 \cap M'$ has a non-empty interior. Take now a $\sigma' \varepsilon V'$ such that there is $A' \varepsilon \sigma'$ with $A' \subset F'_1 \cap M'$, which is possible because V' is cofinal in the set of all open coverings of Y. Note this particular σ' by σ'_1 and A' by A'_1. We have

$$n(A'_1, \sigma'_1, \alpha'_1) = 1 \quad \text{and} \quad n(A_1, \sigma_1, \alpha_1) \geq 1$$

where, according to ((1); II, 4), $n(A, \sigma, \alpha)$ in general means the number of sets of $\alpha \varepsilon \mathfrak{F}$ intersecting $A \varepsilon \sigma \varepsilon V$. Now to any other $A \varepsilon \sigma_1$ we associate the same A'_1 and also for any $A \varepsilon \sigma_1$ we have $n(A, \sigma_1, \alpha_1) \geq 1$. Let us study the limit

$$\overline{\lim_{\alpha_1 < \alpha;\, \alpha \varepsilon \mathfrak{F}}} \frac{n(A'_1, \sigma'_1, \alpha')}{n(A_1, \sigma_1, \alpha)}$$

where $\alpha' = G(\alpha)$. As G is non-increasing we have for any two $\alpha, \beta \varepsilon \mathfrak{F}$ with $\beta < \alpha$ and $\alpha' = G(\alpha)$, $\beta' = G(\beta)$,

$$\frac{n(A'_1, \sigma'_1, \beta')}{n(A_1, \sigma_1, \beta)} \geq \frac{\bar{k}(\beta', \alpha') n(A'_1, \sigma'_1, \beta')}{k(\beta, \alpha) n(A_1, \sigma_1, \beta)} \geq \frac{n(A'_1, \sigma'_1, \alpha')}{n(A_1, \sigma_1, \alpha)}.$$

[1] A is any arbitrary set of $\sigma \in V$, below, witt $A \cap M \neq \emptyset$.

Therefore

$$\overline{\lim_{\alpha_1<\alpha;\, \alpha\varepsilon\mathfrak{F}}} \frac{n(A'_1, \sigma'_1, \alpha')}{n(A_1, \sigma_1, \alpha)} \leq 1.$$

Now let us define

$$f_{\sigma_1} : \sigma_1 \longrightarrow \sigma'_1$$

by

$$f_{\sigma_1}(A) = A'_1, \quad \text{for any} \quad A \varepsilon \sigma_1$$

and put also $\sigma'_1 = r(\sigma_1)$.

We intend to define

$$r : V \longrightarrow V'$$

and

$$f_{\sigma_1} : \sigma_i \longrightarrow \sigma'_i = r(\sigma_i)$$

for the whole family V. Therefore let us suppose that we have already defined f_{σ_i} and r and let us see how we define $f_{\sigma_{i+1}}$ and r for $i+1$.

Take any $A_{i+1} \varepsilon \sigma_{i+1}$ with $A_{i+1} \cap M \neq \emptyset$ as before, and $\alpha_{i+1} \varepsilon \mathfrak{F}$ such that

$$n(A, \sigma_{i+1}, \alpha_{i+1}) \geq i + 1$$

for any $A \varepsilon \sigma_{i+1}$, intersecting M, using conditions a) and c) of the theorem. Select $F'_{i+1} \varepsilon \alpha'_{i+1} = G(\alpha_{i+1})$ such that $F'_{i+1} \cap A'_i$ has a nonempty interior and take a $\sigma' \varepsilon V'$ such that there is $A' \varepsilon \sigma'$ with $A' \subset F'_{i+1} \cap A'_i$ and note this σ' by σ'_{i+1} and A' by A'_{i+1}. Now to each $A \varepsilon \sigma_{i+1}$ put

$$f_{\sigma_{i+1}}(A) = A'_{i+1}$$

and

$$\sigma'_{i+1} = r(\sigma_{i+1}).$$

In this case we have

$$\frac{n(A'_{i+1}, \sigma'_{i+1}, \alpha'_{i+1})}{n(A_{i+1}, \sigma_{i+1}, \alpha_{i+1})} \leq \frac{1}{i+1}.$$

Now, reasoning analogously as we did for σ_1 we conclude that:

$$\overline{\lim_{\alpha_{i+1}<\alpha;\, \alpha\varepsilon\mathfrak{F}}} \frac{n(A'_{i+1}, \sigma'_{i+1}, \alpha')}{n(A_{i+1}, \sigma_{i+1}, \alpha)} \leq \frac{1}{i+1}$$

