Coerciveness inequalities for nonelliptic systems of partial differential equations.

JOHN R. SCHULENBERGER and CALVIN H. WILCOX (*) (**)

Abstract. - This paper deals with first-order matrix partial differential operators of the form

$$(1) \quad L = -i \sum_{j=1}^n L_j(x) D_j + L_0(x)$$

where $x = (x_1, \dots, x_n) \in R^n$, $D_j = \partial/\partial x_j$, and the $L_j(x)$, $j = 0, 1, 2, \dots, n$, are $m' \times m$ matrix-valued functions of x . Let $L_{2,m}$ be the Hilbert space of square integrable m -vector-valued functions on R^n . The operator (1) determines a closed linear operator $L: L_{2,m} \rightarrow L_{2,m'}$. L is said to be coercive on a subspace $V \subset L_{2,m}$ if there is a constant $\mu > 0$ such that

$$(2) \quad \sum_{j=1}^n \|D_j u\|^2 \leq \mu^2 (\|Lu\|^2 + \|u\|^2)$$

for all $u \in D(L) \cap V$, where $D(L)$ denotes the domain of L and $\|\cdot\|$ denotes the norm in $L_{2,m}$. L is said to have constant deficit k in R^n if the symbol $L(p, x) = \sum_{j=1}^n L_j(x) p_j$ has constant rank $m - k$ for all $x \in R^n$ and $p \in R^n - \{0\}$ (L is elliptic if and only if $k = 0$). The paper gives criteria for nonelliptic operators L of constant deficit k to be coercive on subspaces. In particular, operators of the form $\Delta = -iE(x)^{-1} \sum_{j=1}^n A_j D_j$ are considered where $E(x)$ and A_j are $m \times m$ Hermitian matrices and $E(x)$ is positive definite. Δ defines a self-adjoint operator on the Hilbert space H with inner product $(u, v)_E = \int_{R^n} u^* E v \, dx$. It is shown, under suitable hypotheses on $E(x)$ and A_j , that Δ is coercive on the subspace $N(\Delta)^\perp$, the orthogonal complement in H of $N(\Delta)$, the nullspace of Δ .

§. 1 Introduction.

This paper deals with first order matrix partial differential operators of the form

$$(1.1) \quad L = -i \sum_{j=1}^n L_j(x) D_j + L_0(x)$$

where $L_0(x)$, $L_1(x)$, $\dots$, $L_n(x)$ are functions of $x = (x_1, \dots, x_n) \in R^n$ whose values are $m' \times m$ matrices over the complex number field C , and $D_j = \partial/\partial x_j$.

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Such operators operate on functions $u(x) = (u_1(x), \dots, u_m(x))$ whose values are $m \times 1$ matrices⁽¹⁾ over C to produce functions $f(x) = (f_1(x), \dots, f_{m'}(x))$ whose values are $m' \times 1$ matrices over C .

Thus the equation

$$(1.2) \quad Lu = f$$

represents a system of m' first order partial differential equations for the m unknown functions $u_1, \dots, u_m$.

The set of (equivalence classes of) $m \times 1$ matrix-valued functions $u(x)$ which are LEBESGUE measurable on R^n and satisfy⁽²⁾

$$(1.3) \quad \int_{R^n} u(x)^* u(x) dx < \infty$$

form a vector space V_m . Moreover,

$$(1.4) \quad (u, v) = \int_{R^n} u(x)^* v(x) dx$$

defines an inner product on V_m and the pair $(V_m, (\cdot, \cdot))$ is a HILBERT space $L_{2,m}$. More generally, let $E(x)$ be an $m \times m$ HERMITIAN matrix for $x \in R^n$ and assume that $E(x)$ is LEBESGUE measurable and satisfies

$$(1.5) \quad c \xi^* \xi \leq \xi^* E(x) \xi \leq c' \xi^* \xi$$

for all $\xi \in C^m$ and $x \in R^n$ where c and c' are positive constants. Then

$$(1.6) \quad (u, v)_E = \int_{R^n} u(x)^* E(x) v(x) dx$$

defines an inner product on V_m and the pair $(V_m, (\cdot, \cdot)_E)$ is a HILBERT space $L_{2,m}(E)$. In particular $L_{2,m} = L_{2,m}(I)$ where I is the unit matrix. Moreover, (1.5) implies that the norms in the spaces $L_{2,m}(E)$ are all equivalent.

Differential operators L of the form (1.1) determine various unbounded linear operators L from HILBERT spaces $L_{2,m}(E)$ into HILBERT spaces $L_{2,m'}(E')$. Different operators are distinguished by different choices of E and E' and

⁽¹⁾ If M is matrix then ${}^t M$ denotes the transpose of M .

⁽²⁾ If M is a matrix then M^* denotes the HERMITIAN conjugate of M . Thus $M^* = {}^t \bar{M}$ where $\bar{M}$ is the complex conjugate of M .

the domains of definition $D(L)$. It is assumed here that $D(L)$ is a linear subspace of $L_{2,m}(E)$ such that

$$(1.7) \quad C_0^1(R^n; C^m) \subset D(L) \subset L_{2,m}(E)$$

where $C_0^1(R^n; C^m)$ is the set of $m \times 1$ matrix-valued functions which are continuously differentiable in R^n and have compact support.

DEFINITION 1.1. - An operator $L : L_{2,m}(E) \rightarrow L_{2,m'}(E')$ is said to be coercive if and only if for all $u \in D(L)$, $D_j u \in L_{2,m}(E)$ for $j = 1, 2, \dots, n$, and there exists a constant $K > 0$ such that for all $u \in D(L)$ ⁽³⁾

$$(1.8) \quad \sum_{j=1}^n \|D_j u\|^2 \leq K^2 (\|Lu\|^2 + \|u\|^2).$$

DEFINITION 1.2. - The symbol of the differential operator L given by (1.1) is the $m' \times m$ matrix-valued function

$$(1.9) \quad L(p, x) = \sum_{j=1}^n L_j(x) p_j$$

defined for $p \in R^n$ and $x \in R^n$.

DEFINITION 1.3. - The differential operator L given by (1.1) is elliptic at $x_0 \in R^n$ if and only if

$$(1.10) \quad \text{rank } L(p, x_0) = m \quad \text{for all } p \in R^n - \{0\}.$$

L is elliptic in a region $\Omega \subset R^n$ if and only if it is elliptic at each $x_0 \in \Omega$.

The concept of coerciveness for the linear operators associated with boundary value problems for partial differential equations in regions $\Omega \subset R^n$ was introduced by N. ARONSZAJN [5] and has been developed extensively (see, e.g., S. AGMON [1], D. G. de FIGUEIREDO [11], M. SCHECHTER [17], K. T. SMITH [21]). In this paper the concept is used for operators on R^n only.

The definition of ellipticity given above is a natural generalization of the definitions of ellipticity for single equations and $m \times m$ systems used in the literature (see, e.g., [3] and [10]) and of the definition of overdetermined elliptic systems given by S. AGMON [3]. Note that if L is elliptic for some $x_0 \in R^n$ then $m' \geq m$; i.e. there are at least as many equations as unknowns.

It is known that a coercive operator is necessarily elliptic in its region of definition. Indeed, for the case considered here of operators on R^n the

⁽³⁾ The norm in (1.8) is the norm of $L_{2,m} = L_{2,m}(I)$. Since $L_{2,m}(E)$ has an equivalent norm, (1.8) also holds with the norm of $L_{2,m}(E)$.

properties of coerciveness and ellipticity are equivalent under mild assumptions concerning the coefficient matrices. This is proved below in § 3, Theorems 3.5 and 3.9. Thus a nonelliptic operator L cannot be coercive in the sense of Definition 1.1. However, it is possible for it to be coercive on a subspace $V \subset L_{2,m}(E)$ in the sense of the following

DEFINITION 1.4. - An operator $L : L_{2,m}(E) \rightarrow L_{2,m'}(E')$ is said to be coercive on the subspace $V \subset L_{2,m}(E)$ if and only if for all $u \in D(L) \cap V$, $D_j u \in L_{2,m}(E)$ for $j = 1, 2, \dots, n$ and there exists a constant $K > 0$ such that (1.8) holds for all $u \in D(L) \cap V$.

Coerciveness inequalities in this restricted sense are of great interest in connection with the study of the wave propagation problems of classical physics. It was shown in [22] that most of the wave propagation phenomena of classical physics are governed by systems of partial differential equations of the form

$$(1.11) \quad E(x)D_t u + \sum_{j=1}^n A_j D_j u = 0$$

where $E(x)$ is an $m \times m$ HERMITIAN positive definite matrix for all $x \in R^n$, the matrices A_j are constant $m \times m$ HERMITIAN matrices, $u = u(x, t)$ is an $m \times 1$ (column) matrix, $x \in L^n$ (space) and $t \in R^1$ (time). Alternatively, systems (1.11) can be written in the SCHRÖDINGER form

$$(1.12) \quad D_t u = -i\Lambda u$$

where

$$(1.13) \quad \Lambda = -iE(x)^{-1} \sum_{j=1}^n A_j D_j.$$

It is assumed here, as in [22], that $E(x)$ is bounded and uniformly positive definite on R^n ; i.e. satisfies (1.5). Under these assumptions it was shown in [22] that a differential operator of the form (1.13) determines a selfadjoint operator Λ on the HILBERT space $\mathcal{H} = L_{2,m}(E)$. The appropriate definition of $D(\Lambda)$ is given in § 2 below.

Examples of systems (1.11) include MAXWELL'S equations of the electromagnetic field, the equations of acoustics and elasticity and most of the equations of wave propagation of classical physics. It is shown below by examples that the spatial operators Λ which arise in classical physics are usually nonelliptic.

It is the purpose of this paper to prove that the nonelliptic selfadjoint operators Λ associated with systems of the form (1.11) are coercive on the subspaces $N(\Lambda)^\perp$, the orthocomplements in $L_{2,m}(E)$ of the nullspaces $N(\Lambda)$;

i.e. there exists a constant $\mu > 0$ such that

$$(1.14) \quad \sum_{j=1}^n \|D_j u\|^2 \leq \mu^2 (\|\Lambda u\|^2 + \|u\|^2) \quad \text{for all} \\ u \in D(\Lambda) \cap N(\Lambda)^\perp.$$

The coerciveness inequality (1.14) is very important for theories of wave propagation and scattering governed by systems (1.11). A simple application of (1.14) is outlined below. Applications to proofs of asymptotic completeness of the wave operators and unitarity of the scattering operator for systems (1.11) will be given in separate publications [20].

A simple application of (1.14) is to the proof of regularity theorems for solutions of systems (1.11). It is known that (1.11) has a unique solution with values $u(\cdot, t) \in H$ for all initial values $u(x, 0) = u_0(x) \in H$; see [22, p. 47]. Moreover, it is known that if $u_0 \in D(\Lambda)$ then $u(\cdot, t) \in D(\Lambda)$ and $D_t u \in H$ for all t . It follows that if $u_0 \in D(\Lambda) \cap N(\Lambda)^\perp$ then $u(\cdot, t) \in D(\Lambda) \cap N(\Lambda)^\perp$ for all t and hence, by (1.14), $D_1 u(\cdot, t), \dots, D_n u(\cdot, t)$ are in H for all t . On the other hand, if $u_0 \in D(\Lambda) \cap N(\Lambda)$ then $u(x, t) = u_0(x)$ for all t and the regularity of $u(x, t)$ is the same as that of $u_0(x)$. Similar results hold for higher derivatives; cf. § 6 below.

In the remainder of this introductory section the coerciveness inequality (1.14) is proved for the special cases of electromagnetic, acoustic and elastic waves in homogeneous isotropic media. These examples provide insight into the general case and suggest the methods that are used to treat it in subsequent sections.

EXAMPLE 1: *Maxwell's Equations.* - For the simple case of electromagnetic wave propagation in free space, MAXWELL'S equations for the electromagnetic field can be written in the complex form

$$(1.15) \quad D_t u = -i \nabla \times u$$

where the complex vector $u = E + iH$ has the electric field E for its real part and the magnetic field H for its imaginary part. This is a SCHRÖDINGER equation (1.12) with $\Lambda_0 u = \nabla \times u$. Hence the symbol

$$(1.16) \quad \Lambda_0(p) = i \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}.$$

Note that $\Lambda_0(p)^* = \Lambda_0(p)$, i.e. $\Lambda_0(p)$ is HERMITIAN.

The operator $\Lambda_0 = \nabla \times$ is a simple example of a matrix partial differential operator (1.1), with $n = 3$ and $m' = m = 3$. Note that

$$(1.17) \quad \text{rank } \Lambda_0(p) = 2 \quad \text{for all } p \in R^3 - \{0\}.$$

Hence Λ_0 is not elliptic. However, note that $\text{rank } \Lambda_0(p)$ is a constant, independent of p , for all $p \neq 0$.

The coerciveness inequality for the nonelliptic operator $\Lambda_0 = \nabla \times$ is suggested by the well-known LAGRANGE identity for vectors:

$$(1.18) \quad |p \times v|^2 + |p \cdot v|^2 = |p|^2 |v|^2$$

which holds for all $p \in R^3$ and $v \in C^3$. The operators $p \times$ and $p \cdot$ can be related to the differential operators $\times \nabla$ and $\nabla \cdot$ by the PLANCHEREL theory of the FOURIER transform. The transform and its inverse are defined for all $u \in L_{2,m}$ by the equations

$$(1.19) \quad \begin{aligned} \widehat{u}(p) &= \Phi u(p) = (2\pi)^{-n/2} \int_{R^n} e^{-ip \cdot x} u(x) dx, \\ u(x) &= \Phi^* \widehat{u}(x) = (2\pi)^{-n/2} \int_{R^n} e^{ix \cdot p} \widehat{u}(p) dp \end{aligned}$$

where the integrals converge in $L_{2,m}$ (but not necessarily pointwise).

The transform maps $L_{2,m}$ onto itself and is an isometry; i.e. PARSEVAL'S theorem

$$(1.20) \quad \|\widehat{u}\| = \|u\|$$

holds for all $u \in L_{2,m}$.

Define the domains of the operators $\nabla \times$ and $\nabla \cdot$ by

$$(1.21) \quad \begin{aligned} D(\nabla \times) &= L_{2,3} \cap \{u : \nabla \times u \in L_{2,3}\}, \\ D(\nabla \cdot) &= L_{2,3} \cap \{u : \nabla \cdot u \in L_{2,3}\}. \end{aligned}$$

Then take $v = \widehat{u}(p)$ in (1.18) and integrate the identity over $p \in R^3$.

Then the PLANCHEREL theory implies that $u \in D(\nabla \times) \cap D(\nabla \cdot)$ if and only if u , $D_1 u$, $D_2 u$ and $D_3 u$ are in $L_{2,3}$ and the identity

$$(1.22) \quad \|\nabla \times u\|^2 + \|\nabla \cdot u\|^2 = \sum_{j=1}^3 \|D_j u\|^2$$

holds. In particular, $\nabla \times$ is coercive on the subspace $N(\nabla \cdot)$, the nullspace of the divergence operator.

Identity (1.22) suggests the introduction of the 1×3 matrix operator $\Lambda'_0 u = \nabla \cdot u$. Clearly

$$(1.23) \quad \Lambda'_0(p) = i(p_1 \ p_2 \ p_3)$$

and $\text{rank } \Lambda'_0(p) = 1$ for $p \neq 0$. Consider the augmented 4×3 matrix operator Λ''_0 with symbol

$$(1.24) \quad \Lambda''_0(p) = \begin{pmatrix} \Lambda'_0(p) \\ \Lambda_0(p) \end{pmatrix}$$

Lagrange's identity (1.18) can be written

$$(1.25) \quad |\Lambda''_0(p)v|^2 = |\Lambda'_0(p)v|^2 + |\Lambda_0(p)v|^2 = |p|^2 |v|^2.$$

Thus Λ''_0 is elliptic and (1.22) is equivalent to the identity

$$(1.26) \quad \|\Lambda''_0 u\|^2 = \|\Lambda'_0 u\|^2 + \|\Lambda_0 u\|^2 = \sum_{j=1}^3 \|D_j u\|^2$$

for all $u \in D(\Lambda''_0) = D(\Lambda'_0) \cap D(\Lambda_0)$, which implies that Λ''_0 is coercive.

The operator Λ_0 is closed and densely defined on $L_{2,3}$ (see, e.g. [14] for definitions). Hence

$$(1.27) \quad R(\Lambda_0)^\perp = N(\Lambda_0^*);$$

i.e. the orthogonal complement of the range of Λ_0 in $L_{2,3}$ is the null space of Λ_0^* , the adjoint of Λ_0 on $L_{2,3}$ [14, p. 267]. Since $\Lambda_0^* = \Lambda_0$ this implies $R(\Lambda_0)^\perp = N(\Lambda_0)$ and hence $\overline{R(\Lambda_0)} = \overline{N(\Lambda_0)^\perp}$, where the bar denotes closure in $L_{2,3}$.

The symbols $\Lambda'_0(p)$ and $\Lambda_0(p)$ satisfy the identities

$$(1.28) \quad \Lambda'_0(p)\Lambda_0(p) = \Lambda'_0(p)\Lambda_0(p)^* = 0$$

which imply that

$$(1.29) \quad \Lambda'_0 \Lambda_0 = 0,$$

i.e. $u \in D(\Lambda_0)$ implies $v = \Lambda_0 u \in D(\Lambda'_0)$ and $\Lambda'_0 v = 0$. Equation (1.29) is equivalent to the well-known identity «div curl = 0». It implies that

$$(1.30) \quad N(\Lambda_0)^\perp = \overline{R(\Lambda_0)} \subset N(\Lambda'_0).$$

Thus (1.26) implies that $\Lambda_0 = \nabla \times$ is coercive on $N(\Lambda_0)^\perp$, the orthocomplement of its nullspace in $L_{2,3}$: in fact

$$(1.31) \quad \|\Lambda_0 u\|^2 = \sum_{j=1}^3 \|D_j u\|^2 \quad \text{for all } u \in D(\Lambda_0) \cap N(\Lambda_0)^\perp,$$

which implies (1.14) with $K = 1$.

Notice that the identity (1.28), $\Lambda_0'(p)\Lambda_0(p)^* = 0$, implies that the rows of $\Lambda_0'(p)$ and $\Lambda_0(p)$ are orthogonal vectors in the sense of the HERMITIAN inner product $(\xi, \eta) = \sum_{j=1}^m \xi_j \bar{\eta}_j$. Hence, the rows of $\Lambda_0'(p)$ and $\Lambda_0(p)$ are linearly independent and the conditions $\text{rank } \Lambda_0'(p) = 1$ and $\text{rank } \Lambda_0(p) = 2$ for all $p \neq 0$ imply that $\text{rank } \Lambda_0''(p) = 3$, i.e. the augmented operator Λ_0'' is elliptic. It is (1.28) rather than (1.18) that is extended below to more general operators Λ_0 .

EXAMPLE 2: *The Equations of Acoustics.* - Two-dimensional acoustic waves in a homogeneous fluid can be described by solutions of a SCHRÖDINGER equation (1.10) with

$$(1.32) \quad \Lambda_0(p) = \begin{bmatrix} 0 & 0 & p_1 \\ 0 & 0 & p_2 \\ p_1 & p_2 & 0 \end{bmatrix}.$$

(See [22, p. 75] for a derivation). In this example $n = 2$ and $m' = m = 3$. Note that $\Lambda_0(p)$ is real symmetric and hence $\Lambda_0(p)^* = \Lambda_0(p)$. Moreover,

$$(1.33) \quad \text{rank } \Lambda_0(p) = 2 \quad \text{for all } p \in R^2 - \{0\},$$

whence Λ_0 is not elliptic.

A left null vector for $\Lambda_0(p)$ is

$$(1.34) \quad \Lambda_0'(p) = (p_2, -p_1, 0).$$

This suggests consideration of the augmented operator Λ_0'' with symbol given by (1.24) in which $\Lambda_0'(p)$ is defined by (1.32) and $\Lambda_0(p)$ by (1.34). A simple computation shows that identity (1.25) also holds in this case. Thus Λ_0'' is elliptic and satisfies the identity (1.26). Finally, by the same arguments as for MAXWELL'S equations, Λ_0 is coercive on the orthocomplement of its nullspace in $L_{2,3}$ and (1.31) holds.

The two examples given above of wave propagation systems (1.11) are very special because the associated operator Λ_0 given by (1.13) can be aug-

mented by a first-order operator Λ'_0 in such a way that Λ'_0 is elliptic. This is because electromagnetic and acoustic waves propagate with a single non-zero speed. For more general systems (1.11) there will be several propagation speeds (varying with p if the medium is anisotropic). The situation is illustrated by the example of two-dimensional elastic waves in a homogeneous isotropic solid for which there are two nonzero speeds, corresponding to pressure and shear waves.

EXAMPLE 3: *The Equations of Elasticity.* - Two-dimensional elastic waves in a homogeneous isotropic solid are described by a displacement vector $w = w(x, t) = w_1(x, t)\vec{i} + w_2(x, t)\vec{j}$ which satisfies the second order vector equation [15]

$$(1.35) \quad D^2 w = \mu \Delta w + (\lambda + \mu) \nabla \nabla \cdot w$$

where $x = (x_1, x_2) \in R^2$ and λ and μ are positive constants. The components Σ_{ij} of the stress tensor are related to the displacement by [15]

$$(1.36) \quad \begin{aligned} \Sigma_{11} &= (\lambda + 2\mu) D_1 w_1 + \lambda D_2 w_2 \\ \Sigma_{12} &= \Sigma_{21} = \mu (D_1 w_2 + D_2 w_1) \\ \Sigma_{22} &= (\lambda + 2\mu) D_2 w_2 + \lambda D_1 w_1 \end{aligned}$$

and the velocity is given by

$$(1.37) \quad v_1 = D_t w_1, \quad v_2 = D_t w_2.$$

Introduce now dependent variables $u = (\Sigma_{11}, \Sigma_{22}, \Sigma_{12}, v_1, v_2)$. Then equations (1.35) through (1.37) imply that u is a solution of a first-order system (1.11). The energy density for (1.35) is [15]

$$(1.38) \quad \eta = 2^{-1}[(\lambda + 2\mu)(\sigma_{11}^2 + \sigma_{22}^2) + 2\lambda\sigma_{11}\sigma_{22} + 4\mu\sigma_{12}^2 + v_1^2 + v_2^2]$$

where

$$(1.39) \quad \sigma_{ik} = 2^{-1}(D_j w_k + D_k w_j).$$

When expressed in terms of the components of u , this takes the form

$$(1.40) \quad \eta = 2^{-1}[a u_1^2 + 2b u_1 u_2 + a u_2^2 + (1/\mu) u_3^2 + u_4^2 + u_5^2],$$

where

$$(1.41) \quad a = (\lambda + 2\mu)/[4\mu(\lambda + \mu)], \quad b = -\lambda/[4\mu(\lambda + \mu)].$$

It follows that u satisfies an equation of the form (1.11) with

$$(1.42) \quad E_0 = \begin{bmatrix} a & b & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ 0 & 0 & 1/\mu & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$(1.43) \quad A(p) = \sum_{j=1}^2 A_j p_j = \begin{bmatrix} 0 & 0 & 0 & -p_1 & 0 \\ 0 & 0 & 0 & 0 & -p_2 \\ 0 & 0 & 0 & -p_2 & -p_1 \\ -p_1 & 0 & -p_2 & 0 & 0 \\ 0 & -p_2 & -p_1 & 0 & 0 \end{bmatrix}$$

The operator Λ_0 of (1.13) has the symbol $\Lambda_0(p) = E_0^{-1}A(p)$. It is an example of a system (1.1) with $n = 2$ and $m' = m = 5$. It is easy to verify that

$$(1.44) \quad \text{rank } \Lambda_0(p) = \text{rank } A(p) = 4 \quad \text{for all } p \in \mathbb{R}^2 - \{0\},$$

whence Λ_0 is not elliptic.

A left null vector for $A(p)$ is

$$(1.45) \quad A'(p) = (p_2^2 \ p_1^2 - p_1 p_2 \ 0 \ 0)$$

Thus if

$$(1.46) \quad \Lambda_0'(p) = A'(p)E_0$$

then $\Lambda_0'(p)\Lambda_0(p) = 0$ and also

$$(1.47) \quad \Lambda_0'(p)E_0^{-1}\Lambda_0(p)^* = A'(p)A(p)E_0^{-1} = 0.$$

The last equation implies that the rows of $\Lambda_0'(p)$ and $\Lambda_0(p)$ are orthogonal with respect to the inner product $(\xi, \eta) = \xi E_0^{-1} \eta$. It follows that the augmented form

$$(1.48) \quad \Lambda_0''(p) = \begin{pmatrix} \Lambda_0'(p) \\ \Lambda_0(p) \end{pmatrix}$$

has rank $\Lambda_0''(p) = 5$.

In the present case it is easy to verify that there is no first-order operator Λ'_0 such that Λ''_0 is elliptic. Accordingly the coerciveness of Λ_0 on the orthocomplement of its nullspace is more difficult to prove. Several alternatives present themselves. One is to keep the definitions (1.45), (1.46), (1.48) above and work with the concept of ellipticity due to A. DOUGLIS and L. NIRENBERG [9] where the equations and unknowns are allowed to have different orders. A second alternative is to define

$$(1.49) \quad \Lambda''_0(p) = \begin{pmatrix} \Lambda'_0(p) \\ \Lambda_0(p)^2 \end{pmatrix}$$

where $\Lambda'_0(p)$ is given by (1.45), (1.46) and use the theory of elliptic systems of the second order. This approach gives the coerciveness of Λ''_0 on $N(\Lambda_0)$. A third alternative is to give up the requirement that Λ'_0 be a differential operator and define

$$(1.50) \quad \Lambda'_0(p) = |p|(\eta_2^2 - \eta_1^2 - \eta_1\eta_2 \ 0 \ 0)E_0, \quad \eta = p/|p|.$$

The corresponding operator can be written

$$(1.51) \quad \Lambda'_0 = (-\Delta)^{1/2}(-D_2^2 - D_1^2 \ D_1 D_2 \ 0 \ 0)E_0$$

It is the third method that is followed here and for more general systems in § 5 below. In the present case notice that since $\Lambda''_0(p)$ has rank 5 for all $p \in R^2 - \{0\}$, there exists a positive constant μ such that

$$(1.52) \quad |\Lambda''_0(\eta)\tilde{u}| \geq \mu^2 > 0$$

for all unit vectors η and $\tilde{u}$. Now $\Lambda''_0(p)$ is homogeneous of degree one if $\Lambda'_0(p)$ is defined by (1.50). Thus putting $\eta = p/|p|$ and $\tilde{u} = \widehat{u(p)}/|\widehat{u(p)}|$ in (1.52) gives

$$(1.53) \quad |\Lambda''_0(p)\widehat{u(p)}|^2 \geq \mu^2 |p|^2 |\widehat{u(p)}|^2 \quad \text{for all } p \in R^2$$

and all $u \in L_{2,5}$. Since E_0 is constant it is natural to define

$$(1.54) \quad \begin{aligned} D(\Lambda_0) &= L_{2,5}(E_0) \cap \{u : \Lambda_0(p)\widehat{u(p)} \in L_{2,5}(E_0)\} \text{ and} \\ D(\Lambda'_0) &= L_{2,5}(E_0) \cap \{u : \Lambda'_0(p)\widehat{u(p)} \in L_{2,1}\}, \end{aligned}$$

and $D(\Lambda_0'') = D(\Lambda_0) \cap (D\Lambda_0')$. With these definitions, (1.53) and the PLANCHEREL theory imply that

$$(1.55) \quad \|\Lambda_0'' u\|^2 = \|\Lambda_0 u\|^2 + \|\Lambda_0' u\|^2 \geq \mu^2 \sum_{j=1}^2 \|D_j u\|^2$$

for all $u \in D(\Lambda_0'')$.

The operator Λ_0 is selfadjoint on the HILBERT space $L_{2,5}(E_0)$. Hence $\overline{R(\Lambda_0)} = N(\Lambda_0)^\perp$, where the orthocomplement is taken in $L_{2,5}(E_0)$. Moreover, $\Lambda_0' \Lambda_0 = 0$ and hence $\overline{R(\Lambda_0)} \subset N(\Lambda_0')$. Thus, (1.55) implies that

$$(1.56) \quad \sum_{j=1}^2 \|D_j u\|^2 \leq \mu^{-2} \|\Lambda_0 u\|^2 \quad \text{for all } u \in D(\Lambda_0) \cap N(\Lambda_0)^\perp.$$

Hence Λ_0 is coercive on the orthocomplement of its nullspace in $L_{2,5}(E_0)$.

The operator Λ_0 for each of the three examples has the property that $\text{rank } \Lambda_0(p)$ is constant, independent of $p \in R^n - \{0\}$. This suggests

DEFINITION 1.5. - The differential operator L given by (1.1) has constant deficit k at $x_0 \in R^n$ ($1 \leq k \leq m$) if and only if

$$(1.57) \quad \text{rank } L(p, x_0) = m - k \quad \text{for all } p \in R^n - \{0\}.$$

L has constant deficit k in a region $\Omega \subset R^n$ if and only if (1.57) holds for each $x_0 \in \Omega$.

For operators

$$(1.58) \quad \Lambda_0 = -iE_0^{-1} \sum_{j=1}^n A_j D_j$$

with constant coefficients and constant deficit the coerciveness inequality (1.14) can be proved via the PLANCHEREL theory in the same way as for the equations of elasticity. A proof is given in § 5 below.

The class of uniformly propagative systems

$$(1.59) \quad E_0 D_t u + \sum_{j=1}^n A_j D_j u = 0$$

was introduced in [22]. In essence, these are characterized by the property that the roots $\lambda_j(p)$ of the characteristic polynomial

$$(1.60) \quad P(p, \lambda) = \det(E_0 \lambda - \sum_{j=1}^n A_j p_j)$$

have constant algebraic sign and constant multiplicity for all $p \in R^n - \{0\}$. In particular, a root $\lambda_j(p) = 0$ if and only if it is identically zero and the

root $\lambda_j(p) \equiv 0$ (if there is one) has constant multiplicity k for all $p \in R^n - \{0\}$. It follows that the operator Δ_0 for a uniformly propagative system has constant deficit k .

For operators Δ with variable coefficients the FOURIER transform method fails and the proof of coerciveness inequalities is much more complicated, just as in the case of GÄRDING's inequality and coerciveness inequalities for elliptic operators.

The remainder of this paper is organized as follows. In § 2 the linear operators from $L_{2,m}(E)$ into $L_{2,m'}(E')$ determined by a differential operator L of the form (1.1) are discussed. It is shown that L , restricted to $C_0^1(R^n; C^m)$, has a unique extension to a closed operator from $L_{2,m}(E)$ into $L_{2,m'}(E')$. In § 3 coerciveness inequalities for elliptic systems are discussed. In § 4 the basic construction of an elliptic augmentation L'' for a nonelliptic operator L of constant deficit is given, and a coerciveness theorem is proved for the case where L'' is a first-order operator. Sections 5 and 6 contain the principal new results in the paper. In § 5 it is proved that a class of operators Δ of the form (1.13) which are perturbations of uniformly propagative systems Δ_0 of the form (1.58) are coercive on $N(\Delta)^\perp$. In § 6 coerciveness inequalities for higher order derivatives are proved for perturbations of uniformly propagative systems. Section 7 contains a brief discussion of unsolved problems concerning coerciveness inequalities for nonelliptic operators.

§ 2. - Operators on Hilbert Spaces Determined by a Partial Differential Operator.

This preparatory section contains a discussion of the various linear operators between the HILBERT spaces $L_{2,m}(E)$ and $L_{2,m'}(E')$ determined by a first-order differential operator L of the form (1.1). The main result states that, under mild assumptions concerning the coefficients, the restriction of L to $C_0^1(R^n; C^m)$ has a unique extension to a closed linear operator from $L_{2,m}(E)$ to $L_{2,m'}(E')$. This is a consequence of K. O. FRIEDRICHS' famous theorem [12] concerning the identity of weak and strong extensions of differential operators.

In this section the following assumptions are made concerning the coefficients of the differential operator L :

$$(2.1) \quad L_j(x) \text{ and } D_k L_j(x) \text{ are continuous and bounded on } R^n \\ \text{for } 1 \leq j, k \leq n;$$

$$(2.2) \quad L_0(x) \text{ is continuous and bounded on } R^n.$$

Some of the results given below can be proved under weaker assumptions than these. However, for simplicity assumptions (2.1) and (2.2) are retained throughout the paper.

A differential operator L can be used to define various linear operators from $L_{2,m}(E)$ to $L_{2,m'}(E')$. Three of these are considered here, namely the minimal operator and its strong and weak extensions.

DEFINITION 2.1. - The minimal operator for L is the operator $\tilde{L}: L_{2,m}(E) \rightarrow L_{2,m'}(E')$ with domain

$$(2.3) \quad D(\tilde{L}) = C_0^1(R^n; C^m)$$

whose action is defined by

$$(2.4) \quad \tilde{L}u = Lu = -i \sum_{j=1}^n L_j(x) D_j u + L_0(x)u \text{ for all } u \in D(\tilde{L}).$$

Clearly, if $E(x)$ and $E'(x)$ satisfy (1.5) and L satisfies (2.1) and (2.2) then $\tilde{L}u \in L_{2,m'}(E')$ for all $u \in D(\tilde{L})$. Hence $\tilde{L}$ is a well-defined linear operator from $L_{2,m}(E)$ to $L_{2,m'}(E')$.

The definitions of the strong and weak extensions of $\tilde{L}: L_{2,m}(E) \rightarrow L_{2,m'}(E')$ make use of a differential operator $L_{E,E'}^\dagger$, the formal adjoint of L with respect to the inner products $(u, v)_E$ and $(u', v')_{E'}$ which is defined by the identity

$$(2.5) \quad (Lu, v)_{E'} = (u, L_{E,E'}^\dagger v)_E$$

for all $u \in C_0^1(R^n; C^m)$ and $v \in C_0^1(R^n; C^{m'})$. A detailed discussion of the special case where $E = I$, $E' = I'$ is given below. Then the extension to more general E and E' is indicated and an application is given to the operators Δ that arise in the theory of wave propagation.

The simplified notation $L^\dagger = L_{I,I'}^\dagger$ will be used in what follows. With this convention integration by parts gives

$$(2.6) \quad (Lu, v) = (u, L^\dagger v)$$

for all $u \in C_0^1(R^n; C^m)$ and $v \in C_0^1(R^n; C^{m'})$ where

$$(2.7) \quad L^\dagger v = -i \sum_{j=1}^n D_j \{ L_j^* v \} + L_0^* v.$$

Note that under hypotheses (2.1) and (2.2) L^+ is an $m \times m'$ matrix differential operator of the same form as L :

$$(2.8) \quad L^+ = -i \sum_{j=1}^n L_j^*(x) D_j + (L_0^*(x) - i \sum_{j=1}^n D_j L_j^*(x)).$$

Moreover, $L^{++} = (L^+)^+ = L$.

Note that (2.6) is equivalent to

$$(2.9) \quad (\tilde{L}u, v) = (u, \tilde{L}^+v)$$

for all $u \in D(\tilde{L})$ and $v \in D(\tilde{L}^+)$. It follows that

$$(2.10) \quad \tilde{L}^+ \subset \tilde{L}^*$$

where $\tilde{L}^*$ denotes the adjoint of the HILBERT space operator $\tilde{L}: L_{2,m} \rightarrow L_{2,m'}$. In particular, $\tilde{L}^*$ is densely defined on $L_{2,m'}$ since $C_0^1(R^n; C^{m'}) = D(L^+)$ is dense in $L_{2,m'}$. It follows that $\tilde{L}$ is a closable operator whose closure is the operator $\tilde{L}^{**}$ [14, p. 267].

DEFINITION 2.2. - The strong extension $L_s: L_{2,m} \rightarrow L_{2,m'}$ of the minimal operator $\tilde{L}: L_{2,m} \rightarrow L_{2,m'}$ is the closure of $\tilde{L}$; i.e.

$$(2.11) \quad L_s = \tilde{L}^{**}.$$

Thus $u \in D(L_s)$ and $L_s u = v$ if and only if there exists a sequence $\{u_k\} \subset D(\tilde{L})$ such that $u_k \rightarrow u$ in $L_{2,m}$ and $\tilde{L}u_k \rightarrow v$ in $L_{2,m'}$.

DEFINITION 2.3. - The weak extension L_w of the minimal operator $\tilde{L}: L_{2,m} \rightarrow L_{2,m'}$ is the operator $L_w: L_{2,m} \rightarrow L_{2,m'}$ defined by the conditions that $u \in D(L_w)$ and $L_w u = v$ if and only if $v \in L_{2,m'}$ and

$$(2.12) \quad (u, \tilde{L}^+ \varphi) = (v, \varphi) \quad \text{for all } \varphi \in C_0^1(R^n; C^{m'}) = D(\tilde{L}^+).$$

Each $u \in D(L_w)$ has a unique image $v = L_w u$ because $C_0^1(R^n; C^{m'})$ is dense in $L_{2,m'}$. Moreover, (2.9) implies that

$$(2.13) \quad \tilde{L} \subset L_w$$

i.e. L_w is indeed an extension of $\tilde{L}$. Moreover, it is obvious from the definition that L_w is a closed operator. Hence (2.13) implies

$$(2.14) \quad L_s \subset L_w.$$

Note that Definition 2.3 is equivalent to defining L_w to be the HILBERT-space adjoint of $\tilde{L}^+$, the minimal operator for L^+ :

$$(2.15) \quad L_w = (\tilde{L}^+)^*.$$

It is a remarkable fact, discovered by K. O. FRIEDRICHS [12], that the weak and strong extensions coincide for first-order operators whose coefficients satisfy (2.1) and (2.2).

THEOREM 2.1. (*Friedrichs' Theorem*). - If L is a differential operator whose coefficients satisfy (2.1) and (2.2) then

$$(2.16) \quad L_s = L_w$$

L_s , the closure of $\tilde{L}$, is clearly the minimal closed extension of $\tilde{L}$. Moreover, $D(L_w)$ can be described in terms of distribution theory as the set of all distributions $u \in L_{2,m}$ for which the distribution Lu is in $L_{2,m'}$. Thus L_w is the maximal closed extension of $\tilde{L}$. Hence, Theorem 2.1 implies that if L satisfies (2.1) and (2.2) then the minimal operator $\tilde{L} : L_{2,m} \rightarrow L_{2,m'}$ has a unique closed extension $L_s = L_w$.

PROOF OF THEOREM 2.1. - The proof is, in essence, contained in FRIEDRICHS' original paper. Hence it is only sketched here.

Because of (2.14) it suffices to show that $L_w \subset L_s$. For this it is enough to show that if $u \in D(L_w)$ then there exists a sequence $\{u_k\} \subset D(\tilde{L}) = C_0^1(R^n; C^m)$ such that $u_k \rightarrow u$ in $L_{2,m}$ and $\tilde{L}u_k \rightarrow L_w u$ in $L_{2,m'}$. The approximation is carried out in two steps. First, let $\phi(t) \in C^1(R)$, $0 \leq \phi(t) \leq 1$, $\phi(t) \equiv 1$ for $t \leq 0$ and $\phi(t) \equiv 0$ for $t \geq 1$ and define $\phi_R(x) = \phi(|x| - R)$. Then it is easy to check that if $u \in D(L_w)$ then $u_R = \phi_R u \in D(L_w)$, $u_R \rightarrow u$ in $L_{2,m}$ and $L_w u_R \rightarrow L_w u$ in $L_{2,m'}$ when $R \rightarrow \infty$. Hence, it suffices to construct an approximating sequence $\{u_k\} \subset D(\tilde{L})$ for vectors $u \in D(L_w)$ with compact support.

To construct the sequence $\{u_k\}$, FRIEDRICHS introduced his well-known « mollifier », as follows. Let $j(t) \in C_0^1(R)$, $j(t) \equiv 0$ for $|t| \geq 1$, $j(t) \geq 0$ and

$$(2.17) \quad \int_{-1}^1 j(t) dt = 1.$$

Define

$$(2.18) \quad j_k(x) = k^n j(kx_1) j(kx_2) \dots j(kx_n)$$

and define the sequence $\{u_k\}$ and operator J_k by

$$(2.19) \quad u_k(x) = J_k u(x) = \int_{R^n} j_k(x - y) u(y) dy.$$

The basic approximation result for J_k is (see [12] and [13] for proofs).

LEMMA 2.3. - The operator J_k is bounded on $L_{2,m}$ with bound $\|J_k\| \leq 1$ for all k . Moreover, for all $u \in L_{2,m}$,

$$(2.20) \quad \lim_{k \rightarrow \infty} u_k = u \text{ in } L_{2,m}.$$

If $u \in D(L_w)$ and has compact support then clearly $u_k \in C_0^1(R^n; C^m)$ and $u_k \rightarrow u$ in $L_{2,m}$ by Lemma 2.3. Thus, the proof of Theorem 2.1 will be complete if it is shown that $\tilde{L}u_k \rightarrow L_w u$ in $L_{2,m'}$. Now

$$(2.21) \quad \tilde{L}u_k = \tilde{L}J_k u = J_k L_w u + (\tilde{L}J_k - J_k L_w)u.$$

and $J_k L_w u \rightarrow L_w u$ by Lemma 2.3. Hence it suffices to show that the last term in (2.21) tends to zero in $L_{2,m'}$ when $k \rightarrow \infty$. FRIEDRICHS showed that the operator $\tilde{L}J_k - J_k L_w$ can be extended to be a bounded operator from $L_{2,m}$ to $L_{2,m'}$ (in fact, an integral operator) and that

$$(2.22) \quad \lim_{k \rightarrow \infty} \|\tilde{L}J_k - J_k L_w\| = 0$$

when the coefficients of L satisfy (2.1) and (2.2). This completes the sketch of the proof of Theorem 2.1.

In the remainder of this paper, if L is a first-order partial differential operator whose coefficients satisfy (2.1) and (2.2) then the closed linear operator $L_s = L_w$ will also be denoted by L . In each case it will be clear from the context which meaning is intended.

A corollary of FRIEDRICHS' theorem that is needed below in § 5 is

COROLLARY 2.2. - Let L be a differential operator whose coefficients satisfy (2.1) and (2.2) and let $L = L_s = L_w$ denote the unique closed extension of L . Then if $u \in D(L)$ also $J_k u \in D(L)$ and $\lim_{k \rightarrow \infty} L J_k u = L u$ in $L_{2,m'}$.

PROOF. - Let $u \in D(L)$ and let $\{u_j\} \subset C_0^1(R^n; C^m)$ be such that $u_j \rightarrow u$ and $L u_j \rightarrow L u$ in $L_{2,m}$ when $j \rightarrow \infty$. This is possible because $L = L_s$ is the closure of $\tilde{L}$. Then $L J_k u_j = J_k L u_j + (L J_k - J_k L) u_j = J_k L u_j + (\tilde{L} J_k - J_k L) u_j$. Now J_k and $L J_k - J_k L$ are bounded operators on $L_{2,m}$ for fixed k . Thus $\{L J_k u_j\}$ is a CAUCHY sequence in $L_{2,m}$ for $j \rightarrow \infty$. Moreover, $J_k u_j \rightarrow J_k u$ for $j \rightarrow \infty$. Thus, since L is a closed operator, $J_k u \in D(L)$ and $L J_k u_j \rightarrow L J_k u$ in $L_{2,m}$. This implies that if K_k is the bounded extension of $\tilde{L} J_k - J_k L$ on $L_{2,m}$ then the restriction of K_k to $D(L)$ is $L J_k - J_k L$. To show that $L J_k u \rightarrow L u$ when $k \rightarrow \infty$, note that since $J_k u \in D(L)$, $L u - L J_k u = (L u - L \phi_R u) + (L \phi_R u - L J_k \phi_R u) + (L J_k \phi_R u - J_k L \phi_R u) + (J_k L \phi_R u - J_k L u) + (J_k L u - L J_k u) = (L u - L \phi_R u) + (L \phi_R u - L J_k \phi_R u) - L J_k \phi_R u + (L J_k - J_k L)(\phi_R u - u) + J_k(L \phi_R u - L u)$ where $\phi_R(x) = \phi(|x| - R)$ is

the function defined in the proof of Theorem 2.1. Let $\|LJ_k - J_kL\| \leq M$, where M is independent of k . (This is possible by (2.22)). Then given any $\varepsilon > 0$ choose R such that $\|Lu - L\psi_R u\| < \varepsilon/4$ and $\|\psi_R u - u\| \leq \varepsilon/4M$. (R does not depend on k). Then $\|Lu - LJ_k u\| \leq \varepsilon/4 + \|L\psi_R u - LJ_k \psi_R u\| + \varepsilon/4 + \varepsilon/4 = 3\varepsilon/4 + \|L\psi_R u - LJ_k \psi_R u\|$. The last term tends to zero (for fixed R) when $k \rightarrow \infty$ by the proof of Theorem 2.1 ((2.21) and the argument following it).

The strong and weak extensions of the minimal operator $\tilde{L}: L_{2,m}(E) \rightarrow L_{2,m}(E')$ can be defined by means of the formal adjoint $L_{E,E'}^+$ in strict analogy with Definitions 2.2 and 2.3. Both the coefficients of $L_{E,E'}^+$ and the scalar products $(u, v)_E$ and $(u', v')_{E'}$ involve the matrix-valued functions $E(x)$ and $E'(x)$. If $E(x)$ and $E'(x)$ and their first derivatives are continuous and if (1.5) holds for both, then it can again be shown that the operators L_s and L_w coincide and define the same linear operator $L (= L_s = L_w)$ as in the case where $E = I$ and $E' = I'$. This result is needed below only for differential operators Λ of the form (1.13). Therefore, this section is concluded with a discussion of this case.

Consider the differential operators

$$(2.23) \quad \Lambda = -iE(x)^{-1} \sum_{j=1}^n A_j D_j$$

and

$$(2.24) \quad \Lambda_0 = -iE_0^{-1} \sum_{j=1}^n A_j D_j$$

where, as in § 1, $E(x)$, E_0 , $A_1, \dots, A_n$ are $n \times n$ HERMITIAN matrices, E_0 , $A_1, \dots, A_n$ are constant, $E(x)$ and E_0 are positive definite and $E(x)$ satisfies (1.5). In the paper [22] the notation

$$(2.25) \quad H = L_{2,m}(E), \quad H_0 = L_{2,m}(E_0)$$

was used and the differential operators Λ and Λ_0 were used to define self-adjoint operators Λ and Λ_0 on H and H_0 respectively. The operator $\Lambda_0: H_0 \rightarrow H_0$ was defined in [22] by means of the FOURIER transform (1.19) and the symbol

$$(2.26) \quad \Lambda_0(p) = E_0^{-1} \sum_{j=1}^n A_j p_j$$

as follows.