In conclusion, we have defined by induction, the following:

1) To each $\sigma_i \varepsilon V$ there corresponds *a* $\sigma'_i \varepsilon V'$ and therefore a function

$$r : V \longrightarrow V';$$

2) Given any $\sigma_i \varepsilon V$ and any $A \varepsilon \sigma_i$ we associated with it the set $A'_i \varepsilon \sigma'_i$ as defined above and so we have a function:

$$f_{\sigma_i} : \sigma_i \to \sigma'_i = r(\sigma_i)$$

given by

$$f_{\sigma_i}(A) = A'_i .$$

3) With the notation of ((1); II, 4) for g-derivatives we have

$$D_{\sigma_i} f(A) = (a_i, b_i), \qquad (i = 1, 2, 3, \dots),$$

where (a_i, b_i) is an interval in the real numbers with

$$0 \leq a_i \leq b_i \leq \frac{1}{i}$$

for any $A \varepsilon \sigma_i$, with $A \cap M \neq \emptyset$.

In this way we have defined a g-function

$$f: (X, V) \to (Y, V')$$

and also its g-derivative

$$Df: (X, V) \to [R, V_R],$$

where V_R is defined as usual by all collections of sets (a_i, b_i) for each integer $i \geq 1$.

Now it is easy to see that f satisfies all our requirements. Indeed:

1) f is continuous; this is because for any two σ_i, $\sigma_j \varepsilon V$ with $\sigma_i < \sigma_j$, by construction we have

$$\forall A \varepsilon \sigma_i, \quad B \varepsilon \sigma_j, \quad B \subset A \Rightarrow f_{\sigma_j}(B) \subset f_{\sigma_i}(A);$$

2) f is a solution of (3) in M, because given any $\varepsilon > 0$ we can select an integer i such that $1/i < \varepsilon$ and so by construction

$$D_{\sigma_j} f(A) \subset [0, 1/i] \subset [0, \varepsilon]$$

for any $A \varepsilon \sigma_j$, with $\sigma_i > \sigma_j$ and $A \cap M \neq \emptyset$;

3) f satisfies initial conditions relative to the pair (M, M'), because $\forall A \varepsilon \sigma_i$ and $A \cap M \neq \emptyset \Rightarrow A' = f_{\sigma_i}(A) \subset M'$, for any $\sigma_i \varepsilon V$.

The existence theorem is therefore completely proved.

5. I finish this paper with some remarks and outlines of possible further developments. I begin by discussing some typical cases where the existence theorem would have a meaning.

a) Let X and Y be the real line. Take as $\mathfrak{F}$ and $\mathfrak{F}'$ the canonical standard families ((1); II, 7) and as $G: \mathfrak{F} \rightarrow \mathfrak{F}'$ the canonical Gauss transformation; namely, the one which associates to each $\alpha \varepsilon \mathfrak{F}$ that $\alpha' \varepsilon \mathfrak{F}'$ whose sets have the same length as those of α. Take as M any bounded subset of X and as M' any open set of Y. It is easiiy seen that $\mathfrak{F}$ and $\mathfrak{F}'$ are of finite type relative to M and M' respectively and that G is non-increasing. Actually for any α, $\beta \varepsilon \mathfrak{F}$, with $\alpha < \beta$ we have $k(\alpha, \beta) = \bar{k}(\alpha, \beta)$ and G is also non-decreasing, so $k(\alpha, \beta) = k[G(\alpha), G(\beta)] = \bar{k}[G(\alpha), G(\beta)]$. Take as V the family of open coverings σ_i, where σ_i is of order 2 and is made up of open sets of length $1/2^i$ and such that $i < j$ implies $\sigma_i < \sigma_j$. Take as V' any cofinal family of open coverings in Y. Now it is easily seen that all conditions of the existence theorem are satisfied and therefore the equation

$$Df = 0$$

has a solution in M satisfying initial conditions relative to the pair (M, M').