DEFINITION 2.4. - $\Lambda: H_0 \rightarrow H_0$ is the linear operator defined by the conditions that

$$(2.27) \quad D(\Lambda_0) = H_0 \cap \{u: \Lambda_0(p)\widehat{u}(p) \in H_0\}$$

and $\Lambda_0 u = v$ if and only if $\widehat{v}(p) = \Lambda_0(p)\widehat{u}(p)$.

Note that multiplication by $E(x)$ defines an operator E on H which is bounded by (1.5). Moreover, multiplication by $E(x)^{-1}$ defines the inverse operator E^{-1} which is also bounded because $E(x)^{-1}$ satisfies (1.5) when $E(x)$ does. The same remarks apply to E_0 and E_0^{-1} . With this notation the operator Λ was defined in [22] as follows.

DEFINITION 2.5. - $\Lambda : H \rightarrow H$ is the linear operator defined by

$$(2.28) \quad D(\Lambda) = D(\Lambda_0)$$

and

$$(2.29) \quad \Lambda = E^{-1}E_0\Lambda_0.$$

Let J denote the identification map from H_0 onto H :

$$(2.30) \quad J : H_0 \rightarrow H, \quad Ju = u \quad \text{for all } u \in H_0.$$

Then J and J^{-1} are bounded operators because the norms in H_0 and H are equivalent. Since Λ and E are regarded as operators on H and Λ_0 and E_0 as operators on H_0 , it would be correct to define Λ in terms of Λ_0 by

$$(2.31) \quad D(\Lambda) = JD(\Lambda_0)$$

and

$$(2.32) \quad \Lambda = E^{-1}JE_0\Lambda_0J^{-1}$$

rather than (2.31) and (2.32). In many cases the operator J can be omitted, as in Definition 2.5, without confusion. However, it is sometimes important to recognize that J^* , the adjoint of J , is not the identification map but

$$(2.33) \quad J^* = E_0^{-1}J^{-1}E.$$

This fact is used below in § 5.

Definitions 2.4 and 2.5 were given in [22] where it was shown that the operators Λ_0 and Λ so defined are selfadjoint on H_0 and H respectively. These definitions differ from Definitions 2.2 and 2.3 of the strong extension Λ_s and the weak extension Λ_w . Nevertheless, these operators all coincide if $E(x) \in C^1(R^n)$ and satisfies (1.5).

This is proved below, following two preliminary lemmas.

LEMMA 2.4. - Let Λ_0 be defined by Definition 2.4. Then $u \in D(\Lambda_0)$ and $\Lambda_0 u = v$ if and only if

$$(2.34) \quad (u, \Lambda_0 \varphi)_{E_0} = (v, \varphi)_{E_0} \quad \text{for all } \varphi \in C_0^1(R^n; C^m).$$

The proof of this is immediate from Definition 2.4 and the PLANCHEREL theory and is omitted.

LEMMA 2.5. - Let Λ be defined by Definition 2.5. Then $u \in D(\Lambda)$ and $\Lambda u = v$ and only if

$$(2.35) \quad (u, \Lambda \varphi)_E = (v, \varphi)_E \quad \text{for all } \varphi \in C_0^1(R^n; C^m).$$

PROOF. - Note that

$$(2.36) \quad (u, \Lambda \varphi)_E = (u, \Lambda_0 \varphi)_{E_0}$$

for all $u \in H$ and $\varphi \in C_0^1(R^n; C^m)$. Moreover,

$$(2.37) \quad \begin{aligned} (v, \varphi)_E &= (E_0^{-1} E v, \varphi)_{E_0} \text{ and} \\ (w, \varphi)_{E_0} &= (E^{-1} E_0 w, \varphi)_E \end{aligned}$$

for all u, v and φ in H . Now assume that $u \in D(\Lambda) = D(\Lambda_0)$ and $\Lambda u = E^{-1} E_0 \Lambda_0 u = v$. Then, by (2.36) and (2.37)

$$(2.38) \quad (u, \Lambda \varphi)_E = (u, \Lambda_0 \varphi)_{E_0} = (\Lambda_0 u, \varphi)_{E_0} = (E^{-1} E_0 \Lambda_0 u, \varphi)_E = (v, \varphi)_E$$

for all $\varphi \in C_0^1(R^n; C^m)$. Conversely, if (2.35) holds then (2.36) and (2.37) imply

$$(2.39) \quad (u, \Lambda_0 \varphi)_{E_0} = (u, \Lambda \varphi)_E = (v, \varphi)_E = (E_0^{-1} E v, \varphi)_{E_0}$$

for all $\varphi \in C_0^1(R^n; C^m)$. Thus Lemma 2.4 implies that $u \in D(\Lambda_0) = D(\Lambda)$ and $\Lambda_0 u = E_0^{-1} E v$, whence $\Lambda u = E^{-1} E_0 \Lambda_0 u = v$, which completes the proof.

Definition 2.3 of Λ_w makes use of operator $\Lambda^\dagger$, the formal adjoint of Λ as an operator on $L_{2,m}(I)$. By (2.7) and (2.23)

$$(2.40) \quad \Lambda^\dagger \varphi(x) = -i \sum_{j=1}^n A_j D_j [E(x)^{-1} \varphi(x)]$$

since A_j is constant and A_j and $E(x)$ are HERMITIAN. Thus $u \in D(\Lambda_w)$ and $\Lambda_w u = v$ if and only if

$$(2.41) \quad (u, \Lambda^\dagger \varphi) = (v, \varphi) \quad \text{for all } \varphi \in C_0^1(R^n; C^m).$$

This implies

THEOREM 2.6. - Assume that $E(x) \in C^1(R^n)$ and satisfies (1.5) Then $\Lambda_w = \Lambda$.

PROOF. - Note that E defines a mapping of $C_0^1(R^n; C^m)$ onto itself with inverse E^{-1} ; i.e. $\varphi = E\psi \in C_0^1(R^n; C^m)$ if and only if $\psi = E^{-1}\varphi \in C_0^1(R^n; C^m)$. Thus if $u \in D(\Lambda_w)$ and $\Lambda_w u = v$ then

$$(v, \psi)_E = (v, E^{-1}\varphi)_E = (v, \varphi) = (u, \Lambda^+\varphi) =$$

$$\int_{R^n} u(x)^* (-i \sum_{j=1}^n A_j D_j \{ E^{-1}(x) \varphi(x) \}) dx = \int_{R^n} u(x)^* (-i \sum_{j=1}^n A_j D_j \psi(x)) dx =$$

(2.42)

$$\int_{R^n} u(x)^* E(x) (-i E(x)^{-1} \sum_{j=1}^n A_j D_j \psi(x)) dx = (u, \Lambda \psi)_E$$

for all $\psi \in C_0^1(R^n; C^m)$. Thus $u \in D(\Lambda)$ and $\Lambda u = v$; i.e. $\Lambda_w \subset \Lambda$.

Conversely, if $u \in D(\Lambda)$ and $\Lambda u = v$ then as above

$$(v, \varphi) = (v, \psi)_E = (u, \Lambda \psi)_E = (u, \Lambda^+\varphi)$$

for all $\varphi \in C_0^1(R^n; C^m)$. Thus $u \in D(\Lambda_w)$ and $\Lambda_w u = v$, i.e. $\Lambda \subset \Lambda_w$. Thus $\Lambda = \Lambda_w$.

COROLLARY 2.7. - $\Lambda_0 = \Lambda_{0, w}$.

PROOF. - This follows immediately from Theorem 2.6 with $E(x) = E_0$.

§ 3. - Coerciveness Inequalities for Elliptic Operators.

In this section coerciveness inequalities are derived for a class of elliptic operators on R^n . The results given here are either known or very similar to known results. The purpose of the section is to establish the results and techniques for elliptic systems that are used in the study of nonelliptic systems in the following sections.

The derivation of coerciveness inequalities for elliptic systems is based on a reformulation of the ellipticity condition contained in the following two lemmas.

LEMMA 3.1. - The differential operator L is elliptic at $x_0 \in R^n$ if and only if it has the property that for all $p \in R^n - \{0\}$ and $\xi \in C^m$,

$$(3.1) \quad L(p, x_0)\xi = 0 \quad \text{implies that} \quad \xi = 0.$$

Condition (3.1) is clearly equivalent to condition (1.10) of Definition 1.3.

LEMMA 3.2. - The differential operator L is elliptic at $x_0 \in R^n$ if and only if there exists a constant $\mu(x_0) > 0$ such that

$$(3.2) \quad |L(p, x_0)u|^2 \geq \mu(x_0)^2 |p|^2 |u|^2 \quad \text{for all } p \in R^n \text{ and } u \in C^m$$

(The norm $|\cdot|$ can be any matrix norm).

PROOF. - Write $p = |p|\eta$ and $u = |u|\tilde{u}$ where η and $\tilde{u}$ are unit vectors. Then $|L(\eta, x_0)\tilde{u}|$ is continuous in η and $\tilde{u}$, and hence it has a lower bound $\mu(x_0) \geq 0$ which is achieved for some η_0 and $\tilde{u}_0$, since the unit spheres are compact. If $\mu(x_0) = 0$ then $L(\eta_0, x_0)\tilde{u}_0 = 0$ and L is not elliptic at x_0 by Lemma 3.1. Thus if L is elliptic at x_0 then $|L(\eta, x_0)\tilde{u}|^2 \geq \mu(x_0) > 0$. This implies (3.2), since $L(p, x_0)u$ is homogeneous of degree one in p and u . Conversely, if (3.2) holds with $\mu(x_0) > 0$ then $L(p, x_0)u = 0$ implies $u = 0$ for all $p \in R^n - \{0\}$. Hence L is elliptic at x_0 by Lemma 3.1.

DEFINITION 3.1. - The differential operator L is uniformly elliptic in a region $\Omega \subset R^n$ if and only if there exists a constant $\mu > 0$ such that

$$(3.3) \quad |L(p, x)u|^2 \geq \mu^2 |p|^2 |u|^2 \quad \text{for all } p \in R^n, u \in C^m \text{ and } x \in \Omega.$$

A sufficient condition for uniform ellipticity in R^n is given by

LEMMA 3.3. - Let L be a differential operator which satisfies (2.1), (2.2). Assume that L is elliptic in R^n and that

$$(3.4) \quad \lim_{|x| \rightarrow \infty} L_j(x) = L_j^\infty, \quad j = 1, \dots, n,$$

uniformly in $x/|x|$ where

$$(3.5) \quad L^\infty = -i \sum_{j=1}^n L_j^\infty D_j$$

is an elliptic operator. Then L is uniformly elliptic in R^n .

PROOF. - Let r be any positive number. Then $|L(\eta, x)\tilde{u}|$ is continuous and never zero on the compact set $\{(\eta, x, \tilde{u}) : |\eta| = 1, |x| \leq r, |\tilde{u}| = 1\}$. Hence there is a positive constant $\mu(r)$ such that

$$(3.6) \quad |L(\eta, x)\tilde{u}| \geq \mu(r) \quad \text{for } |\eta| = 1, |x| \leq r, |\tilde{u}| = 1.$$

Next

$$(3.7) \quad L(\eta, x)\tilde{u} - L^\infty(\eta)\tilde{u} = \sum_{j=1}^n [L_j(x) - L_j^\infty]\eta_j\tilde{u}$$

whence

$$(3.8) \quad |L(\eta, x)\tilde{u} - L^\infty(\eta)\tilde{u}| \leq \sum_{j=1}^n |[L_j(x) - L_j^\infty]\tilde{u}| \leq \sum_{j=1}^n |L_j(x) - L_j^\infty|$$

for all $(\eta, x, \tilde{u})$ with $|\eta| = 1$, $x \in R^n$ and $|\tilde{u}| = 1$. Thus (3.4) implies that

$$(3.9) \quad \lim_{|x| \rightarrow \infty} L(\eta, x)\tilde{u} = L^\infty(\eta)\tilde{u} \text{ uniformly in } x/|x|, \eta \text{ and } \tilde{u}.$$

Now, since L^∞ is elliptic there is a $\mu_1 > 0$ such that

$$(3.10) \quad |L^\infty(\eta)\tilde{u}| \geq \mu_1 \quad \text{for } |\eta| = 1, |\tilde{u}| = 1.$$

Thus (3.9) implies that there is an $r_0 > 0$ such that

$$(3.11) \quad \left| |L(\eta, x)\tilde{u}| - |L^\infty(\eta)\tilde{u}| \right| \leq 2^{-1}\mu_1$$

for $|\eta| = 1, |x| \geq r_0, |\tilde{u}| = 1$.

Hence

$$(3.12) \quad |L(\eta, x)\tilde{u}| = |L^\infty(\eta)\tilde{u}| + (|L(\eta, x)\tilde{u}| - |L^\infty(\eta)\tilde{u}|) \geq$$

$$\mu_1 - 2^{-1}\mu_1 = 2^{-1}\mu_1$$

for $|\eta| = 1, |x| \geq r_0$ and $|\tilde{u}| = 1$. Combining (3.6) with $r = r_0$ and $\mu(r_0) = \mu_0$ and (3.12) gives

$$(3.13) \quad |L(\eta, x)\tilde{u}| \geq \mu = \min(\mu_0, 2^{-1}\mu_1) > 0$$

for $|\eta| = 1, x \in R^n$ and $|\tilde{u}| = 1$. Finally, since $L(p, x)u$ is homogeneous of degree one in (p, u) (3.13) implies that (3.3) holds for all $x \in R^n, p \in R^n$ and $u \in C^n$.

The coerciveness property for an operator $L : L_{2,m} \rightarrow L_{2,w'}$ can be reformulated in terms of the SOBOLEV space

$$(3.14) \quad L_{2,m}^1 = L_{2,m} \cap \{u : D_j u \in L_{2,m} \text{ for } j = 1, 2, \dots, n\}$$

$L_{2,m}^1$ is a HILBERT space with inner product

$$(3.15) \quad (u, v)_1 = \int_{R^n} (u^* v + \sum_{j=1}^n D_j u^* D_j u) dx$$

and norm $\|u\|_1 = (u, u)_1^{1/2}$.

LEMMA 3.4. - The operator $L : L_{2,m} \rightarrow L_{2,m'}$ is coercive if and only if $D(L) = L_{2,m}^1$ and there exists a constant $K > 0$ such that for all $u \in D(L)$

$$(3.16) \quad \|u\|_1^2 \leq K^2(\|Lu\|^2 + \|u\|^2).$$

Condition (3.16) is obviously equivalent to (1.8), with a different constant K , since

$$(3.17) \quad \|u\|_1^2 = \|u\|^2 + \sum_{j=1}^n \|D_j u\|^2.$$

The principal result of this section is

THEOREM 3.5. - Let the differential operator L have the properties

- (1) L is uniformly elliptic in R^n ,
- (2) The coefficients $L_0(x), L_1(x), \dots, L_n(x)$ satisfy conditions (2.1) and (2.2).

Then the closed linear operator $L : L_{2,m} \rightarrow L_{2,m'}$ associated with L is coercive.

PROOF. - Since $L = L_s$ is the closure of the minimal operator $\tilde{L}$ it suffices to prove that there exists a $K > 0$ such that (3.16) holds for all $u \in C_0^1(R^n; C^m)$. Indeed, if this true and $u \in D(L)$ then there exists a sequence $\{u_k\} \subset C_0^1(R^n; C^m)$ such that $u_k \rightarrow u$ in $L_{2,m}$ and $Lu_k \rightarrow Lu$ in $L_{2,m'}$. (3.16) holds for each u_k and for the differences $u_k - u_l$. It follows that $\{u_k\}$ is a CAUCHY sequence in $L_{2,m}^1$ and hence $u \in L_{2,m}^1$. Thus $D(L) \subset L_{2,m}^1$. Finally, writing (3.16) for u_k and making $k \rightarrow \infty$ gives (3.16) for $u \in D(L)$.

The proof of (3.16) for $u \in C_0^1(R^n; C^m)$ is exactly analogous to a proof of GÄRDING'S inequality due to S. AGMON [3]. The main steps are given here to show how the method applies to elliptic systems with m' equations and m unknowns ($m' \geq m$).

Let

$$(3.18) \quad L_p = -i \sum_{j=1}^n L_j(x) D_j$$

denote the principal part of L . Then if $u \in C_0^1(R^n; C^m)$,

$$(3.19) \quad \begin{aligned} \|L_p u\|^2 &= \int_{R^n} (L_p u)^* L_p u dx = \int_{R^n} \left(\sum_{j=1}^n L_j D_j u \right)^* \sum_{k=1}^n L_k D_k u dx = \\ &= \int_{R^n} \left(\sum_{j=1}^n \sum_{k=1}^n D_j u^* L_j^* L_k D_k u \right) dx = \int_{R^n} \left(\sum_{j=1}^n \sum_{k=1}^n D_j u^* T_{jk}(x) D_k u \right) dx \end{aligned}$$

where $T_{jk}(x) = L_j^*(x) L_k(x)$ is an $m \times m$ matrix.

LEMMA 3.6. - If L_p has constant coefficients $L_j (j = 1, \dots, n)$ then for all $u \in C_0^1(R^n; C^m)$,

$$(3.20) \quad \|L_p u\|^2 \geq \mu^2 \sum_{j=1}^n \|D_j u\|^2$$

where μ is the ellipticity constant from (3.3).

PROOF. - The proof is the same as the proof of (1.55) for the equations of elasticity. By the PLANCHEREL theory

$$(3.21) \quad \begin{aligned} \|L_p u\|^2 &= \int_{R^n} \tilde{u}(p)^* \sum_{j=1}^n \sum_{k=1}^n L_j^* p_j L_k p_k \tilde{u}(p) dp = \\ &= \int_{R^n} |L(p) \tilde{u}(p)|^2 dp \geq \mu^2 \int_{R^n} |p|^2 |\tilde{u}(p)|^2 dp = \mu^2 \sum_{j=1}^n \|D_j u\|^2. \end{aligned}$$

LEMMA 3.7. - Under the hypotheses of Theorem 3.5, there exists a constant $\rho > 0$ such that

$$(3.22) \quad \|L_p u\|^2 \geq 2^{-1} \mu^2 \sum_{j=1}^n \|D_j u\|^2$$

for all $u \in C_0^1(R^n; C^m)$ such that $\text{supp } u$, the support of u , has diameter less than ρ . ρ depends only on n and the moduli of continuity of $L_1(x), \dots, L_n(x)$.

PROOF. - It is easy to show that (2.1) implies that $L_1(x), \dots, L_n(x)$ are uniformly continuous on R^n . Hence each of the matrices $T_{jk}(x) = L_j^*(x) L_k(x)$ is uniformly continuous on R^n . Thus, gives any $\varepsilon > 0$ there exists a $\rho > 0$ such that

$$(3.23) \quad |T_{jk}(x) - T_{jk}(x_0)| \leq \varepsilon \quad \text{for } 1 \leq j, k \leq n,$$

provided that $|x - x_0| \leq \rho$. ρ depends on ε and the moduli of continuity of the $T_{jk}(x)$. Let $u \in C_0^1(R^n; C^m)$ and let $\text{supp } u$ have diameter less than ρ . Let $x_0 \in \text{supp } u$ and define

$$(3.24) \quad L_{p,0} = -i \sum_{j=1}^n L_j(x_0) D_j.$$

Then

$$(3.25) \quad \begin{aligned} \|L_p u\|^2 - \|L_{p,0} u\|^2 &= \\ &= \int_{\text{supp } u} \sum_{j=1}^n \sum_{k=1}^n (D_j u)^* [T_{jk}(x) - T_{jk}(x_0)] D_k u dx \end{aligned}$$

and hence

$$\begin{aligned}
 & \left| \|L_p u\|^2 - \|L_{p,0} u\|^2 \right| \leq \\
 & \int_{\text{supp } u} \sum_{j=1}^n \sum_{k=1}^n |T_{jk}(x) - T_{jk}(x_0)| \cdot |D_j u| \cdot |D_k u| dx \leq \\
 (3.26) \quad & \varepsilon \sum_{j=1}^n \sum_{k=1}^n \int_{R^n} |D_j u| \cdot |D_k u| dx \leq \varepsilon/2 \sum_{j=1}^n \sum_{k=1}^n \int_{R^n} (|D_j u|^2 + |D_k u|^2) dx = \\
 & n\varepsilon \sum_{j=1}^n \|D_j u\|^2.
 \end{aligned}$$

Inequality (3.26) implies

$$(3.27) \quad \|L_p u\|^2 \geq \|L_{p,0} u\|^2 - n\varepsilon \sum_{j=1}^n \|D_j u\|^2.$$

Hence, by Lemma 3.6,

$$(3.28) \quad \|L_p u\|^2 \geq (\mu^2 - n\varepsilon) \sum_{j=1}^n \|D_j u\|^2 = 2^{-1}\mu^2 \sum_{j=1}^n \|D_j u\|^2$$

provided $n\varepsilon = (1/2)\mu^2$. This choice of ε determines ρ .

LEMMA 3.8. - Under the hypothesis of Theorem 3.5, there exists a constant K_1 such that

$$(3.29) \quad \|Lu\|^2 \geq \|L_p u\|^2 - K_1 \|u\| \cdot \|u\|_1,$$

for all $u \in C_0^1(R^n; C^m)$. K_1 depends only on n and the bounds for the coefficients $L_0(x)$, $L_1(x)$, ..., $L_n(x)$.

PROOF. - $L = L_p + L_0$. Hence

$$\begin{aligned}
 \|Lu\|^2 &= \int_{R^n} (L_p u + L_0 u)^* (L_p u + L_0 u) dx = \int_{R^n} \{ (L_p u)^* (L_p u) + \\
 (3.30) \quad &+ (L_p u)^* (L_0 u) + (L_0 u)^* (L_p u) + (L_0 u)^* (L_0 u) \} dx \\
 &= \|L_p u\|^2 + R_1(u)
 \end{aligned}$$

where

$$(3.31) \quad R_1(u) = \int_{R^n} \{ i \sum_{j=1}^n D_j u^* L_p^* L_0 u - i \sum_{j=1}^n u^* L_0^* L^j D_j u + u^* L_0^* L_0 u \} dx.$$

Now

$$(3.32) \quad \left| \int_{R^n} D_j u^* L_p^* L_0 u dx \right| \leq \int_{R^n} |D_j u| \cdot |u| \cdot |L_p^* L_0| dx \leq K_2 \|u\| \cdot \|u\|_1$$

where K_2 depends on the bounds for the L_j . The second term in (3.31) can be estimated in the same way, and

$$(3.33) \quad \left| \int_{R^n} u^* L_0^* L_0 u dx \right| \leq K_3 \|u\|^2.$$

Thus

$$(3.34) \quad |R_1(u)| \leq \sum_{j=1}^n 2K_2 \|u\| \cdot \|u\|_1 + K_3 \|u\|^2 \leq K_1 \|u\| \cdot \|u\|_1$$

where $K_1 = 2nK_2 + K_3$. (3.30) and (3.34) imply (3.29).

Proof of Theorem 3.5, concluded. - The proof makes use of Lemmas 3.6, 3.7 and 3.8 and a special partition of unit $\{\zeta_\alpha(x)\}$, due to S. AGMON [3], with the following properties: $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n -tuple of integers, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and

$$(3.35) \quad \sum_{|\alpha|=0}^{\infty} \zeta_\alpha(x)^2 = 1 \quad \text{for all } x \in R^n$$

The functions ζ_α are defined by

$$(3.36) \quad \zeta_\alpha = \zeta'_\alpha \left[\sum_{\beta} \zeta_\beta'^2 \right]^{-1/2}, \quad \zeta'_\beta(x) = \zeta'(x - d\beta)$$

where

$$(3.37) \quad \zeta'(x) \in C_0^\infty(R^n), \quad 0 \leq \zeta(x) \leq 1, \quad \text{and} \quad \text{supp } \zeta \subset Q_0$$

where

$$(3.38) \quad Q_\alpha = \{x : d\alpha_k - 1 < x_k < d(\alpha_k + 1), \quad k = 1, 2, \dots, n\}$$

is the cube with center at $d\alpha$ and side $2d$. Note that $\text{supp } \zeta_\alpha \subset Q_\alpha$.

To prove the coerciveness inequality (3.16) for arbitrary $u \in C_0^1(R^n; C^n)$, choose d so that the diameter of Q_α is less than the number ρ of Lemma 3.7; i.e. $2d\sqrt{n} < \rho$, so that Lemma 3.7 is applicable to the function $\zeta_\alpha u$. Then, by (3.19) and (3.35)

$$\begin{aligned}
 \|L_p u\|^2 &= \int_{R^n} \sum_{|\alpha|=0}^{\infty} \sum_{j=1}^n \sum_{k=1}^n \zeta_{\alpha}(x)^2 D_j u(x)^* T_{jk}(x) D_k u(x) dx \\
 (3.39) \quad &= \int_{R^n} \sum_{|\alpha|=0}^{\infty} \sum_{j=1}^n \sum_{k=1}^n D_j(\zeta_{\alpha} u)^* T_{jk}(x) D_k(\zeta_{\alpha} u) dx + R_2(u) \\
 &= \sum_{|\alpha|=0}^{\infty} \|L_p(\zeta_{\alpha} u)\|^2 + R_2(u)
 \end{aligned}$$

where

$$\begin{aligned}
 (3.40) \quad R_2(u) &= - \sum_{|\alpha|=0}^{\infty} \sum_{j=1}^n \sum_{k=1}^n \int_{Q_{\alpha}} \{ (\zeta_{\alpha} D_j \zeta_{\alpha}) u^* T_{jk} D_k u \\
 &\quad + (\zeta_{\alpha} D_k \zeta_{\alpha}) D_j u^* T_{jk} u + (D_j \zeta_{\alpha} D_k \zeta_{\alpha}) u^* T_{jk} u \} dx.
 \end{aligned}$$

Notice that all the sums in (3.39) and (3.40) are finite because $\text{supp } u$ meets only a finite number of the cubes Q_{α} . Let

$$(3.41) \quad |D_j \zeta_{\alpha}| \leq K_4 \text{ for } j = 1, 2, \dots, n \text{ and all } \alpha.$$

(K_4 depends only on d ; i.e. ρ and n). Then

$$\begin{aligned}
 (3.42) \quad |R_2(u)| &\leq \sum_{|\alpha|=0}^{\infty} \sum_{j=1}^n \sum_{k=1}^n \int_{Q_{\alpha}} \{ K_4 |u^* T_{jk} D_k u| + K_4 |D_j u^* T_{jk} u| + \\
 &\quad K_4^2 |u^* T_{jk} u| \} dx.
 \end{aligned}$$

Note that, apart from a null set, every $x \in R^n$ is in precisely 2^n of the cubes Q_{α} . Thus

$$\begin{aligned}
 (3.43) \quad |R_2(u)| &\leq 2^n \sum_{j=1}^n \sum_{k=1}^n \int_{R^n} \{ K_4 |u^* T_{jk} D_k u| + K_4 |D_j u^* T_{jk} u| + \\
 &\quad K_4^2 |u^* T_{jk} u| \} dx \leq 2^n \sum_{j=1}^n \sum_{k=1}^n \int_{R^n} \{ K_4 K_5 \|u\| \cdot \|D_k u\| + \\
 &\quad K_4 K_5 \|D_j u\| \cdot \|u\| + K_4^2 K_5 \|u\|^2 \} dx
 \end{aligned}$$

where K_5 is a bound for $\|T_{jk}\|$. Thus, by SCHWARZ'S inequality

$$(3.44) \quad \begin{aligned} |R_2(u)| &\leq 2^n (K_4 K_5 n \|u\| \sum_{j=1}^n \|D_j u\| + K_4 K_5 n \|u\| \sum_{k=1}^n \|D_k u\| + \\ &+ K_4^2 K_5 n^2 \|u\|^2) \leq K_6 \|u\| \cdot \|u\|_1 \end{aligned}$$

where K_6 depends only on n , ρ and the bounds for the coefficients $L_j(x)$.
Thus

$$(3.45) \quad \|L_p u\|^2 \geq \sum_{|\alpha|=0}^{\infty} \|L_p(\zeta_\alpha u)\|^2 - K_6 \|u\| \cdot \|u\|_1.$$

Combining this with Lemma 3.8 gives

$$\|Lu\|^2 \geq \sum_{|\alpha|=0}^{\infty} \|L_p(\zeta_\alpha u)\|^2 - K_7 \|u\| \cdot \|u\|_1,$$

where $K_7 = K_1 + K_6$. Now, application of Lemma 3.7 to the function $\zeta_\alpha u$ gives

$$(3.47) \quad \|L_p(\zeta_\alpha u)\|^2 \geq (1/2) \mu^2 \sum_{j=1}^n \|D_j(\zeta_\alpha u)\|^2.$$

Moreover,

$$(3.48) \quad \begin{aligned} \|D_j(\zeta_\alpha u)\|^2 &= \int_{R^n} \|\zeta_\alpha D_j u + (D_j \zeta_\alpha) u\|^2 dx \\ &= \int_{R^n} \{ \zeta_\alpha^2 D_j u^* D_j u + \zeta_\alpha (D_j \zeta_\alpha) (D_j u^*) u + (D_j \zeta_\alpha) \zeta_\alpha u^* D_j u + \\ &+ (D_j \zeta_\alpha)^2 u^* u \} dx \end{aligned}$$

whence

$$(3.49) \quad \sum_{|\alpha|=0}^{\infty} \sum_{j=1}^n \|D_j(\zeta_\alpha u)\|^2 = \sum_{j=1}^n \|D_j u\|^2 + R_3(u)$$

where

$$(3.50) \quad \begin{aligned} R_3(u) &= \sum_{|\alpha|=0}^{\infty} \sum_{j=1}^n \int_{Q_\alpha} \{ \zeta_\alpha D_j \zeta_\alpha (D_j u^*) u + (D_j \zeta_\alpha) \zeta_\alpha u^* D_j u + \\ &+ (D_j \zeta_\alpha)^2 u^* u \} dx. \end{aligned}$$

This is similar in structure to $R_2(u)$ and can be estimated in the same way:

$$(3.51) \quad |R_3(u)| \leq K_8 \|u\| \cdot \|u\|_1.$$

Combining (3.46), (3.47), (3.49) and (3.51) gives

$$\begin{aligned}
 \|Lu\|^2 &\geq 2^{-1}\mu^2 \left(\sum_{j=1}^n \|D_j u\|^2 - K_8 \|u\| \|u\|_1 \right) - K_7 \|u\| \cdot \|u\|_1 \\
 (3.52) \quad &= 2^{-1}\mu^2 (\|u\|_1^2 - \|u\|^2) - (K_7 + 2^{-1}\mu^2 K_8) \|u\| \cdot \|u\|_1 \\
 &= 2^{-1}\mu^2 \|u\|_1^2 - K_9 \|u\| \cdot \|u\|_1 - 2^{-1}\mu^2 \|u\|^2
 \end{aligned}$$

where $K_9 = K_7 + (1/2)\mu^2 K_8$. Now, the inequality $2ab \leq \delta a^2 + \delta^{-1}b^2$, applied to the term $\|u\| \cdot \|u\|_1$ gives

$$(3.53) \quad \|Lu\|^2 \geq (2^{-1}\mu^2 - 2^{-1}K_9\delta) \|u\|_1^2 - (2^{-1}\mu^2 + 2^{-1}K_9\delta^{-1}) \|u\|^2.$$

Choose δ so that $2^{-1}K_9\delta = 4^{-1}\mu^2$. Then

$$\|Lu\|^2 \geq (1/4)\mu^2 \|u\|_1^2 - K_{10} \|u\|^2.$$

This is equivalent to (3.16), which completes the proof of Theorem 3.5.

Theorem 3.5 states that a differential operator L is coercive if it is uniformly elliptic and its coefficients satisfy hypotheses (2.1) and (2.2). The converse is also true under the same hypotheses on the coefficients. This assertion is implied by

THEOREM 3.9. - Let the coefficients $L_0(x), L_1(x), \dots, L_n(x)$ satisfy conditions (2.1) and (2.2), and let the corresponding closed linear operator $L: L_{2,m} \rightarrow L_{2,m'}$ be coercive. Then L is uniformly elliptic in R^n .

PROOF. - The proof is exactly analogous to a proof of a converse of GARDING's inequality due to S. AGMON [3, p. 86]. Therefore the proof is only sketched here, to show how it applies to systems.

By assumption, there exists a constant K such that (3.16) holds for all $u \in D(L)$. AGMON's method is to take, in (3.16),

$$(3.55) \quad u(x) = e^{itp} \cdot {}^x\phi(x)$$

where $t \in R$, $p \in R^n$ and $\phi \in C_0^1(R^n; C^m)$, and then make $t \rightarrow \infty$. A short calculation gives

$$\begin{aligned}
 \|u\|_1^2 &\sim t^2 |p|^2 \int_{R^n} |\phi(x)|^2 dx, \quad t \rightarrow \infty, \\
 (3.56) \quad \|Lu\|^2 &\sim t^2 \int_{R^n} |L(p, x)\phi(x)|^2 dx, \quad t \rightarrow \infty,
 \end{aligned}$$

which, with (3.16), implies

$$(3.57) \quad |p|^2 \int_{R^n} |\phi(x)|^2 dx \leq K^2 \int_{R^n} |L(p, x)\phi(x)|^2 dx$$

for all $\phi \in C_0^1(R^n; C^m)$. Now take

$$(3.58) \quad \phi(x) = j_k(x - x_0)^{1/2} v$$

where $j_k(x)$ is the mollifier (2.18) and v is a constant vector. Making $k \rightarrow \infty$ and using the continuity of $L(p, x)$ gives

$$|p|^2 |v|^2 \leq K^2 |L(p, x_0)v|^2.$$

Since x_0 is an arbitrary point in R^n , this proves that L is uniformly elliptic.

§ 4. - Coerciveness Inequalities for Nonelliptic Operators of Constant Deficit.

In this section and the next coerciveness inequalities on subspaces (Definition 1.4) are derived for nonelliptic differential operators L of constant deficit k on R^n (Definition 1.5). The examples in § 1 show how such inequalities can be found by constructing an augmented operator

$$(4.1) \quad L'' = \begin{pmatrix} L' \\ L \end{pmatrix}$$

which is coercive. It is shown below that such augmented operators L'' always exist, although L' may not be a first-order differential operator. The case in which L' is a first-order differential operator is treated in this section. The more general case is treated in § 5 for the case of operators

$$(4.2) \quad \Lambda = -iE(x)^{-1} \sum_{j=1}^n A_j D_j$$

which are perturbations of uniformly propagative operators.

DEFINITION 4.1. - Let $L' : L_{2,m} \rightarrow L_{2,k}$ be an (unbounded) linear operator (not necessarily a differential operator). Let L'' be the operator with domain $D(L'') = D(L') \cap D(L)$ defined by (4.1). Then L'' is called an augmentation of the operator L .

DEFINITION 4.2. - An augmentation L'' of an operator L is said to be L -coercive if and only if $D(L'') = L_{2,m}^1$ and there exists a constant $K > 0$ such that

$$(4.3) \quad \|u\|_1^2 \leq K^2(\|L''u\|^2 + \|u\|^2) \quad \text{for all } u \in D(L'');$$

that is

$$(4.4) \quad \begin{aligned} \|u\|_1^2 &\leq K^2(\|L'u\|^2 + \|Lu\|^2 + \|u\|^2) \\ &\text{for all } u \in D(L') \cap D(L). \end{aligned}$$

Note that if L has an augmentation which is L -coercive then L is coercive on the subspace $N(L')$:

$$(4.5) \quad \|u\|_1^2 \leq K^2(\|Lu\|^2 + \|u\|^2) \quad \text{for all } u \in D(L) \cap N(L').$$

Let L' be a $k \times m$ matrix differential operator of order α :

$$(4.6) \quad L' = \sum_{|\alpha|=\alpha} L'_\alpha(x)(-iD)^\alpha.$$

The standard multi-index notation is used in (4.6); i.e. $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n -tuple of non-negative integers and $|\alpha| = \alpha_1 + \dots + \alpha_n$, while $(-iD)^\alpha = (-iD_1)^{\alpha_1} \dots (-iD_n)^{\alpha_n}$. The symbol of L' is defined by

$$(4.7) \quad L'(p, x) = \sum_{|\alpha|=\alpha} L'_\alpha(x)p^\alpha, \quad p \in R^n,$$

where $p^\alpha = p_1^{\alpha_1} \dots p_n^{\alpha_n}$. The symbol of the augmented operator L'' is

$$(4.8) \quad L''(p, x) = \begin{pmatrix} L'(p, x) \\ L(p, x) \end{pmatrix}.$$

DEFINITION 4.3. - The augmented operator L'' is elliptic at $x_0 \in R^n$ if and only if

$$(4.9) \quad \text{rank } L''(p, x_0) = m \quad \text{for all } p \in R^n - \{0\},$$

L'' is elliptic in a region $\Omega \subset R^n$ if and only if it is elliptic at each $x_0 \in R^n$.

This definition generalizes Definition 1.3. It is also a generalization to overdetermined systems of a definition of ellipticity due to A. DOUGLIS and L. NIRENBERG [9] which has been used as a basis for *a priori* estimates for elliptic systems by S. AGMON, A. DOUGLIS and L. NIRENBERG [4]. It is shown here that every constant deficit operator L has an elliptic augmentation in

this sense. In order to achieve this, L' is required to satisfy the following conditions which are suggested by the examples of § 1:

$$(4.10) \quad \text{rank } L'(p, x) = k \quad \text{for all } p \in R^n - \{0\} \text{ and } x \in R^n,$$

where k is the deficit of L .

$$(4.11) \quad L'(p, x)M(x)L(p, x)^* = 0 \quad \text{for all } p \in R^n \text{ and } x \in R^n,$$

where, for each $x \in R^n$, $M(x)$ is a HERMITIAN positive definite matrix.

LEMMA 4.1. - If L is an operator of constant deficit k in R^n and L' satisfies (4.10) and (4.11) then the augmented operator L'' is elliptic in R^n .

PROOF. - (4.11) implies that the rows of $L'(p, x)$ and $L(p, x)$ are orthogonal in the sense of the HERMITIAN inner product $(\xi, \eta) = \xi^* M(x) \eta$. Hence the rows of $L'(p, x)$ are linearly independent of the rows of $L(p, x)$, and therefore

$$(4.12) \quad \text{rank } L''(p, x) = \text{rank } L'(p, x) + \text{rank } L(p, x) = m.$$

THEOREM 4.2. - Let L have constant deficit k in R^n and for each $x \in R^n$ let $M(x)$ be an $m \times m$ HERMITIAN positive definite HERMITIAN matrix. Then there exists an operator L' of the form (4.6) which satisfies (4.10) and (4.11). In particular, the augmented operator L'' is elliptic in R^n .

PROOF. - A simple computation shows that (4.11) is equivalent to the condition

$$(4.13) \quad L(p, x)M(x)L'(p, x)^* = 0 \quad \text{for all } p \in R^n \text{ and } x \in R^n.$$

This equation states that the rows of $L'(p, x)$ are solutions of the homogeneous linear system

$$(4.14) \quad (L(p, x)M(x))u = 0.$$

Since $M(x)$ is nonsingular, $\text{rank } L(p, x)M(x) = \text{rank } L(p, x) = m - k$.

Thus the nullspace of $L(p, x)M(x)$ has dimension k . Hence (4.14) has k linearly independent solutions. Moreover, such solutions can be constructed as homogeneous rational functions of the components of $L(p, x)M(x)$; e.g. by GAUSSIAN elimination. Multiplying these solutions by their least common denominators gives solutions which are homogeneous polynomials in the components of $L(p, x)M(x)$. Hence there are k linearly independent solutions of this form. Define $L'(p, x)$ to be a $k \times m$ matrix whose rows are these solutions. Then $L'(p, x)$ has properties (4.10) and (4.11) and is the symbol of an operator L' of the form (4.6).

COROLLARY 4.3. - If L has coefficients $L_1(x), \dots, L_n(x) \in C^s(R^n)$ and $M(x) \in C^s(R^n)$ then L' has coefficients $L'_x(x) \in C^s(R^n)$.

PROOF. - The coefficients $L'_x(x)$ are polynomials in the coefficients of L and $M(x)$.

For operators L of constant deficit having an elliptic augmentation L'' which is a first-order operator the following coerciveness theorem holds.

THEOREM 4.4. - Let L satisfy hypotheses (2.1), (2.2) and have constant deficit k in R^n . Let L have an elliptic augmentation L'' which is a first-order operator and satisfies (4.11) where $M(x)$ is a given function. Assume that

$$(4.15) \quad \lim_{|x| \rightarrow \infty} L_j(x) = L_j^\infty, \quad j = 1, 2, \dots, n,$$

uniformly in $x/|x|$ where

$$(4.16) \quad L_\infty = i \sum_{j=1}^n L_j^\infty D_j$$

has constant deficit k . Moreover, assume that $M(x)$ and $D_j M(x)$ are continuous and bounded in R^n and

$$(4.17) \quad \lim_{|x| \rightarrow \infty} M(x) = M_\infty$$

uniformly in $x/|x|$ where M_∞ is positive definite. Then L'' is uniformly elliptic on R^n and hence L is coercive on $N(L')$.

PROOF. - The components of $L'(p, x)$ are polynomials in the components of $L(p, x)$ and $M(x)$. Thus L'' satisfies the hypotheses of Lemma 3.3 and hence is uniformly elliptic on R^n . The last conclusion follows from Theorem 3.5 and Definition 4.2.

If a bounded and uniformly positive definite matrix-valued function $E(x)$ is given on R^n then the HILBERT space $H = L_{2,m}(E)$ is defined and it is meaningful to ask if L is coercive on the sub-space $N(L)^\perp$, the orthogonal complement of $N(L)$ in H . Note that the augmentation L'' and the subspace $N(L')$ of Theorem 4.2 depend on the choice of the matrix-valued function $M(x)$, and it may be possible to choose $M(x)$ so that $N(L)^\perp \subset N(L')$. This is carried out in § 5 for perturbations of uniformly propagative systems. To see how it can be done recall that (cf. (1.27))

$$(4.18) \quad \overline{R(L^*)} = N(L)^\perp$$

where L^* is the adjoint of L on H . Thus, $N(L)^\perp \subset N(L')$ provided that

$$(4.19) \quad L'L^* = 0,$$

which implies that $\overline{E(L^*)} \subset N(L')$. In particular, if L is selfadjoint the condition becomes $L'L = 0$. In § 5,

$$(4.20) \quad \Lambda = E(x)^{-1}A \quad \text{where} \quad A = -i \sum_{j=1}^n A_j D_j$$

has constant coefficients. Define A' by the condition $A'(p)A(p) = 0$, as in Theorem 4.2, so that $A'A = 0$. Then defining Λ' by

$$(4.21) \quad \Lambda' = A'E(x)$$

gives $\Lambda'\Lambda = A'A = 0$ and $N(\Lambda)^\perp \subset N(\Lambda')$.

§ 5. - Coerciveness Inequalities for Perturbations of Uniformly Propagative System.

This section deals with operators of the form

$$(5.1) \quad \Lambda = iE(x)^{-1} \sum_{j=1}^n A_j D_j$$

which occur in wave propagation theories of classical physics. It is assumed, as in § 1, that $E(x)$, $A_1, \dots, A_n$ are HERMITIAN, the A_j are constant and $E(x)$ is bounded and uniformly positive definite (i.e., satisfies (1.5)). It is also assumed that Λ is homogeneous at infinity in the sense of the following

DEFINITION 5.1. - Λ is said to be homogeneous at infinity if and only if there exists a positive definite E_0 such that

$$(5.2) \quad \lim_{|x| \rightarrow \infty} E(x) = E_0, \quad \text{uniformly in } x/|x|.$$

The operator

$$(5.3) \quad \Lambda_0 = iE_0^{-1} \sum_{j=1}^n A_j D_j,$$

which has constant coefficients, is called the unperturbed operator and Λ is referred to as a perturbation of Λ_0 .

The purpose of this section is to prove that the operator Λ on H is coercive on the subspace $N(\Lambda)^\perp$ provided that the unperturbed system Λ_0 is

uniformly propagative. To begin, the definitions and properties of these systems are reviewed briefly. Proofs of the properties are contained in [6, 22, 23, 24] and are not repeated here.

Δ_0 has the symbol

$$(5.4) \quad \Delta_0(p) = E_0^{-1} A(p) \quad \text{where} \quad A(p) = \sum_{j=1}^n A_j p_j.$$

The characteristic polynomial of $\Delta_0(p)$ is

$$(5.5) \quad P(p, \lambda) = \det(\lambda I - \Delta_0(p))$$

and has a factorization

$$(5.6) \quad P(p, \lambda) = Q_1^{m_1}(p, \lambda) \dots Q_n^{m_n}(p, \lambda)$$

where the factors $Q_j(p, \lambda)$ are distinct homogeneous polynomials in (p, λ) , irreducible over the complex number field. The polynomial

$$(5.7) \quad Q(p, \lambda) = Q_1(p, \lambda) \dots Q_N(p, \lambda)$$

is known to be the minimal polynomial for $\Delta_0(p)$ (see [22, p. 56] and [6, p. 87]).

DEFINITION 5.2. - The operator Δ_0 is said to be uniformly propagative if and only if

- (1) The locus $S = \{p : Q(p, 1) = 0\}$ is bounded in R^n , and
- (2) $p \cdot \nabla_p Q(p, 1) = p_1 D_1 Q(p, 1) + \dots + p_n D_n Q(p, 1) \neq 0$ when $p \in S$.

$Q(p, \lambda)$ can be written as a polynomial in λ ⁽⁴⁾

$$(5.8) \quad Q(p, \lambda) = \lambda^l + Q_1(p) \lambda^{l-1} + \dots + Q_{l-1}(p) \lambda + Q_l(p).$$

The roots $\lambda_j(p)$ of $Q(p, \lambda) = 0$ are the propagation speeds of plane waves governed by the operator Δ_0 . Their behavior is described by

THEOREM 5.1. - For uniformly propagative systems

- (1) $Q(p, \lambda) = 0$ has l distinct real roots $\lambda_j(p)$ for each $p \in R^n - \{0\}$.
- (2) The roots can be enumerated to form l distinct branches $\lambda_j(p)$ which are analytic for all $p \in R^n - \{0\}$.
- (3) Each $\lambda_j(p)$ has constant algebraic sign for all $p \in R^n - \{0\}$. In particular, a root $\lambda_j(p)$ vanishes for some $p \in R^n - \{0\}$ if and only if it vanishes identically.