b) Let X and Y be metric spaces. Suppose M is any compact sub-set of X and M', any open sub-set of Y with compact closure. As it is known ((1); II, 3, th. 1) X and Y can be supplied with a structure of Gauss space, indicated by $(X, \mathfrak{F})$ and $(Y, \mathfrak{F}')$. Now given any $\alpha \varepsilon \mathfrak{F}$, there is only a finite number of sets of α intersecting M. Indeed, to each $x \varepsilon M$ associate a neighborhood $V(x)$ intersecting only a finite number of elements of α ((1); II, 2, Def. I). As we can cover M with only a finite number of $V(x)$ our assertion is proved. So, with our previous notations α_M is finite and if we take any two α, $\beta \varepsilon \mathfrak{F}$, with $\alpha < \beta$, then, as α_M and β_M are both finite, the numbers $k(\alpha, \beta)$ and $\bar{k}(\alpha, \beta)$ are well defined and $\mathfrak{F}$ is of finite type relative to M. Analogously $\mathfrak{F}'$ is of finite type relative to M'. Take now a finite number of balls of radius 1 covering M and consider also the set $X - M$. All these sets together make up a covering σ_1 of X. Look now to the Lebesgue number λ_1 of σ_1 relative to M and consider the covering σ_2 of X made up of a finite number of balls of radius less than λ_1 covering M and the set $X - M$. So by induction we define a sequence V of coverings having the conditions given by the existence theorem. Take V' any cofinal covering of Y and $G: \mathfrak{F} \rightarrow \mathfrak{F}'$ any non-increasing Gauss transformation. Finally we have to suppose that condition c) of the existence theorem is true, because, obviously this is not always the case, as we see when X is a finite space. Again the g-differential equation (3) will have a solution satisfying initial conditions relative to the pair (M, M').

Going now to a more general situation, if we know how to handle the equation

$$Df = g$$

where g is a given special g-function, we know also how to handle the case

$$D^n f = g \qquad (n > 1).$$

Indeed, by definition of g-derivatives of order n((1); II, 7, def. IV) we put

$$D^{n-1} f = g_{n-1}, \qquad D^{n-2} f = g_{n-2} \dots Df = g_1 .$$

Now, for initial conditions we must give the pairs $(M_1, M_1') \dots (M_{n-1}, M'_{n-1})$, where $M_1, M_2 \dots M_{n-1}$ are subsets of X and $M_1', M_2' \dots M'_{n-1}$ are open subsets of the non-negative real numbers, and also the pair (M, M') is such that M is a subset of X, and M' an open subset of Y. After that Cauchy's problem will be to find a solution of $D^n f = g$ in M satisfyng the initial conditions

a) $A \varepsilon \sigma$ and $A \cap M \neq \emptyset => A' = f_\sigma(A) \subset M'$.

b) $A \varepsilon \sigma$ and $A \cap M_i \neq \emptyset => A'_i = D^i_\sigma f(A) \subset M'_i$ $(i = 1, 2, \dots, n-1)$.

An important case is the equation

$$D^2 f = \Phi \tag{4}$$

where Φ is a uniform g-field, that is,

$$\Phi\colon (Y, V') \longrightarrow [R, V_\Phi]$$

such that, for every $\sigma' \varepsilon V'$ the value $\Phi(A')$ is always the same for all $A' \varepsilon \sigma'$. If we give initial conditions (M, M') for f and (M_1, M_1') for Df, this means physically, that a particle entering the g-field Φ ((*i*); III, *D*, 2, def. VI) with velocity Df, known relative to the pair (M_1, M_1') and with a position f, known relative to the pair (M, M'), must follow some g-path ((1); III, *B*, 2, def. *V*) satisying equation (4) and initial conditions relative to the pairs (M, M') and (M_1, M_1').

It should be extemely interesting to study the general case

$$D^2 f = \Phi \circ f$$

for a given g-field due to its physical meaning ((1); III, *B*), but it looks very difficult for me.

Naturally at this point one could ask something about *unicity* of solutions. It is not easy to introduce such a concept in non-determinist mathematics, but maybe something can be done, up to some equivalence relation; for instance, do not distinguish two g-functions if they generate the same usual continuous function in compact metric spaces. This leads us to the general question: given a differential equation in the usual sense, is it possible to find a g-differential equation whose solutions generate solutions of the classical equation? This should be a very interesting problem to think about.

Finally we can ask if it is possible to introduce some kind of « integration » process for g-functions in such a way that by this process we could define some g-function g « integral » of some g-function f satisfying $Dg = f$?

This should be a « generalized fundamental theorem of calculus » and also very interesting for further research.

REFERENCES

[1] R. G. LINTZ, ***Introduction to General Analysis***, Mimeograph notes, Department of Mathematics, McMaster University, Hamilton, Ontario, Canada. To be published elsewhere.