(4) Note that the coefficients $Q_j(p)$ in (5.8) are different from the factors $Q_j(p, \lambda)$ in (5.7).

COROLLARY 5.2. - The roots $\lambda_j(p)$ can be enumerated as follows:

Case 1. - $l = 2\rho$. The roots are indexed by the 2ρ indices $\{\pm 1, \pm 2, \dots, \pm \rho\}$ and

$$(5.9) \quad \lambda_\rho(p) > \lambda_{\rho-1}(p) > \dots > \lambda_1(p) > 0 > \lambda_{-1}(p) > \dots > \lambda_{-\rho}(p)$$

for all $p \in R^n - \{0\}$.

Case 2. - $l = 2\rho + 1$. The roots are indexed by the $2\rho + 1$ indices $\{0, \pm 1, \pm 2, \dots, \pm \rho\}$ and

$$(5.10) \quad \lambda_\rho(p) > \dots > \lambda_1(p) > 0 \equiv \lambda_0(p) > \lambda_{-1}(p) > \dots > \lambda_{-\rho}(p)$$

for all $p \in R^n - \{0\}$. Moreover, with this enumeration the roots satisfy the relations

$$(5.11) \quad \lambda_{-j}(p) = \lambda_j(-p) \quad \text{for } j = \pm 1, \dots, \pm \rho.$$

The roots $\lambda_j(p)$ induce an orthogonal direct sum decomposition of the HILBERT space H_0 which reduces the selfadjoint operator Λ_0 on H_0 associated with (5.3). The decomposition is defined as follows (see [24] for details).

$$(5.12) \quad (\lambda I - \Lambda_0(p))^{-1} = \sum_{|k|=\pi(l)}^{\rho} \widehat{P}_k(p) [\lambda - \lambda_k(p)]^{-1}$$

where $\pi(l) = 1$ if $l = 2\rho$ and $\pi(l) = 0$ if $l = 2\rho + 1$. The matrices $\widehat{P}_k(p)$ are homogeneous functions of degree zero which are analytic for $|p| = 1$ and satisfy $\sum_{|k|=\pi(l)}^{\rho} \widehat{P}_k(p) = I$, $\widehat{P}_j(p)\widehat{P}_k(p) = \delta_{jk}\widehat{P}_k(p)$, and $(E_0\widehat{P}_k(p))^* = E_0\widehat{P}_k(p)$. These properties imply that the $\widehat{P}_k(p)$ form a complete family of orthogonal projections in the unitary space $C_{E_0}^n$ with inner product $(\xi, \eta) = \xi^*E_0\eta$, and $\widehat{P}_k(p)$ is the projection onto the eigenspace of $\Lambda_0(p)$ for the eigenvalue $\lambda_k(p)$. Corresponding operators P_k on H_0 are defined by means of the FOURIER transform Φ defined by (1.19) above:

$$(5.13) \quad P_k = \Phi^* \widehat{P}_k \Phi.$$

The properties of the P_k are described by

LEMMA 5.3. - The operators P_k , $|k| = \pi(l)$, $\dots$, ρ , form a complete set of orthogonal projections on H_0 ; i.e.,

$$(5.14) \quad \sum_{|k|=\pi(l)}^{\rho} P_k = I,$$

$$(5.15) \quad P_j P_k = \delta_{jk} P_k \quad \text{for} \quad \pi(l) \leq |j|, |k| \leq \rho,$$

and

$$(5.16) \quad P_k^* = P_k \quad \text{for} \quad \pi(l) \leq |k| \leq \rho.$$

COROLLARY 5.4. - If $H_0^k = P_k H_0$, then

$$(5.17) \quad H_0 = \bigoplus_{|k| \leq \pi(l)}^{\rho} H_0^k.$$

Moreover Λ_0 commutes with the P_k and hence is reduced by the decomposition (5.17).

In Case 2 of Corollary 5.2 ($l = 2\rho + 1$) the projection P_0 exists and is the orthogonal projection onto $N(\Lambda_0)$, the null space of Λ_0 . P_0 plays an important part in the proof of the coerciveness inequality for perturbations of uniformly propagative systems. Since $\widehat{P}_0(p)$ is homogeneous of degree zero and $\widehat{P}_0(\eta)$ is analytic on the unit sphere $\Omega = \{\eta : |\eta| = 1\}$ it follows from results of A. CALDERON and A. ZYGMUND [8, p. 909] and S. G. MIKHLIN [16, p. 116] that $P_0 = \Phi^* \widehat{P}_0 \Phi$ is a singular integral operator of the form

$$(5.18) \quad P_0 = \kappa_0 + \tilde{P}_0$$

where κ_0 is the constant matrix operator

$$(5.19) \quad \kappa_0 = \Omega_n^{-1} \int_{\Omega} \widehat{P}_0(\eta) d\Omega_{\eta}$$

($d\Omega_{\eta}$ is the element of hyperarea on Ω and Ω_n is the hyperarea of Ω), and $\tilde{P}_0 : H_0 \rightarrow H_0$ is a singular integral operator of the form

$$(5.20) \quad \tilde{P}_0 f(x) = \int_{R^n} \Gamma_0(\eta) |x - y|^{-n} f(y) dy, \quad \eta = \frac{x - y}{|x - y|}.$$

The right side of (5.20) is a singular integral, defined as the limit in H_0 of

$$(5.21) \quad \int_{|x-y| \geq \varepsilon} \Gamma_0(\eta) |x - y|^{-n} f(y) dy.$$

$\Gamma_0(\eta)$ is a matrix-valued analytic function with mean zero on Ω :

$$(5.22) \quad \int_{\Omega} \Gamma_0(\eta) d\Omega_{\eta} = 0.$$

The defining equation for $\Gamma_0(\eta)$, (5.20), is equivalent to the condition that $|x|^{-n}\Gamma_0(x/|x|)$ is the inverse FOURIER transform of $\widehat{P}_0(p) - \kappa_0 I$ in the sense of distribution theory:

$$(5.23) \quad \frac{\Gamma_0(x/|x|)}{|x|^n} = \mathfrak{F}^* \widehat{P}_0 - \kappa_0 I(x).$$

A discussion of P_0 and its relation to the singularity of the resolvent kernel for Λ_0 may be found in [19].

The differential operators (5.1) and (5.3) determine unique selfadjoint extensions Λ and Λ_0 , on H and H_0 respectively, whose domains may be defined as in [22] or as in § 2 above (Theorem 2.6 and Corollary 2.7). The perturbed and unperturbed SCHRÖDINGER equations (1.12) corresponding to Λ and Λ_0 have static solutions if and only if zero is in the point spectrum of Λ and Λ_0 . This is true precisely in Case 2 of Corollary 5.2, when zero is a root of $Q(p, \lambda)$. The operators Λ and Λ_0 are nonelliptic if and only if zero is in their point spectra. The situation is described by

LEMMA 5.5. - Let Λ_0 be uniformly propagative. Then

Case 1. - Λ and Λ_0 are elliptic if and only if $\lambda = 0$ is not a root of $Q(p, \lambda)$;

Case 2. - Λ and Λ_0 are nonelliptic operators of constant deficit k in R^n if and only if $\lambda = 0$ is a root of $Q(p, \lambda)$ and k is the multiplicity of $\lambda = 0$ as a root of $P(p, \lambda)$.

PROOF. - In Case 1, $\lambda = 0$ is not a root of $Q(p, \lambda)$ or $P(p, \lambda)$. Hence $\Lambda_0(p)$ is nonsingular which implies that $\text{rank } \Lambda_0(p) = m$; i.e., Λ_0 is elliptic. Moreover, the symbol for Λ can be written

$$(5.24) \quad \Lambda(p, x) = E(x)^{-1} \sum_{j=1}^n A_j p_j = E(x)^{-1} E_0 \Lambda_0(p).$$

Thus, since $E(x)$ and E_0 are nonsingular, $\text{rank } \Lambda(p, x) = m$ and Λ is elliptic.

In Case 2, $\lambda = 0$ is a simple root of $Q(p, \lambda)$; i.e., $Q_j(p, \lambda) = \lambda$ is one of the irreducible factors of $P(p, \lambda)$. Thus $P(p, \lambda)$ has a factor λ^k where $k \geq 1$. Hence $\lambda = 0$ is an eigenvalue of $\Lambda_0(p)$ of multiplicity k . It is easy to verify that $\Lambda_0(p)$ is selfadjoint on the inner product space with product $(\xi, \eta) = \xi^* E_0 \eta$. Hence the nullspace of $\Lambda_0(p)$ has dimension k and it follows that

$$(5.25) \quad \text{rank } \Lambda(p, x) = \text{rank } \Lambda_0(p) = m - k;$$

i.e., Λ and Λ_0 are nonelliptic operators of constant deficit k in R^n .

Coerciveness inequalities for perturbations of uniformly propagative systems will now be derived, on the basis of the preliminary results given above. Two special cases are treated in Theorems 5.6 and 5.7. The general case is given in Theorem 5.8.

THEOREM 5.6. - Let Λ be an operator of the form (5.1) such that $E(x)$ and $D_j E(x)$, $j = 1, \dots, n$, are continuous and bounded on R^n and $E(x)$ satisfies (1.5). Assume that Λ is homogeneous at infinity with a limit operator Λ_0 which is uniformly propagative and does not have zero as a point eigenvalue. Then Λ is coercive on H .

PROOF. - Under the hypotheses of the theorem, Lemma 3.3 is applicable to Λ . Hence Λ is uniformly elliptic on R^n and it follows by Theorem 3.5 that Λ is coercive on H .

Next let $\Lambda_0 = -iE_0^{-1} \sum_{j=1}^n A_j D_j$ be a uniformly propagative operator with constant deficit $k > 0$. Then the operator

$$(5.26) \quad A = -i \sum_{j=1}^n A_j D_j$$

has constant deficit k . Let A' be the corresponding operator defined by Theorem 4.2 with $M(x) = I$, such that

$$(5.27) \quad A'' = \begin{pmatrix} A' \\ A \end{pmatrix}$$

is elliptic and (condition (4.11))

$$(5.28) \quad A'(p)A(p)^* = A'(p)A(p) = 0 \quad \text{for all } p \in R^n.$$

It follows that the operators A and A' satisfy

$$(5.29) \quad A'A = 0.$$

In the special case that A' is a first order operator, i.e.,

$$(5.30) \quad A'(p) = \sum_{j=1}^n A'_j p_j,$$

the coerciveness of Λ on $N(\Lambda)^\perp$ can be deduced from Theorem 4.4. To see this define an augmenting operator Λ' for Λ by

$$(5.31) \quad \Lambda' = A'E;$$

i.e.,

$$\begin{aligned} \Lambda' u(x) &= -i \sum_{j=1}^n A_j' D_j \{ E(x) u(x) \} \\ (5.32) \quad &= -i \sum_{j=1}^n A_j' E(x) D_j u(x) - i \sum_{j=1}^n A_j' (D_j E(x)) u(x). \end{aligned}$$

Note that

$$(5.33) \quad \Lambda'(p, x) = \sum_{j=1}^n A_j' E(x) p_j = A'(p) E(x)$$

and

$$(5.34) \quad \Lambda(p, x) = E(x)^{-1} A(p) \quad \text{whence} \quad \Lambda(p, x)^* = A(p) E(x)^{-1}.$$

Thus

$$(5.35) \quad \Lambda'(p, x) E(x)^{-1} \Lambda(p, x)^* = A'(p) A(p) E(x)^{-1} = 0.$$

Thus condition (4.11) holds with $M(x) = E(x)^{-1}$. Moreover,

$$(5.36) \quad \Lambda' \Lambda = A' E (E^{-1} A) = A' A = 0.$$

This leads to

THEOREM 5.7. - Let Λ be an operator of the form (5.1) such that $E(x)$ and $D_j E(x)$ are continuous and bounded on R^n and $E(x)$ satisfies (1.5). Assume that Λ is homogeneous at infinity with a limit operator Λ_0 which is uniformly propagative and has zero as a point eigenvalue, so that Λ and Λ_0 are non-elliptic operators of constant deficit k in R^n . Moreover, assume that there is a first order operator A' which satisfies (5.28). Then Λ is coercive on $N(\Lambda)^\perp$.

Note that this result is applicable to a number of wave propagation equations of classical physics, including MAXWELL'S equations and the equations of acoustics.

PROOF OF THEOREM 5.7. - The hypotheses imply that Λ satisfies the hypotheses of Theorem 4.4 with $M(x) = E(x)^{-1}$ and augmentation Λ' defined by (5.31). Hence, Λ is coercive on $N(\Lambda')$. Moreover (5.36) implies that $N(\Lambda)^\perp = \overline{R(\Lambda)} \subset N(\Lambda')$, whence Λ is coercive on $N(\Lambda)^\perp$.

The principle result in this paper is

THEOREM 5.8. - Let Λ be an operator of the form (5.1) such that $E(x)$ and $D_j E(x)$ are continuous and bounded on R^n and $E(x)$ satisfies (1.5). Assume that Λ is homogeneous at infinity with limiting operator Λ_0 which is uniformly propagative and has zero as a point eigenvalue, so that Λ and Λ_0 are nonelliptic operators of constant deficit k in R^n . Then Λ is coercive on $N(\Lambda)^\perp$.

The remainder of this section is devoted to the proof of Theorem 5.8. The scheme of the proof is the same as for Theorem 5.7. The proof is much more difficult than for Theorem 5.7 because it is impossible, in general, to find a first order differential operator Δ' such that Δ' is elliptic. Instead, a Δ' which is not a differential operator is used, as was done for the equations of elasticity in § 1. Consequently, Theorem 3.5 for elliptic differential operators cannot be used. Instead, the plan of the proof of Theorem 3.5 is carried out directly for the new operator Δ'' .

The norm in the SOBOLEV space $L_{2,m}^1$ was defined by (cf. (3.15))

$$(5.37) \quad \|u\|_1^2 = \|u\|^2 + \sum_{j=1}^n \|D_j u\|^2$$

where $\|u\|$ denotes the norm in $L_{2,m}$. In what follows it is convenient to use the semi-norm $|u|_1$ defined by

$$(5.38) \quad |u|_1^2 = \sum_{j=1}^n \|D_j u\|^2.$$

Note that the coerciveness inequality (1.8) can be written

$$(5.39) \quad |u|_1^2 \leq K^2(\|Lu\|^2 + \|u\|^2).$$

The construction of augmented operators for Δ_0 and Δ is modeled after the treatment of the equations of elasticity in § 1. To begin, the scalar operator $(-\Delta)^{1/2}$ is defined as follows.

DEFINITION 5.3. - $(-\Delta)^{1/2} : H_0 \rightarrow H_0$ is the operator with domain $D((-\Delta)^{1/2}) = H_0 \cap \{u : |p| |\widehat{u}(p)| \in H_0\}$ such that $((-\Delta)^{1/2} u)(p) = |p| \widehat{u}(p)$.

Note that $u \in D((-\Delta)^{1/2})$ if and only if $u \in H_0$ and $|p|^2 |\widehat{u}(p)|^2 = (p_1^2 + \dots + p_n^2) |\widehat{u}(p)|^2$ is integrable. It follows by the PLANCHEREL theory that

$$(5.40) \quad D((-\Delta)^{1/2}) = L_{2,m}^1 \text{ and } \|(-\Delta)^{1/2} u\|^2 = \sum_{j=1}^n \|D_j u\|^2$$

for all $u \in D((-\Delta)^{1/2})$.

DEFINITION 5.4. - $\Delta'_0 : H_0 \rightarrow H_0$ is the operator with domain

$$(5.41) \quad D(\Delta'_0) = H_0 \cap \{u : P_0 u \in L_{2,m}^1\}$$

such that

$$(5.42) \quad \Delta'_0 = (-\Delta)^{1/2} P_0.$$

DEFINITION 5.5. - $\Lambda' : H \rightarrow H$ is the operator with domain

$$(5.43) \quad D(\Lambda') = H \cap \{u : P_0 J^* u \in L_{2,m}^1\}$$

such that

$$(5.44) \quad \Lambda' = J \Lambda_0' J^*.$$

Note that (5.40) implies that if $u \in D(\Lambda_0)$ then $P_0 u \in D((-\Delta)^{1/2})$ so that $\Lambda_0' u$ is defined.

In (5.44), $J^* = E_0^{-1} J^{-1} E : H \rightarrow H_0$ is the adjoint of the identification map $J : H_0 \rightarrow H$ defined by (2.30).

DEFINITION 5.6. - The augmented operator $\Lambda'' : H \rightarrow L_{2,2m}$ is the operator with domain

$$(5.45) \quad D(\Lambda'') = D(\Lambda) \cap D(\Lambda')$$

such that

$$(5.46) \quad \Lambda'' u = \begin{pmatrix} \Lambda' u \\ \Lambda u \end{pmatrix}$$

for all $u \in D(\Lambda'')$.

The notation (5.46) means that $\Lambda'' u$ is a $2m \times 1$ matrix-valued function for which the first m components are those of $\Lambda' u \in H$ and the last m components are those of $\Lambda u \in H$. Since H is equivalent to $L_{2,m}$, $\Lambda'' u$ can clearly be identified with a vector in $L_{2,2m}$.

Theorem 5.8 will be proved by showing that Λ'' is Λ -coercive (Definition 4.2) and coincides with Λ on $N(\Lambda)^\perp$ in H . The second property follows from

LEMMA 5.9. - $N(\Lambda_0)^\perp \subset N(\Lambda_0')$ and $N(\Lambda)^\perp \subset N(\Lambda')$.

PROOF. - P_0 is the orthogonal projection onto $N(\Lambda_0)$ in H_0 .

Thus $N(\Lambda_0)^\perp = N(P_0) \subset N(-\Delta)^{1/2} P_0 = N(\Lambda_0')$. To prove the second relation note that (2.32) implies $\Delta J = E^{-1} J E_0 \Lambda_0$, whence $J N(\Lambda_0) = N(\Lambda)$. Moreover, $(u, Jv)_E = (J^* u, v)_{E_0}$ for all $u \in H$ and $v \in H_0$. Combining these facts gives

$$(5.47) \quad J^* N(\Lambda)^\perp = N(\Lambda_0)^\perp.$$

Thus $J^* N(\Lambda)^\perp = N(P_0)$, whence $N(\Lambda)^\perp \subset \{u : J^* u \in N(P_0)\} = \{u : P_0 J^* u = 0\} \subset N(\Lambda')$.

The symbols of the operators Λ_0 and Λ are $\Lambda_0(p) = E_0^{-1} A(p)$ and $\Lambda(p, x) = E(x)^{-1} A(p)$, respectively, where $A(p) = \sum_{j=1}^n A_j p_j$. The symbols of Λ_0' and Λ' are defined by

$$(5.48) \quad \Lambda_0'(p) = {}^t p \bar{P}_0(p)$$

and

$$(5.49) \quad \Lambda'(p, x) = \Lambda_0'(p)E_0^{-1}E(x),$$

in keeping with (5.42) and (5.44). Finally, the symbol of Λ'' is defined by

$$(5.50) \quad \Lambda''(p, x) = \begin{pmatrix} \Lambda'(p, x) \\ \Lambda(p, x) \end{pmatrix}.$$

Λ'' is not a differential operator. However, it may be said to be elliptic in the sense of the following lemma (cf. Lemma 3.1).

LEMMA 5.10. - If $x \in R^n$, $p \in R^n - \{0\}$ and $\xi \in C^m$ then

$$(5.51) \quad \Lambda''(p, x)\xi = 0 \quad \text{implies that} \quad \xi = 0.$$

PROOF. - $\Lambda''(p, x)\xi = 0$ implies that

$$(5.52) \quad \Lambda'(p, x)\xi = |p| \widehat{P}_0(p)E_0^{-1}E(x)\xi = 0$$

and

$$(5.53) \quad \Lambda(p, x)\xi = E(x)^{-1}E_0\Lambda_0(p)\xi = 0.$$

Now if $p \in R^n - \{0\}$, $\widehat{P}_0(p)$ is the orthogonal projection onto $N(\Lambda_0(p))$ in the unitary space $C_{E_0}^m$ with inner product $(\xi, \eta) = \xi^*E_0\eta$ (cf. Lemma 5.3 and [24]). Thus (5.52) implies that $E_0^{-1}E(x)\xi \in N(\widehat{P}_0(p)) = N(\Lambda_0(p))^\perp$ in $C_{E_0}^m$. Moreover, (5.53) implies that $\xi \in N(\Lambda(p))$. Thus

$$(5.54) \quad 0 = \xi^*E_0(E_0^{-1}E(x)\xi) = \xi^*E(x)\xi,$$

which implies that $\xi = 0$ because $E(x)$ is positive definite.

LEMMA 5.11. - Under the hypotheses of Theorem 5.8 there exists a constant $\mu > 0$ such that

$$(5.55) \quad |\Lambda''(p, x)u|^2 \geq \mu^2 |p|^2 |u|^2 \quad \text{for all} \\ p \in R^n, u \in C^m \quad \text{and} \quad x \in R^n.$$

PROOF. - Lemma 5.10 implies that for each $x_0 \in R^n$ there exists a constant $\mu(x_0)$ such that $|\Lambda''(p, x_0)u|^2 \geq \mu(x_0)^2 |p|^2 |u|^2$ for all $p \in R^n$ and $u \in C^m$. The proof is the same as that of Lemma 3.2. The existence of a constant $\mu > 0$, independent of x , such that (5.55) holds then follows as in the proof of Lemma 3.3, because $E(x) \rightarrow E_0$ when $|x| \rightarrow \infty$, uniformly in $x/|x|$.

It will now be shown that Λ'' is Λ -coercive under the hypotheses of Theorem 5.8. This will be done by a series of steps paralleling the proof of Theorem 3.5.

For any fixed $x_0 \in R^n$, Λ''_{x_0} is used to denote the operator with constant matrix coefficients obtained by making $x = x_0$ in the coefficients of Λ'' . The domain of Λ''_{x_0} is defined by means of the PLANCHEREL theory, as for Λ_0 and Λ'_0 .

LEMMA 5.12. - For each fixed $x_0 \in R^n$, Λ''_{x_0} is Λ_{x_0} -coercive and satisfies the inequality

$$(5.56) \quad \|\Lambda''_{x_0} u\|^2 \geq \mu^2 \|u\|_1^2$$

for all $u \in D(\Lambda''_{x_0})$ where μ is the constant of Lemma 5.11.

PROOF. - Λ''_{x_0} has the symbol $\Lambda''(p, x_0)$. Hence (5.56) follows from (5.55) with $x = x_0$ by the PLANCHEREL theory (cf. the proof of Lemma 3.6).

Next it is shown that Λ'' is coercive for all functions $u \in C_0^1(R^n; C^m)$ whose support is sufficiently small (the analogue of Lemma 3.7).

THEOREM 5.13. - Under the hypotheses of Theorem 5.8, there exist constant $\rho > 0$ and $\mu_0 > 0$, depending only on $E(x)$ and Λ_0 , such that

$$(5.57) \quad \|\varphi\|_1^2 \leq \mu_0^2 (\|\Lambda'' \varphi\|^2 + \|\varphi\|^2)$$

for all $\varphi \in C_0^1(R^n; C^m)$ such that $\text{supp } \varphi$ has diameter less than ρ .

PROOF. - It follows from (5.2) and the continuity of $E(x)$ that $E(x)$ is uniformly continuous on R^n . Define $c = \max\{1, \max_{\eta \in \Omega} |\lambda_{\pm\eta}(\eta)|^2\}$ and define ρ by the condition that

$$(5.58) \quad \begin{aligned} & \max\{2 \max_{|x-x_0| \leq \rho} |E_0^{-1}[E(x) - E(x_0)]|^2, \\ & c \max_{|x-x_0| \leq \rho} |[E(x)^{-1} - E(x_0)^{-1}]E_0|^2\} < \mu^2/8 \end{aligned}$$

where μ is the constant of Lemma 5.11. Let $\text{supp } \varphi \subset \{x : |x - x_0| \leq \rho\}$.

Then

$$(5.59) \quad \begin{aligned} \|\Lambda'' \varphi\|^2 &= \|\Lambda''_{x_0} \varphi + (\Lambda'' - \Lambda''_{x_0}) \varphi\|^2 \\ &\geq \frac{1}{2} \|\Lambda''_{x_0} \varphi\|^2 - \|(\Lambda'' - \Lambda''_{x_0}) \varphi\|^2 \\ &\geq (\mu^2/2) \|\varphi\|_1^2 - \|(\Lambda'' - \Lambda''_{x_0}) \varphi\|^2 \end{aligned}$$

by Lemma 5.13. Now

$$(5.60) \quad \|(\Lambda'' - \Lambda'_{x_0})\varphi\|^2 = \|(\Lambda' - \Lambda'_{x_0})\varphi\|^2 + \|(\Lambda - \Lambda_{x_0})\varphi\|^2$$

and it remains to estimate the two terms on the right. First, since $\Lambda = E^{-1}E_0\Lambda_0$,

$$(5.61) \quad \begin{aligned} \|(\Lambda - \Lambda_{x_0})\varphi\|^2 &= \|(E^{-1} - E_{x_0}^{-1})E_0\Lambda_0\varphi\|^2 \\ &\leq \max_{|x-x_0| \leq \rho} |[E(x)^{-1} - E(x_0)^{-1}]E_0|^2 \|\Lambda_0\varphi\|^2. \end{aligned}$$

Moreover, (cf. [24])

$$(5.62) \quad \begin{aligned} \|\Lambda_0\varphi\|^2 &= \sum_{|k|=1}^{\rho} \|\lambda_k(\cdot)\widehat{P}_k\widehat{\varphi}\|^2 \leq \| |\cdot| \widehat{P}_0 \widehat{\varphi} \|^2 + \sum_{|k|=1}^{\rho} \|\lambda_k(\cdot)\widehat{P}_k\widehat{\varphi}\|^2 \\ &\leq \max\{1, \max_{\eta \in \Omega} |\lambda_{\pm\rho}(\eta)|^2\} \sum_{|k|=0}^{\rho} \| |\cdot| \widehat{P}_k \widehat{\varphi} \|^2 = c \|\varphi\|_1^2. \end{aligned}$$

Combining (5.61) and (5.62) and using (5.48) gives

$$(5.63) \quad \|(\Lambda - \Lambda_{x_0})\varphi\|^2 \leq (\mu^2/8) \|\varphi\|_1^2.$$

Next, since $P_0(-\Delta)^{1/2} \subset (-\Delta)^{1/2}P_0$ and $\|(-\Delta)^{1/2}u\|^2 = \|u\|_1^2$ by (5.40),

$$(5.64) \quad \begin{aligned} \|(\Lambda' - \Lambda'_{x_0})\varphi\|^2 &= \|\Lambda'_0(E_0^{-1}E - E_0^{-1}E_{x_0})\varphi\|^2 \\ &= \|(-\Delta)^{1/2}P_0E_0^{-1}(E - E_{x_0})\varphi\|^2 = \|P_0(-\Delta)^{1/2}E_0^{-1}(E - E_{x_0})\varphi\|^2 \\ &\leq \|(-\Delta)^{1/2}E_0^{-1}(E - E_{x_0})\varphi\|^2 = \sum_{j=1}^n \|D_jE_0^{-1}(E - E_{x_0})\varphi\|^2 \\ &\leq 2\left[\sum_{j=1}^n \|E_0^{-1}(D_jE)\varphi\|^2 + \sum_{j=1}^n \|E_0^{-1}(E - E_{x_0})D_j\varphi\|^2\right] \\ &\leq 2n \max_{1 \leq j \leq n} \|E_0^{-1}D_jE\|_{\infty}^2 \|\varphi\|^2 \\ &\quad + 2 \max_{|x-x_0| \leq \rho} |E_0^{-1}(E(x) - E(x_0))|^2 \|\varphi\|_1^2 \\ &\leq \mu_1^2 \|\varphi\|^2 + (\mu^2/8) \|\varphi\|_1^2 \end{aligned}$$

by (5.58), where $\mu_1^2 = 2n \max_{1 \leq j \leq n} \|E_0^{-1}D_jE\|_{\infty}^2$ and $\|f\|_{\infty}$ denotes the maximum of $|f(x)|$ on R^n . Combining (5.59), (5.60), (5.63) and (5.64) gives

$$(5.65) \quad \|\Lambda'\varphi\|^2 \geq (\mu^2/4) \|\varphi\|_1^2 - \mu_1^2 \|\varphi\|^2.$$

Finally, if $\mu_0^2 = \max(4\mu^{-2}, 4\mu_1^2\mu^{-2})$ then (5.65) implies (5.57), which completes the proof.

The next step is to remove the restriction that $\text{supp } \varphi$ has diameter less than ρ . The result is stated as

THEOREM 5.14. - Under the hypotheses of Theorem 5.8, there exists a constant $\mu_1 > 0$, depending only on $E(x)$ and Δ_0 such that

$$(5.66) \quad \|\varphi\|_1^2 \leq \mu_1^2 (\|\Delta'' \varphi\|^2 + \|\varphi\|^2)$$

for all $\varphi \in C_0^1(R^n; C^m)$.

The proof makes use of the same partition of unity $\{\zeta_\alpha(x)\}$ used in the proof of Theorem 3.5 (see (3.35)-(3.38)). However, the proof is more complicated than for Theorem 3.5 because Δ' is not a differential operator. The following theorem is needed for the proof.

THEOREM 5.15. - Let $B = E_0^{-1}E$. Then, under the hypotheses of Theorem 5.8, there exists a constant $c_0 > 0$ such that

$$(5.67) \quad \left| \sum_{\alpha} \{ \|\zeta_{\alpha} P_0 B D_j \varphi\|^2 - \|P_0 \zeta_{\alpha} B D_j \varphi\|^2 \} \right| \leq c_0 \|\varphi\| \cdot \|D_j \varphi\|$$

for all $\varphi \in C_0^1(R^n; C^m)$ and $j = 1, 2, \dots, n$.

PROOF. - First note that

$$(5.68) \quad \begin{aligned} \sum_{\alpha} \|\zeta_{\alpha} P_0 B D_j \varphi\|^2 &= \sum_{\alpha} \int_{R^n} \zeta_{\alpha}^2 (P_0 B D_j \varphi)^* E_0 (P_0 B D_j \varphi) dx \\ &= \int_{R^n} \left(\sum_{\alpha} \zeta_{\alpha}^2 (P_0 B D_j \varphi)^* E_0 (P_0 B D_j \varphi) \right) dx = \|P_0 B D_j \varphi\|^2 < \infty \end{aligned}$$

where the interchange of summation and integration is justified by LEBESGUE'S dominated convergence theorem. The other sum on the left in (5.67) is finite because φ has compact support, so that $\zeta_{\alpha} B D_j \varphi$ is zero for all but a finite number of α . Now, since P_0 is an orthogonal projection on U_0 ,

$$\begin{aligned} & \sum_{\alpha} \{ \|\zeta_{\alpha} P_0 B D_j \varphi\|^2 - \|P_0 \zeta_{\alpha} B D_j \varphi\|^2 \} \\ &= \sum_{\alpha} (\zeta_{\alpha} P_0 B D_j \varphi, \zeta_{\alpha} P_0 B D_j \varphi)_{E_0} - (\zeta_{\alpha} B D_j \varphi, P_0 \zeta_{\alpha} B D_j \varphi)_{E_0} \\ &= \sum_{\alpha} (\zeta_{\alpha} (P_0 - I) B D_j \varphi, \zeta_{\alpha} P_0 B D_j \varphi)_{E_0} + (\zeta_{\alpha} B D_j \varphi, \zeta_{\alpha} P_0 B D_j \varphi - P_0 \zeta_{\alpha} B D_j \varphi)_{E_0} \\ &= \left(\sum_{\alpha} \zeta_{\alpha}^2 (P_0 - I) B D_j \varphi, P_0 B D_j \varphi \right)_{E_0} \end{aligned}$$

$$\begin{aligned}
& + \sum_{\alpha} (\zeta_{\alpha} B D_j \varphi, \zeta_{\alpha} P_0 B D_j \varphi - P_0 \zeta_{\alpha} B D_j \varphi)_{E_0} \\
& = (P_0 - I) B D_j \varphi, P_0 B D_j \varphi)_{E_0} + \sum_{\alpha} (\zeta_{\alpha} B D_j \varphi, \zeta_{\alpha} P_0 B D_j \varphi - P_0 \zeta_{\alpha} B D_j \varphi)_{E_0} \\
(5.69) \quad & = \sum_{\alpha} (\chi_{\alpha} B D_j \varphi, \zeta_{\alpha} [\zeta_{\alpha} P_0 B D_j \varphi - P_0 \zeta_{\alpha} B D_j \varphi])_{E_0}.
\end{aligned}$$

The first term in the next to the last expression is zero because $P_0^*(P_0 - I) = P_0(P_0 - I) = 0$. χ_{α} is used to denote the characteristic function of $\text{supp } \zeta_{\alpha} \subset Q_{\alpha}$. Now $P_0 = \kappa_0 + \tilde{P}_0$ (see (5.18)) where $\tilde{P}_0$ is a singular integral operator (5.20). When this is substituted in the last term of (5.69) the terms involving κ_0 drop out because the scalar operator ζ_{α} commutes with κ_0 . Thus, if $\Pi_{\alpha j}$ is defined by

$$(5.70) \quad \Pi_{\alpha j} \varphi = \zeta_{\alpha} \tilde{P}_0 B D_j \varphi - \tilde{P}_0 \zeta_{\alpha} B D_j \varphi$$

then (5.20) implies that

$$(5.71) \quad \Pi_{\alpha j} \varphi(x) = \int_{R^n} \frac{\Gamma_0(\eta)}{|x - y|^n} [\zeta_{\alpha}(x) - \zeta_{\alpha}(y)] B(y) D_j \varphi(y) dy$$

and, from (5.69),

$$\begin{aligned}
& \left| \sum_{\alpha} \left(\|\zeta_{\alpha} P_0 B D_j \varphi\|^2 - \|P_0 \zeta_{\alpha} B D_j \varphi\|^2 \right) \right| \\
& = \left| \sum_{\alpha} (\chi_{\alpha} B D_j \varphi, \zeta_{\alpha} [\zeta_{\alpha} \tilde{P}_0 B D_j \varphi - \tilde{P}_0 \zeta_{\alpha} B D_j \varphi])_{E_0} \right| \\
& = \left| \sum_{\alpha} (\chi_{\alpha} B D_j \varphi, \zeta_{\alpha} \Pi_{\alpha j} \varphi)_{E_0} \right| \\
& \leq \int_{R^n} \sum_{\alpha} |(\chi_{\alpha} B D_j \varphi)^* E_0| |\zeta_{\alpha} \Pi_{\alpha j} \varphi| dx \\
(5.72) \quad & \leq \int_{R^n} \left\{ \sum_{\alpha} |(\chi_{\alpha} B D_j \varphi)^* E_0|^2 \right\}^{1/2} \left\{ \sum_{\alpha} |\zeta_{\alpha} \Pi_{\alpha j} \varphi|^2 \right\}^{1/2} dx \\
& \leq \left\{ \int_{R^n} \sum_{\alpha} |\chi_{\alpha} (B D_j \varphi)^* E_0|^2 dx \right\}^{1/2} \left\{ \int_{R^n} \sum_{\alpha} |\zeta_{\alpha} \Pi_{\alpha j} \varphi|^2 dx \right\}^{1/2} \\
& = \left\{ \sum_{\alpha} \int_{R^n} \chi_{\alpha} |(B D_j \varphi)^* E_0|^2 dx \right\}^{1/2} \left\{ \sum_{\alpha} \int_{R^n} |\zeta_{\alpha} \Pi_{\alpha j} \varphi|^2 dx \right\}^{1/2} \\
& = \left\{ 2^n \int_{R^n} |(B D_j \varphi)^* E_0|^2 dx \right\}^{1/2} \left\{ \sum_{\alpha} \int_{R^n} |\zeta_{\alpha} \Pi_{\alpha j} \varphi|^2 dx \right\}^{1/2} \\
& \leq c_1 \|D_j \varphi\| \left\{ \sum_{\alpha} \int_{R^n} |\zeta_{\alpha} \Pi_{\alpha j} \varphi|^2 dx \right\}^{1/2}
\end{aligned}$$

where c_1 depends on n and bounds for E and E_0 only. Thus (5.67) will be proved if there exists a constant $c_2 > 0$ such that

$$(5.73) \quad \sum_{\alpha} \int_{R^n} |\zeta_{\alpha} \Pi_{\alpha j} \varphi|^2 dx \leq c_2^2 \|\varphi\|^2$$

for all $\varphi \in C_0^1(R^n; C^m)$.

To this end a new representation of $\Pi_{\alpha j}$ as a singular integral will be obtained. Consider the integral

$$(5.74) \quad \int_{|x-y|>\varepsilon} \Gamma_0(\eta) |x-y|^{-n} [\zeta_{\alpha}(x) - \zeta_{\alpha}(y)] B(y) D_j \varphi(y) dy$$

which converges in H_0 to $\Pi_{\alpha j} \varphi(x)$ when $\varepsilon \rightarrow 0$. Note that if (5.74) is broken into two integrals containing $\zeta_{\alpha}(x)$ and $\zeta_{\alpha}(y)$ then each converges in H_0 and also pointwise almost everywhere to one of the terms on the right in (5.70) when $\varepsilon \rightarrow 0$ [7, p. 290]. Moreover, (5.74) actually converges pointwise because the integrand is $O(r^{-n+1})$ for $r = |x-y| \rightarrow 0$. Thus the pointwise limit of (5.74) is equal a.e. to $\Pi_{\alpha j} \varphi(x)$.

Integration by parts in (5.74) gives

$$\begin{aligned} & \int_{r>\varepsilon} \Gamma_0(\eta) r^{-n} [\zeta_{\alpha}(x) - \zeta_{\alpha}(y)] B(y) D_j \varphi(y) dy \\ &= \int_{r>\varepsilon} (D_{x_j} \Gamma_0(\eta) r^{-n}) [\zeta_{\alpha}(x) - \zeta_{\alpha}(y)] B(y) \varphi(y) dy \\ & - \zeta_{\alpha}(x) \int_{r>\varepsilon} \Gamma_0(\eta) r^{-n} [D_j B(y)] \varphi(y) dy \\ & + \int_{r>\varepsilon} \Gamma_0(\eta) r^{-n} [D_j \zeta_{\alpha}(y) B(y)] \varphi(y) dy \\ & + \int_{r=\varepsilon} \eta \Gamma_0(\eta) r^{-n} [\zeta_{\alpha}(x) - \zeta_{\alpha}(y)] B(y) \varphi(y) dS_y \\ & \equiv I_1^{\varepsilon}(x) + I_2^{\varepsilon}(x) + I_3^{\varepsilon}(x) + I_4^{\varepsilon}(x) \end{aligned} \quad (5.75)$$

in an obvious notation. Now $I_2^{\varepsilon}(x)$ and $I_3^{\varepsilon}(x)$ converge in H_0 and pointwise a.e. [7]. It will be shown that $I_1^{\varepsilon}(x)$ and $I_4^{\varepsilon}(x)$ converge pointwise, from which it will follow that the right-hand side of (5.75) converges to $\Pi_{\alpha j}^{\varepsilon} \varphi(x)$ a.e. (I_1^{ε} and I_4^{ε} also converge in H_0 , but this fact is not needed here). First it is shown that the pointwise limit of $I_1^{\varepsilon}(x)$ exists. This is the content of the next two lemmas.

LEMMA 5.16. - For $k = 1, 2, \dots, n$

$$(5.76) \quad \int_{\Omega} r^{n+1} \eta_k (D_{x_j} \Gamma_0(\eta) r^{-n}) d\Omega_{\eta} = 0.$$

PROOF. - Number the coordinates so that $x_j = x_1$ and introduce spherical coordinates as follows: $y_1 = x_1 + r \cos \theta_1$, $y_2 = x_2 + r \sin \theta_1 \cos \theta_2$, ..., $y_{n-1} = x_{n-1} + r \sin \theta_1 \dots \sin \theta_{n-2} \cos \theta_{n-1}$, $y_n = x_n + r \sin \theta_1 \dots \sin \theta_{n-2} \sin \theta_{n-1}$ where $-\pi \leq \theta_{n-1} \leq \pi$ and $0 \leq \theta_k \leq \pi$ for $k = 1, 2, \dots, n-2$. Also note that $d\Omega = \sin^{n-2} \theta_1 \dots \sin \theta_{n-2} d\theta_1 \dots d\theta_{n-1}$. Only r and θ_1 depend on x_1 and $D_{x_1} r = -\cos \theta_1$, $D_{x_1} \theta_1 = r^{-1} \sin \theta_1$. Hence

$$(5.77) \quad r^{n+1} D_{x_1} (\Gamma_0(\eta) r^{-n}) = n \cos \theta_1 \Gamma_0(\eta) + \sin \theta_1 D_{\theta_1} \Gamma_0(\eta).$$

Assume first that η_k does not contain $\sin \theta_1$. Then $\eta_k = \eta_1 = \cos \theta_1$ and

$$(5.78) \quad r^{n+1} \eta_k D_{x_1} (\Gamma_0(\eta) r^{-n}) = \cos \theta_1 [n \cos \theta_1 \Gamma_0(\eta) + \sin \theta_1 D_{\theta_1} \Gamma_0(\eta)].$$

Now a short calculation gives

$$(5.79) \quad \begin{aligned} & D_{\theta_1} [\sin^{n-1} \theta_1 \cos \theta_1 \Gamma_0(\eta)] \\ &= \cos \theta_1 [n \cos \theta_1 \Gamma_0(\eta) + \sin \theta_1 D_{\theta_1} \Gamma_0(\eta)] \sin^{n-2} \theta_1 \\ &- \sin^{n-2} \theta_1 \Gamma_0(\eta). \end{aligned}$$

Equations (5.78) and (5.79) imply

$$(5.80) \quad \begin{aligned} & \int_{\Omega} r^{n+1} \eta_1 D_{x_1} (\Gamma_0(\eta) r^{-n}) d\Omega_{\eta} \\ &= \int_{-\pi}^{\pi} d\theta_{n-1} \int_0^{\pi} \sin \theta_{n-2} d\theta_{n-2} \dots \int_0^{\pi} \sin^{n-3} \theta_2 d\theta_2 \\ &\cdot \int_0^{\pi} \sin^{n-2} \theta_1 \{ \cos \theta_1 [n \cos \theta_1 \Gamma_0(\eta) + \sin \theta_1 D_{\theta_1} \Gamma_0(\eta)] \} d\theta_1 \\ &= \int_{-\pi}^{\pi} d\theta_{n-1} \dots \int_0^{\pi} \{ D_{\theta_1} [\sin^{n-1} \theta_1 \cos \theta_1 \Gamma_0(\eta)] + \sin^{n-2} \theta_1 \Gamma_0(\eta) \} d\theta_1 \\ &= 0 + \int_{\Omega} \Gamma_0(\eta) d\Omega_{\eta} = 0, \end{aligned}$$

because $\Gamma_0(\eta)$ has mean zero on Ω . This proves (5.76) for $\eta_k = \eta_1$.

If $k \neq 1$ then η_k contains a factor $\sin \theta_1$ and the result follows easily. Suppose, for example, that $k = 3$. Then $\eta_3 = \sin \theta_1 \sin \theta_2 \cos \theta_3$ and

$$\begin{aligned} & r^{n+1} \eta_k D_{x_1}(\Gamma_0(\eta) r^{-n}) \\ (5.81) \quad & = \cos \theta_3 \sin \theta_2 \sin \theta_1 [n \cos \theta_1 \Gamma_0(\eta) + \sin \theta_1 D_{\theta_1} \Gamma_0(\eta)]. \end{aligned}$$

But

$$\begin{aligned} & \sin \theta_1 [n \cos \theta_1 \Gamma_0(\eta) + \sin \theta_1 D_{\theta_1} \Gamma_0(\eta)] \sin^{n-2} \theta_1 \\ (5.82) \quad & = D_{\theta_1} [\sin^n \theta_1 \Gamma_0(\eta)] \end{aligned}$$

whence

$$\begin{aligned} & \int_{\Omega} r^{n+1} \eta_k D_{x_1}(\Gamma_0(\eta) r^{-n}) d\Omega_{\eta} \\ (5.83) \quad & = \int_{-\pi}^{\pi} d\theta_{n-1} \dots \int_0^{\pi} \cos \theta_3 \sin \theta_2 D_{\theta_1} [\sin^n \theta_1 \Gamma_0(\eta)] d\theta_1 = 0. \end{aligned}$$

It is clear that a similar proof can be given for any $k \neq 1$.

LEMMA 5.17. - Let $\psi(y) = B(y)\varphi(y)$. Then the singular integral

$$(5.84) \quad \int_{R^n} r D_{x_j}(\Gamma_0(\eta) r^{-n}) \nabla \zeta_{\alpha}(x) \cdot \eta \psi(y) dy$$

exists pointwise for $x \in R^n$.

PROOF.

$$\begin{aligned} & \int_{\varepsilon < r < 1} r D_{x_j}(\Gamma_0(\eta) r^{-n}) \nabla \zeta_{\alpha}(x) \cdot \eta \psi(y) dy \\ & = \left\{ \int_{\Omega} \int_{\varepsilon}^1 [r^{n+1} D_{x_j}(\Gamma_0(\eta) r^{-n}) \nabla \zeta_{\alpha}(x) \cdot \eta] r^{-1} dr d\Omega_{\eta} \right\} \psi(x) \\ (5.85) \quad & + \int_{\varepsilon < r < 1} r D_{x_j}(\Gamma_0(\eta) r^{-n}) \nabla \zeta_{\alpha}(x) \cdot \eta [\psi(y) - \psi(x)] dy \\ & = \log \varepsilon^{-1} \left\{ \sum_{k=1}^n D_k \zeta_{\alpha}(x) \int_{\Omega} r^{n+1} \eta_k D_{x_j}(\Gamma_0(\eta) r^{-n}) d\Omega_{\eta} \right\} \psi(x) \\ & + \int_{\varepsilon < r < 1} r D_{x_j}(\Gamma_0(\eta) r^{-n}) \nabla \zeta_{\alpha}(x) \cdot \eta [\psi(y) - \psi(x)] dy. \end{aligned}$$

Now the coefficient of $\log \varepsilon^{-1}$ is zero by Lemma 5.16. Moreover, the integrand of the last integral is $O(r^{-n+1})$ for $r \rightarrow 0$. This completes the proof of Lemma 5.17.

COROLLARY 5.18. - The singular integral

$$(5.86) \quad \int_{R^n} (D_{x_j} \Gamma_0(\eta) r^{-n}) [\zeta_\alpha(x) - \zeta_\alpha(y)] B(y) \varphi(y) dy$$

exists as a pointwise limit.

This is an immediate consequence of Lemma 5.17 and the TAYLOR expansion of $\zeta_\alpha(y)$ about the point x . Next it will be shown that the pointwise limit of $I_4^\varepsilon(x)$ exists.

LEMMA 5.19. - The pointwise limit of $I_4^\varepsilon(x)$ exists and

$$(5.87) \quad \lim_{\varepsilon \rightarrow 0} I_4^\varepsilon(x) = - \left(\sum_{k=1}^n D_k \zeta_\alpha(x) \beta_{kj} \right) B(x) \varphi(x)$$

where

$$(5.88) \quad \beta_{kj} = \int_{\Omega} \eta_k \eta_j \Gamma_0(\eta) d\Omega_\eta.$$

PROOF.

$$(5.89) \quad \begin{aligned} I_4^\varepsilon(x) &= \int_{r=\varepsilon} \eta_j \Gamma_0(\eta) r^{-n} [\zeta_\alpha(x) - \zeta_\alpha(y)] dS_y B(x) \varphi(x) \\ &+ \int_{r=\varepsilon} \eta_j \Gamma_0(\eta) r^{-n} [\zeta_\alpha(x) - \zeta_\alpha(y)] [B(y) \varphi(y) - B(x) \varphi(x)] dS_y. \end{aligned}$$

The last term tends to zero with ε . Moreover, if the TAYLOR series for $\zeta_\alpha(y)$ about x is inserted in the first term on the right in (5.89) the result is

$$(5.90) \quad \begin{aligned} &- \int_{r=\varepsilon} \eta_j \Gamma_0(\eta) r^{-n} \nabla \zeta_\alpha(x) \cdot r \eta dS_y B(x) \varphi(x) \\ &- \int_{r=\varepsilon} \eta_j \Gamma_0(\eta) r^{-n+2} \left[\sum_{i,j=1}^n D_i D_j \zeta_\alpha(z_{x,y}) \right] \eta_i \eta_j dS_y B(x) \varphi(x), \end{aligned}$$

where $z_{x,y}$ lies on the segment joining x and y . The last term tends to zero with ε and the first term equals the right-hand side of (5.87), which proves the lemma.

Combining (5.71), (5.75) with Corollary 5.18 and Lemma 5.19 gives

COROLLARY 5.20 - The operator $\Pi_{\alpha j}$ has the representation

$$\begin{aligned}
 \Pi_{\alpha j} \varphi(x) = & \int_{R^n} (D_{x_j} \Gamma_0(\eta) r^{-n}) [\zeta_\alpha(x) - \zeta_\alpha(y)] B(y) \varphi(y) dy \\
 & - \zeta_\alpha(x) \int_{R^n} \Gamma_0(\eta) r^{-n} [D_j B(y)] \varphi(y) dy \\
 & + \int_{R^n} \Gamma_0(\eta) r^{-n} [D_j \zeta_\alpha(y) B(y)] \varphi(y) dy \\
 & - \left(\sum_{k=1}^n D_k \zeta_\alpha(x) \beta_{kj} \right) B(x) \varphi(x)
 \end{aligned}
 \tag{5.91}$$

for almost every $x \in R^n$ if $\varphi \in C_0^1(R^n; C^m)$.

Corollary 5.20 is now used to prove inequality (5.73). For this purpose a partition of unity $\{\zeta_\alpha\}$ is used which is the same as the partition $\{\zeta_\alpha\}$ of Theorem 3.5 but with cubes of twice the size ($d \rightarrow 2d$). Thus $\xi_\alpha = \xi'_\alpha [\sum_\beta \xi'_\beta]^2]^{-1/2}$ where $\xi'_\beta(x) = \xi'(x - d\beta)$ and $\xi'(x) \in C_0^\infty(R^n)$, $0 \leq \xi'(x) \leq 1$, and $\text{supp } \xi' \subset 2Q_0$ where

$$2Q_\alpha = \{x : d(\alpha_k - 2) < x_k < d(\alpha_k + 2), k = 1, 2, \dots, n\}.
 \tag{5.92}$$

$\xi'(x)$ is chosen so that

$$\xi'(x) = \begin{cases} 1 & \text{for } -\frac{3}{2}d < x_k < \frac{3}{2}d, k = 1, \dots, n, \text{ and} \\ 0 & \text{for } x \notin 2Q_0. \end{cases}
 \tag{5.93}$$

Clearly, $\text{supp } \xi_\alpha \subset 2Q_\alpha$. Moreover, apart from a null set every $x \in R^n$ lies in precisely 4^n of the cubes $2Q_\alpha$. Also, note that $\zeta_\alpha(y)[1 - \xi'_\alpha(y)] = 0$ for all y and $\zeta_\alpha(x)[1 - \xi'_\alpha(y)] = 0$ unless $|x - y| > d/2$. Finally, the derivatives $D^r \zeta_\alpha$ are bounded by a fixed constant for all α .

Returning to Corollary 5.20, the function ξ'_α can be used to represent $\Pi_{\alpha j} \varphi$ as follows:

$$\begin{aligned}
 \Pi_{\alpha j} \varphi(x) = & \int_{R^n} (D_{x_j} \Gamma_0(\eta) r^{-n}) [\zeta_\alpha(x) - \zeta_\alpha(y)] \xi'_\alpha(y) B(y) \varphi(y) dy \\
 & + \int_{R^n} (D_{x_j} \Gamma_0(\eta) r^{-n}) [\zeta_\alpha(x) - \zeta_\alpha(y)] (1 - \xi'_\alpha(y)) B(y) \varphi(y) dy
 \end{aligned}$$

$$\begin{aligned}
 (5.94) \quad & - \zeta_{\alpha'}(x) \int_{R^n} \Gamma_0(\eta) r^{-n} [D_j B(y)] \varphi(y) dy \\
 & + \int_{R^n} \Gamma_0(\eta) r^{-n} [D_j \zeta_{\alpha}(y) B(y)] \varphi(y) dy - \left(\sum_{k=1}^n D_k \zeta_{\alpha}(x) \beta_{kj} \right) B(x) \varphi(x).
 \end{aligned}$$

Using the TAYLOR series for $\zeta_{\alpha}(y)$ about x and $\zeta_{\alpha}(y)[1 - \xi'(y)] = 0$ gives

$$\begin{aligned}
 (5.95) \quad & \Pi_{\alpha j} \varphi(x) = - \sum_{k=1}^n D_k \zeta_{\alpha}(x) \int_{R^n} r \eta_k (D_{x_j} \Gamma_0(\eta) r^{-n}) \xi'_{\alpha}(y) B(y) \varphi(y) dy \\
 & - 2^{-1} \int_{R^n} (D_{x_j} \Gamma_0(\eta) r^{-n}) r^2 \left[\sum_{k=1}^n D_i D_k \zeta_{\alpha}(z_{x,y}) \eta_i \eta_k \right] \xi'_{\alpha}(y) B(y) \varphi(y) dy \\
 & + \zeta_{\alpha}(x) \int_{R^n} (D_{x_j} \Gamma_0(\eta) r^{-n}) (1 - \zeta'_{\alpha}(y)) B(y) \varphi(y) dy \\
 & - \zeta_{\alpha}(x) \int_{R^n} \Gamma_0(\eta) r^{-n} [D_j B(y)] \varphi(y) dy \\
 & + \int_{R^n} \Gamma_0(\eta) r^{-n} [D_j \zeta_{\alpha}(y) B(y)] \varphi(y) dy \\
 & - \left(\sum_{k=1}^n D_k \zeta_{\alpha}(x) \beta_{kj} \right) B(x) \varphi(x) \\
 & = \sum_{k=1}^n S_{j\alpha}^k \psi(x) + W_{j\alpha} \psi(x) + U_{j\alpha} \psi(x) + V_{j\alpha}^1 \varphi(x) + V_{j\alpha}^2 \varphi(x) + Z_{j\alpha} \varphi(x)
 \end{aligned}$$

where $\psi = B\varphi$ and the operators $S_{j\alpha}^k, \dots, Z_{j\alpha}$ are defined by terms of the sum (5.95) in an obvious way. The left-hand side of inequality (5.73) can now be written

$$\begin{aligned}
 & \sum_{\alpha} \int_{R^n} |\zeta_{\alpha} \Pi_{\alpha j} \varphi|^2 dx \\
 & = \sum_{\alpha} \int_{R^n} |\zeta_{\alpha}(x) [\sum_{k=1}^n S_{j\alpha}^k \psi(x) + W_{j\alpha} \psi(x) + U_{j\alpha} \psi(x) + V_{j\alpha}^1 \varphi(x) \\
 & \quad + V_{j\alpha}^2 \varphi(x) + Z_{j\alpha} \psi(x)]|^2 dx \\
 & \leq 6 \sum_{\alpha} \int_{R^n} \left\{ |\zeta_{\alpha}(x) \sum_{k=1}^n S_{j\alpha}^k \psi(x)|^2 + |\zeta_{\alpha}(x) W_{j\alpha} \psi(x)|^2 \right.
 \end{aligned}$$

$$\begin{aligned}
 & + |\zeta_\alpha(x) U_{j\alpha} \psi(x)|^2 + |\zeta_\alpha(x) V_{j\alpha}^1 \varphi(x)|^2 + |\zeta_\alpha(x) V_{j\alpha}^2 \varphi(x)|^2 \\
 & + |\zeta_\alpha(x) Z_{j\alpha} \psi(x)|^2 \Big\} dx \\
 (5.96) \quad & \leq 6n \sum_{k=1}^n \sum_{\alpha} \int_{R^n} |\zeta_\alpha(x) S_{j\alpha}^k \psi(x)|^2 \\
 & + 6 \sum_{\alpha} \int_{R^n} \Big\{ |\zeta_\alpha(x) W_{j\alpha} \psi(x)|^2 + |\zeta_\alpha(x) U_{j\alpha} \psi(x)|^2 + |\zeta_\alpha(x) V_{j\alpha}^1 \varphi(x)|^2 \\
 & + |\zeta_\alpha(x) V_{j\alpha}^2 \varphi(x)|^2 + |\zeta_\alpha(x) Z_{j\alpha} \psi(x)|^2 \Big\} dx.
 \end{aligned}$$

It will now be shown that each term on the right-hand side of this inequality is bounded by a constant times $\|\varphi\|^2$. This will complete the proof of inequality (5.73) and Theorem 5.15.

LEMMA 5.21. - There exists a constant $c_2 > 0$ such that

$$(5.97) \quad \sum_{\alpha} \int_{R^n} |\zeta_\alpha(x) S_{j\alpha}^k \psi(x)|^2 dx \leq c_2 \|\varphi\|^2$$

for all $\psi \in C_0^1(R^n; C^n)$.

PROOF. - Define $\mu_\alpha = \xi'_\alpha \psi$ so that (see (5.95)) $S_{j\alpha}^k \psi(x) = -D_k \zeta_\alpha(x) S_j^k \mu_\alpha(x)$ where

$$(5.98) \quad S_j^k \mu_\alpha(x) = \int_{R^n} r \eta_k D_{x_j} \Gamma_0(\eta) r^{-n} \mu_\alpha(y) dy.$$

Then $|S_{j\alpha}^k \psi(x)|^2 \leq K |S_j^k \mu_\alpha(x)|^2$, where K is a constant, because the derivatives of ζ_α are bounded, uniformly in α . Moreover, S_j^k is a bounded operator on H_0 (cf. [8,16]), and therefore

$$\begin{aligned}
 & \sum_{\alpha} \int_{R^n} |\zeta_\alpha(x) S_{j\alpha}^k \psi(x)|^2 dx \leq K \sum_{\alpha} \int_{R^n} \zeta_\alpha^2(x) |S_j^k \mu_\alpha(x)|^2 dx \\
 & \leq K \sum_{\alpha} \int_{Q_\alpha} |S_j^k \mu_\alpha(x)|^2 dx \leq K \sum_{\alpha} \|S_j^k \mu_\alpha\|^2 \\
 & \leq K \sum_{\alpha} \|S_j^k\|^2 \|\mu_\alpha\|^2 = K \|S_j^k\|^2 \sum_{\alpha} \int_{R^n} [\xi'_\alpha(y)]^2 |\psi(y)|^2 dy
 \end{aligned}$$

$$(5.99) \quad \leq K \|S_j^k\|^2 \sum_{\alpha} \int_{\alpha} |\phi(x)|^2 \leq 4^n K \|S_j^k\|^2 \|\phi\|^2$$

which proves the lemma.

LEMMA 5.22. - There exists a constant $c_3 > 0$ such that

$$(5.100) \quad \sum_{\alpha} \int_{R^n} |\zeta_{\alpha}(x) W_{j\alpha} \phi(x)|^2 dx \leq c_3 \|\phi\|^2$$

for all $\phi \in C_0^1(R^n; C^m)$.

PROOF. - The kernel of the integral defining $W_{j\alpha}$ is $O(r^{1-n})$ for $r = |x - y| \rightarrow 0$; see (5.95). Hence

$$(5.101) \quad \begin{aligned} |\zeta_{\alpha}(x) W_{j\alpha} \phi(x)| &\leq K_1 \zeta_{\alpha}(x) \int_{R^n} |\xi'_{\alpha}(y) \phi(y)| r^{1-n} dy \\ &\leq K_1 \zeta_{\alpha}(x) \int_{2Q_{\alpha}} |\phi(y)| r^{1-n} dy. \end{aligned}$$

Hence

$$(5.102) \quad \begin{aligned} |\zeta_{\alpha}(x) W_{j\alpha} \phi(x)|^2 &\leq K_1^2 \zeta_{\alpha}^2(x) \int_{2Q_{\alpha}} r^{1-n} dy \int_{2Q_{\alpha}} r^{1-n} |\phi(y)|^2 dy \\ &\leq K_1^2 2\sqrt{n\bar{d}} \Omega_n \zeta_{\alpha}^2(x) \int_{2Q_{\alpha}} r^{1-n} dy \int_{2Q_{\alpha}} r^{1-n} |\phi(y)|^2 dy \end{aligned}$$

where $2\sqrt{n\bar{d}}$ is the radius of circumscribed sphere about $2Q_{\alpha}$. Thus

$$(5.103) \quad \begin{aligned} \sum_{\alpha} \int_{R^n} |\zeta_{\alpha}(x) W_{j\alpha} \phi(x)|^2 dx &\leq K_2 \sum_{\alpha} \int_{R^n} \zeta_{\alpha}^2(x) dx \int_{2Q_{\alpha}} r^{1-n} |\phi(y)|^2 dy \\ &\leq K_2 \sum_{\alpha} \int_{Q_{\alpha}} dx \int_{2Q_{\alpha}} r^{1-n} |\phi(x)|^2 dy = K_2 \sum_{\alpha} \int_{2Q_{\alpha}} |\phi(x)|^2 dy \int_{Q_{\alpha}} r^{1-n} dx \\ &\leq K_2 \sqrt{n\bar{d}} \Omega_n \sum_{\alpha} \int_{2Q_{\alpha}} |\phi(y)|^2 = K_2 \sqrt{n\bar{d}} \Omega_n 4^n \|\phi\|^2 \end{aligned}$$

which proves (5.100).

LEMMA 5.23. - There exists a constant $C_4 > 0$ such that

$$(5.104) \quad \sum_{\alpha} \int_{R^n} |\zeta_{\alpha}(x) U_{j\alpha} \psi(x)|^2 dx \leq C_4 \|\psi\|^2$$

for all $\psi \in C_0^1(R^n; C^m)$.

PROOF. - The kernel $D_{x_j} \Gamma_0(\eta) r^{-n} = O(r^{-n-1})$ for $r \rightarrow 0$. Thus

$$(5.105) \quad |U_{j\alpha} \psi(x)| \leq K_3 \zeta_{\alpha}(x) \int_{R^n} r^{-n-1} [1 - \xi'_{\alpha}(y)] |\psi(y)| dy.$$

Moreover, $\zeta_{\alpha}(x)[1 - \xi'_{\alpha}(y)] = 0$ unless $|x - y| > d/2$. Hence for fixed x the integral extends over the set $\{y : r = |x - y| > d/2\}$. Hence, by the SCHWARZ inequality,

$$\begin{aligned} |U\psi(x)|^2 &\leq K_3^2 \zeta_{\alpha}^2(x) \int_{r>d/2} r^{-n-1} dy \int_{r>d/2} [1 - \xi'_{\alpha}(y)]^2 |\psi(y)|^2 r^{-n-1} dy \\ &\leq K_3^2 2d^{-1} \Omega_n \zeta_{\alpha}^2(x) \int_{r>d/2} [1 - \xi'_{\alpha}(y)]^2 |\psi(y)|^2 r^{-n-1} dy \\ (5.106) \quad &\leq K_4 \int_{r>d/2} [1 - \xi'_{\alpha}(y)]^2 |\psi(y)|^2 r^{-n-1} dy. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{\alpha} \int_{R^n} |\zeta_{\alpha}(x) U_{j\alpha} \psi(x)|^2 dx \\ &\leq K_4 \sum_{\alpha} \int_{R^n} dx \int_{r>d/2} \zeta_{\alpha}^2(x) [1 - \xi'_{\alpha}(y)]^2 |\psi(y)|^2 r^{-n-1} dy \\ (5.107) \quad &\leq K_4 \int_{R^n} |\psi(y)|^2 dy \int_{r>d/2} \sum_{\alpha} \zeta_{\alpha}^2(x) r^{-n-1} dx \\ &= K_4 2d^{-1} \Omega \int_{R^n} |\psi(y)|^2 dy = K_4 2d^{-1} \Omega_n \|\psi\|^2, \end{aligned}$$

which proves (5.104).

LEMMA 5.24. - There exists a constant $c_5 > 0$ such that

$$(5.108) \quad \sum_{\alpha} \int_{R^n} |\zeta_{\alpha}(x) V_{j\alpha}^i \varphi(x)|^2 dx \leq c_5 \|\varphi\|^2$$

for all $\varphi \in C_0^1(R^n; C^m)$ and $i = 1, 2$.

PROOF. - The proof will be given for $i = 1$; the case $i = 2$ is essentially the same. Let

$$(5.109) \quad V\mu(x) = \int_{R^n} \Gamma_0(\eta) r^{-n} \mu(y) dy.$$

Then V is bounded on H_0 . Moreover, if $\mu_{j\alpha} = (D_j \zeta_{\alpha} B) \varphi$

$$(5.110) \quad V_{j\alpha}^1 \varphi(x) = V\mu_{j\alpha}(x),$$

whence

$$\begin{aligned} \sum_{\alpha} \int_{R^n} |\zeta_{\alpha}(x) V_{j\alpha}^1 \varphi(x)|^2 dx &= \sum_{\alpha} \int_{R^n} \zeta_{\alpha}^2(x) |V\mu_{j\alpha}(x)|^2 dx \\ &\leq \sum_{\alpha} \int_{R^n} |V\mu_{j\alpha}(x)|^2 dx \leq \sum_{\alpha} \|V\mu_{j\alpha}\|^2 \\ (5.111) \quad &\leq \|V\|^2 \sum_{\alpha} \|\mu_{j\alpha}\|^2 = \|V\|^2 \sum_{\alpha} \int_{R^n} |(D_j \zeta_{\alpha}(y) B(y)) \varphi(y)|^2 dy \\ &\leq 2 \|V\|^2 \left\{ \sum_{\alpha} \int_{R^n} [D_j \zeta_{\alpha}(y)]^2 |B(y) \varphi(y)|^2 dy \right. \\ &\quad \left. + \sum_{\alpha} \int_{R^n} \zeta_{\alpha}^2(y) |(D_j B(y)) \varphi(y)|^2 dy \right\} \\ &\leq 2 \|V\|^2 \left\{ K_5 \sum_{\alpha} \int_{Q_{\alpha}} |B(y) \varphi(y)|^2 dy + \|(D_j B) \varphi\|^2 \right\} \\ &\leq 2 \|V\|^2, 2^n K_5 \|B \varphi\|^2 + \|(D_j B) \varphi\|^2 \leq C_5 \|\varphi\|^2; \end{aligned}$$

i.e., (5.108) holds.

LEMMA 5.25. - There exists a constant $c_6 > 0$ such that

$$(5.112) \quad \sum_{\alpha} \int_{R^n} |\zeta_{\alpha}(x) Z_{j\alpha} \psi(x)|^2 dx \leq c_6 \|\psi\|^2$$

for all $\psi \in C_0^1(R^n; C^m)$.

PROOF. - Clearly

$$(5.113) \quad |Z_{j\alpha} \psi(x)|^2 \leq K_6 |\psi(x)|^2$$

whence

$$(5.114) \quad \sum_{\alpha} \int_{R^n} |\zeta_{\alpha}(x) Z_{j\alpha} \psi(x)|^2 dx \leq K_6 \int_{R^n} |\psi(x)|^2 dx = K_6 \|\psi\|^2.$$

Lemma 5.21-5.25, with (5.96), show that inequality (5.73) holds. This completes the proof of Theorem 5.15.

Returning now to the coerciveness inequality for Λ'' , the proof of Theorem 5.14 will be based on the following two lemmas. The first depends on Theorem 5.15.

LEMMA 5.26. - There exist positive constants α_1 and α_2 such that for all $\varphi \in C_0^1(R^n; C^m)$,

$$(5.115) \quad \|\Lambda' \varphi\|^2 \geq 4^{-1} \sum_{\alpha} \|\Lambda' \zeta_{\alpha} \varphi\|^2 - \alpha_1 \|\varphi\|^2 - \alpha_2 \|\varphi\| \cdot \|\varphi\|_1.$$

PROOF - Definitions 5.4 and 5.5 and (5.40) imply

$$(5.116) \quad \|\Lambda' \varphi\|^2 = \|\Lambda'_0 B \varphi\|^2 = \|(-\Delta)^{1/2} P_0 B \varphi\|^2 = \sum_{j=1}^n \|D_j P_0 B \varphi\|^2.$$

Now $B\varphi = E_0^{-1} E\varphi$ has first derivatives in H_0 , whence $D_j P_0 B \varphi = P_0 D_j B \varphi$. Thus (5.116) implies

$$(5.117) \quad \begin{aligned} \|\Lambda' \varphi\|^2 &= \sum_{j=1}^n \|P_0 D_j B \varphi\|^2 = \sum_{j=1}^n \|P_0 (D_j B) \varphi + P_0 B (D_j \varphi)\|^2 \\ &\geq 2^{-1} \sum_{j=1}^n \|P_0 B D_j \varphi\|^2 - \sum_{j=1}^n \|P_0 (D_j B) \varphi\|^2 \\ &\geq 2^{-1} \sum_{j=1}^n \|P_0 B D_j \varphi\|^2 - c_1 \|\varphi\|^2 \end{aligned}$$

where c_1 is a constant which depend on E and E_0 . Moreover, it is easy to verify that $\|P_0 B D \varphi\|^2 = \sum_{\alpha} \|\zeta_{\alpha} P_0 B D_j \varphi\|^2$. Thus Theorem 5.15 implies

$$\begin{aligned}
 \|\Lambda' \varphi\|^2 &\geq 2^{-1} \sum_{j=1}^n \left(\|P_0 \zeta_{\alpha} B D_j \varphi\|^2 - c_2 \sum_{j=1}^n \|\varphi\| \cdot \|D_j \varphi\| \right. \\
 &\quad \left. - c_1 \|\varphi\|^2 \right) \\
 (5.118) \quad &\geq 2^{-1} \sum_{j=1}^n \sum_{\alpha} \|P_0 \zeta_{\alpha} B D \varphi\|^2 - c_3 \|\varphi\| \cdot \|\varphi\|_1 - c_1 \|\varphi\|^2.
 \end{aligned}$$

Now

$$\begin{aligned}
 \|P_0 B D_j(\zeta_{\alpha} \varphi)\|^2 &= \|P_0 B(D_j \zeta_{\alpha}) \varphi + P_0 B \zeta_{\alpha}^{\sim} D_j \varphi\|^2 \\
 &= \|P_0 B(D_j \zeta_{\alpha}) \varphi\|^2 + \|P_0 \zeta_{\alpha} B D_j \varphi\|^2 \\
 (5.119) \quad &+ 2 \operatorname{Re}(B(D_j \zeta_{\alpha}) \varphi, P_0 B \zeta_{\alpha}^{\sim} D_j \varphi)_{E_0}.
 \end{aligned}$$

Hence, (5.118) can be rewritten

$$\begin{aligned}
 \|\Lambda' \varphi\|^2 &\geq 2^{-1} \sum_{j=1}^n \sum_{\alpha} \left(\|P_0 B D_j(\zeta_{\alpha} \varphi)\|^2 - \|P_0 B(D_j \zeta_{\alpha}) \varphi\|^2 \right. \\
 (5.120) \quad &\quad \left. - 2 \operatorname{Re}(B(D_j \zeta_{\alpha}) \varphi, P_0 B \zeta_{\alpha}^{\sim} D_j \varphi)_{E_0} \right) - c_3 \|\varphi\| \cdot \|\varphi\|_1 - c_1 \|\varphi\|^2.
 \end{aligned}$$

The second and third terms in braces are estimated next. First,

$$\begin{aligned}
 \|P_0 B(D_j \zeta_{\alpha}) \varphi\|^2 &\leq \|B(D_j \zeta_{\alpha}) \varphi\|^2 \leq \|B\|_{\infty}^2 \|(D_j \zeta_{\alpha}) \varphi\|^2 \\
 (5.121) \quad &\leq K_1 \|B\|_{\infty}^2 \int_Q |\varphi(x)|^2 dx
 \end{aligned}$$

whence

$$\begin{aligned}
 &2^{-1} \sum_{j=1}^n \sum_{\alpha} \|P_0 B(D_j \zeta_{\alpha}) \varphi\|^2 \\
 (5.122) \quad &\leq 2^{-1} n 2^n K_1 \|B\|_{\infty}^2 \|\varphi\|^2 = c_4 \|\varphi\|^2.
 \end{aligned}$$

Next,

$$\begin{aligned}
 |(B(D_j \zeta_{\alpha}) \varphi, P_0 B \zeta_{\alpha}^{\sim} D_j \varphi)_{E_0}| &\leq \|B(D_j \zeta_{\alpha}) \varphi\| \cdot \|P_0 B \zeta_{\alpha}^{\sim} D_j \varphi\| \\
 (5.123) \quad &\leq K_1 \|B\|_{\infty}^2 \left\{ \int_{Q_{\alpha}} |\varphi(x)|^2 dx \right\}^{1/2} \left\{ \int_{Q_{\alpha}} |D_j \varphi(x)|^2 dx \right\}^{1/2}
 \end{aligned}$$

whence

$$\begin{aligned}
 & \sum_{\alpha} |(B(D_j \zeta_{\alpha}) \varphi, P_0 B \zeta_{\alpha} D_j \varphi)_{E_0}| \\
 (5.124) \quad & \leq K_1 \|B\|_{\infty}^2 \sum_{\alpha} \left\{ \int_{Q_{\alpha}} |\varphi(x)|^2 dx \right\}^{1/2} \left\{ \int_{Q_{\alpha}} |D_j \varphi(x)|^2 dx \right\}^{1/2} \\
 & \leq K_1 \|B\|_{\infty}^2 \left\{ \sum_{\alpha} \int_{Q_{\alpha}} |\varphi|^2 dx \right\}^{1/2} \left\{ \sum_{\alpha} \int_{Q_{\alpha}} |D_j \varphi|^2 dx \right\}^{1/2} \\
 & = K_1 2^n \|B\|_{\infty}^2 \|\varphi\| \cdot \|D_j \varphi\| = c_5 \|\varphi\| \cdot \|D_j \varphi\|.
 \end{aligned}$$

Thus

$$\begin{aligned}
 & \sum_{j=1}^n \sum_{\alpha} \operatorname{Re}(B(D_j \zeta_{\alpha}) \varphi, P_0 B \zeta_{\alpha} D_j \varphi)_{E_0} \leq c_5 \|\varphi\| \sum_{j=1}^n \|D_j \varphi\| \\
 (5.125) \quad & \leq c_5 n \|\varphi\| \cdot \|\varphi\|_1 = c_6 \|\varphi\| \cdot \|\varphi\|_1.
 \end{aligned}$$

Inequalities (5.120), (5.122) and (5.125) imply

$$\begin{aligned}
 & \|\Lambda' \varphi\|^2 \\
 (5.126) \quad & \geq 2^{-1} \sum_{j=1}^n \sum_{\alpha} \|P_0 B D_j(\zeta_{\alpha} \varphi)\|^2 - c_7 \|\varphi\|^2 - c_8 \|\varphi\| \cdot \|\varphi\|_1.
 \end{aligned}$$

Now (5.116) implies that

$$\begin{aligned}
 \|\Lambda' \zeta_{\alpha} \varphi\|^2 &= \sum_{j=1}^n \|P_0 D_j(B \zeta_{\alpha} \varphi)\|^2 \\
 &= \sum_{j=1}^n \|P_0 B D_j(\zeta_{\alpha} \varphi) + P_0(D_j B) \zeta_{\alpha} \varphi\|^2 \\
 (5.127) \quad & \leq 2 \sum_{j=1}^n (\|P_0 B D_j(\zeta_{\alpha} \varphi)\|^2 + \|P_0(D_j B) \zeta_{\alpha} \varphi\|^2) \\
 & \leq 2 \sum_{j=1}^n \|P_0 B D_j(\zeta_{\alpha} \varphi)\|^2 + 2K_2 \sum_{j=1}^n \int_{Q_{\alpha}} |\varphi(x)|^2 dx
 \end{aligned}$$

whence

$$\sum_{\alpha} \|\Lambda' \zeta_{\alpha} \varphi\|^2 \leq 2 \sum_{j=1}^n \sum_{\alpha} \|P_0 B D_j(\zeta_{\alpha} \varphi)\|^2 + 2K_2 n 2^n \|\varphi\|^2.$$

Combining (5.126) and (5.128) gives

$$(5.129) \quad \|\Lambda' \varphi\|^2 \geq 4^{-1} \sum \|\Lambda' \zeta_\alpha \varphi\|^2 - a_1 \|\varphi\|^2 - a_2 \|\varphi\| \cdot \|\varphi\|_1.$$

LEMMA 5.27. - There exist positive constants a_3 and a_4 such that for all $\varphi \in C_0^1(R^n; C^m)$,

$$(5.130) \quad \|\Lambda \varphi\|^2 \geq \sum_\alpha \|\Lambda \zeta_\alpha \varphi\|^2 - a_3 \|\varphi\|^2 - a_4 \|\varphi\| \cdot \|\varphi\|_1.$$

PROOF. - The partition of unity $\{\zeta_\alpha\}$ gives

$$(5.131) \quad \begin{aligned} \|\Lambda \varphi\|^2 &= \sum_\alpha \|\zeta_\alpha \Lambda \varphi\|^2 \\ &= \sum_\alpha \|\Lambda \zeta_\alpha \varphi\|^2 - \left(\sum_\alpha \|(\Lambda \zeta_\alpha) \varphi\|^2 + 2 \operatorname{Re}(\zeta_\alpha \Lambda \varphi, (\Lambda \zeta_\alpha) \varphi)_{E_0} \right). \end{aligned}$$

The following estimates hold for the terms in braces

$$(5.132) \quad \sum_\alpha \|(\Lambda \zeta_\alpha) \varphi\|^2 \leq \sum_\alpha \int_{Q_\alpha} |(\Lambda \zeta_\alpha) \varphi|^2 dy \leq K^2 \sum_\alpha \int_{Q_\alpha} |\varphi|^2 dx = K_2 2^n \int_{R^n} |\varphi|^2 dx.$$

Also

$$(5.133) \quad \begin{aligned} &|(\zeta_\alpha \varphi, (\Lambda \zeta_\alpha) \varphi)_{E_0}| \\ &\leq c_1 \int_{Q_\alpha} |\Lambda_0 \varphi| \cdot |\varphi| dx \leq c_1 \left\{ \int_{Q_\alpha} |\Lambda_0 \varphi|^2 dx \right\}^{1/2} \left\{ \int_{Q_\alpha} |\varphi|^2 dx \right\}^{1/2}. \end{aligned}$$

Moreover, $\|\Lambda_0 \varphi\| \leq c_2 \|\varphi\|_1$. Thus

$$(5.134) \quad \begin{aligned} &\sum_\alpha 2 \operatorname{Re}(\zeta_\alpha \Lambda \varphi, (\Lambda \zeta_\alpha) \varphi)_{E_0} \\ &\leq c_1 \sum_\alpha \left\{ \int_{Q_\alpha} |\Lambda_0 \varphi|^2 dx \right\}^{1/2} \left\{ \int_{Q_\alpha} |\varphi|^2 dx \right\}^{1/2} \\ &\leq c_1 \left\{ \sum_\alpha \int_{Q_\alpha} |\Lambda \varphi|^2 dx \right\}^{1/2} \left\{ \sum_\alpha \int_{Q_\alpha} |\varphi|^2 dx \right\}^{1/2} \leq 2^n c_1 \|\Lambda_0 \varphi\| \cdot \|\varphi\| \\ &\leq 2^n c_1 c_2 \|\varphi\| \cdot \|\varphi\|_1 \equiv a_5 \|\varphi\| \cdot \|\varphi\|_1. \end{aligned}$$

(5.130) follows from (5.131), (5.132) and (5.134).

Lemma 5.26 and 5.27 imply

COROLLARY 5.28. - There exist positive constants c_1 and c_2 such that

$$(5.135) \quad \|\Lambda''\varphi\|^2 \geq 4^{-1} \sum_{\alpha} \|\Lambda''\zeta_{\alpha}\varphi\|^2 - c_1\|\varphi\|^2 - c_2\|\varphi\| \cdot |\varphi|_1.$$

PROOF OF THEOREM 5.14. - Let ρ be defined as in Theorem 5.13 and construct the partition of unity $\{\zeta_{\alpha}\}$ such that the width $2d$ of the cubes satisfies $2d\sqrt{n} < \rho$. Then Theorem 5.13 implies

$$(5.136) \quad |\zeta_{\alpha}\varphi|_1^2 \leq \mu_0^2 (\|\Lambda''\zeta_{\alpha}\varphi\|^2 + \|\zeta_{\alpha}\varphi\|^2)$$

for all $\varphi \in C_0^1(R^n; C^m)$ and all α . Hence

$$(5.137) \quad \sum_{\alpha} |\zeta_{\alpha}\varphi|_1^2 \leq \mu_0^2 \left(\sum_{\alpha} \|\Lambda''\zeta_{\alpha}\varphi\|^2 + \|\varphi\|^2 \right).$$

Now

$$(5.138) \quad \begin{aligned} |\zeta_{\alpha}\varphi|_1^2 &= \sum_{j=1}^n \|D_j\zeta_{\alpha}\varphi\|^2 \\ &\geq 2^{-1} \sum_{j=1}^n \|\zeta_{\alpha}(D_j\varphi)\|^2 - \sum_{j=1}^n \|(D_j\zeta_{\alpha})\varphi\|^2 \end{aligned}$$

whence

$$(5.139) \quad \begin{aligned} \sum_{\alpha} |\zeta_{\alpha}\varphi|_1^2 &\geq 2^{-1} \sum_{j=1}^n \|D_j\varphi\|^2 - K_1\|\varphi\|^2 = 2^{-1} |\varphi|_1^2 \\ &\quad - K_1\|\varphi\|^2 \end{aligned}$$

where K_1 is independent of φ . Combining (5.135), (5.137) and (5.139) gives

$$(5.140) \quad \begin{aligned} \|\Lambda''\varphi\|^2 &\geq 4^{-1} \sum_{\alpha} \|\Lambda''\zeta_{\alpha}\varphi\|^2 - c_1\|\varphi\|^2 - c_2\|\varphi\| \cdot |\varphi|_1 \\ &\geq 4^{-1}\mu_0^{-2} \sum_{\alpha} |\zeta_{\alpha}\varphi|_1^2 - c_3\|\varphi\|^2 - c_2\|\varphi\| \cdot |\varphi|_1 \\ &\geq 8^{-1}\mu_0^{-2} |\varphi|_1^2 - c_4\|\varphi\|^2 - c_2\|\varphi\| \cdot |\varphi|_1. \end{aligned}$$

Now $2\|\varphi\| \cdot |\varphi|_1 \leq \varepsilon |\varphi|_1^2 + \varepsilon^{-1}\|\varphi\|^2$ for any $\varepsilon > 0$. Thus

$$(5.141) \quad \|\Lambda''\varphi\|^2 \geq (8^{-1}\mu_0^{-2} - 2^{-1}c_2\varepsilon) |\varphi|_1^2 - (c_4 + 2^{-1}c_2\varepsilon^{-1})\|\varphi\|^2.$$

Choosing $\varepsilon = 8^{-1}c_2^{-1}\mu_0^{-2}$ gives an inequality equivalent to (5.66).

This completes the proof of Theorem 5.14.

The next step in proving the coerciveness of Λ'' is to prove

THEOREM 5.29. - Under the hypotheses of Theorem 5.8,

$$(5.142) \quad \|u\|_1^2 \leq \mu_1^2 (\|\Lambda'' u\|^2 + \|u\|^2)$$

for all $u \in D(\Lambda'') \cap C^1(R^n; C^m)$, where μ_1 is the constant of Theorem 5.14.

The proof proceeds via several lemmas. The purpose of the next two lemmas is to show that inequality (5.142) holds for all $u \in L_{2,m}^1 \cap C^1$. To this end let $\varphi(t) \in C^\infty(R)$, $0 \leq \varphi(t) \leq 1$, $\varphi(t) \equiv 0$ for $t > 1$ and $\varphi(t) \equiv 1$ for $t < 0$. Define $\varphi_R(x) = \varphi(|x| - R) \in C_0^\infty(R^n)$ for $(R > 0)$.

LEMMA 5.30. - If $u \in D(\Lambda'') \cap C^1(R^n; C^m)$ then $\varphi_R u \in D(\Lambda'')$.

PROOF. - Since $u \in C^1(R^n; C^m)$ and $E(x)$ is continuously differentiable, $J^* \varphi_R u = E_0^{-1} E \varphi_R u \in C_0^1(R^n; C^m) \subset L_{2,m}^1$. Hence $P_0 J^* \varphi_R u \in L_{2,m}^1$ and therefore $\varphi_R u \in D(\Lambda')$. Moreover it is easy to verify, using Lemma 2.5, that $u \in D(\Lambda)$ implies $\varphi_R u \in D(\Lambda)$ and $\Lambda \varphi_R u = \varphi_R \Lambda u + (\Lambda \varphi_R) u$. Hence $\varphi_R u \in D(\Lambda') \cap D(\Lambda) = D(\Lambda'')$.

LEMMA 5.31. - Under the hypotheses of Theorem 5.8, inequality (5.142) holds for all $u \in L_{2,m}^1 \cap C^1(R^n; C^m)$.

PROOF. - If φ_R is defined as in Lemma 5.30 then $\varphi_R u \in C_0^1(R^n; C^m)$ and hence, by Theorem 5.14

$$(5.143) \quad \|\varphi_R u\|_1^2 \leq \mu_1^2 (\|\Lambda'' \varphi_R u\|^2 + \|\varphi_R u\|^2).$$

Now it is easy to verify that $\varphi_R u \rightarrow u$ and $D_j(\varphi_R u) \rightarrow D_j u$ in H .

Also

$$\begin{aligned} \|\Lambda' \varphi_R u - \Lambda' u\|^2 &= \sum_{j=1}^n \|D_j P_0 J^*(\varphi_R u - u)\|^2 = \\ (5.144) \quad &= \sum_{j=1}^n \|P_0 D_j J^*(\varphi_R u - u)\| \leq \sum_{j=1}^n \|D_j J^*(\varphi_R u - u)\|^2 \\ &\leq 2 \sum_{j=1}^n (\|D_j E_0^{-1} E\|_\infty^2 \|\varphi_R u - u\|^2 + \|E_0^{-1} E\|_\infty^2 \|D_j(\varphi_R u - u)\|^2) \end{aligned}$$

which tends to zero for $R \rightarrow \infty$. Similarly, $\Lambda \varphi_R u \rightarrow \Lambda u$ in H . Hence, inequality (5.142) can be obtained by making $R \rightarrow \infty$ in (5.143).

In order to prove Theorem 5.29 an inequality is needed whose proof is of comparable difficulty with that of Theorem 5.15. The next lemma serves to motivate the inequality.

LEMMA 5.32. - Let $u \in D(\Lambda'') \cap C^1(R^n; C^m)$ and suppose that there exists a positive constant $M = M(u)$ such that

$$(5.145) \quad \|\Lambda' \varphi_R u\| \leq M$$

for all $R > 0$. Then inequality (5.142) holds for u where μ_1 , the constant of Theorem 5.14, does not depend on u .

PROOF. - It will be shown that (5.145) implies $u \in L_{2,m}^1$. The result then follows from Lemma 5.31. Now $\varphi_R u \in C_0^1(R^n; C^m)$ and hence (5.143) holds. Moreover, $\varphi_R u$ and $\Lambda \varphi_R u$ are bounded in H , uniformly for $R > 0$. Hence $\|\varphi_R u\|_1$ is bounded for $R > 0$. Now

$$(5.146) \quad \begin{aligned} \|\varphi_R D_j u\| &= \|D_j(\varphi_R u) - (D_j \varphi_R)u\| \\ &\leq \|D_j(\varphi_R u)\| + \|(D_j \varphi_R)u\| \leq \|\varphi_R u\|_1 + c\|u\| \end{aligned}$$

where c is independent of R . Thus $\lim_{R \rightarrow \infty} \|\varphi_R D_j u\| = \|D_j u\| < \infty$ ($j = 1, \dots, n$) and therefore $u \in L_{2,m}^1$.

Theorem 5.29 is an immediate consequence of Lemma 5.32 and

THEOREM 5.33. - Under the hypotheses of Theorem 5.8, if $u \in D(\Lambda'') \cap C^1(R^n; C^m)$ then there exists a constant $M = M(u)$ such that

$$(5.147) \quad \|D_j P_0 J^* \varphi_R u\| \leq M$$

for all $R > 0$.

PROOF OF THEOREM 5.29 BASED ON THEOREM 5.33 - (5.40) and (5.44) imply that

$$(5.148) \quad \|\Lambda \varphi_R u\|^2 = \|(\cdot - \Lambda)^{1/2} P_0 J^* \varphi_R u\|^2 = \sum_{j=1}^n \|D_j P_0 J^* \varphi_R u\|^2.$$

Thus (5.147) implies (5.145) and hence (5.142).

PROOF OF THEOREM 5.33. - Put $\psi = J^* u$ and note that $D_j P_0 J^* \varphi_R u = D_j P_0 \varphi_R J^* u = D_j P_0 \varphi_R \psi$. This can be written, using the decomposition $P_0 = \kappa_0 + \tilde{P}_0$ of (5.18), as follows.

$$(5.149) \quad \begin{aligned} D_j P_0 \varphi_R \psi &= D_j P_0 \varphi_R \psi - D_j \varphi_R P_0 \psi + D_j \varphi_R P_0 \psi \\ &= D_j \tilde{P}_0 \varphi_R \psi - D_j \varphi_R \tilde{P}_0 \psi + (D_j \varphi_R) P_0 \psi + \varphi_R D_j P_0 \psi. \end{aligned}$$

Now $u \in D(\Lambda')$ implies $P_0 J^* u = P_0 \psi \in L_{2,m}^1$ (Definition 5.5). Hence the last two terms in (5.149) are bounded in H_0 uniformly with respect to R .

Thus, defining

$$(5.150) \quad f_R = \tilde{P}_0 \varphi_R \psi - \varphi_R \tilde{P}_0 \psi,$$

it remains to show that $D_j f_R$ is bounded in H_0 uniformly with respect to R . This will be shown to follow from the next two lemmas.

LEMMA 5.34. - Let $v \in C_0(R^n; C^m)$ and define

$$(5.151) \quad \begin{aligned} g_R(x) &= \tilde{P}_0 \varphi_R v(x) - \varphi_R(x) \tilde{P}_0 v(x) \\ &= \int_{R^n} \Gamma_0(\eta) r^{-n} [\varphi_R(y) - \varphi_R(x)] v(y) dy \end{aligned}$$

Then

$$(5.152) \quad \begin{aligned} D_j g_R(x) &= \int_{R^n} (D_{x_j} \Gamma_0(\eta) r^{-n}) [\varphi_R(y) - \varphi_R(x)] v(y) dy \\ &\quad - \int_{R^n} \Gamma_0(\eta) r^{-n} [D_j \varphi_R(x)] v(y) dy - \left(\sum_{k=1}^n D_k \varphi_R(x) \beta_{jk} \right) v(x) \end{aligned}$$

where

$$(5.153) \quad \beta_{jk} = \int_{\Omega} \eta_j \eta_k \Gamma_0(\eta) \Omega_{\eta}.$$

PROOF. - The integrand in (5.151) has a singularity of order $O(r^{-n+1})$ for $R \rightarrow 0$. Hence (5.152) is just the formula for the differentiation of an integral with a weak singularity (cf. [18, Ch. III], [16, p. 59]). The proof will not be repeated here. Note that the existence of the first integral on the right in (5.152) was proved in Corollary 5.18. (5.152) has nearly the same form as the representation of $\Pi_{\sigma j}$ in Corollary 5.20.

LEMMA 5.35. - Let $D_j g_R$ be given by (5.152) for $v \in C_0^1(R^n; C^m)$, then there is a constant $N > 0$, independent of R and v , such that

$$(5.154) \quad \|D_j g_R\| \leq N \|v\|$$

for all $v \in C_0^1(R^n; C^m)$.

PROOF. - Note that $g_R = \tilde{P}_0 \varphi_R v - \varphi_R \tilde{P}_0 v = P_0 \varphi_R v - \varphi_R P_0 v \in L_{2,m}^1$ so that $D_j g_R \in H_0$. Therefore

$$(5.155) \quad \|D_j g_R\|^2 = \sum_{\alpha} \|\zeta_{\alpha} D_j g_R\|^2.$$

Inequality (5.154) now follows from (5.152) in exactly the same way that (5.73) follows from (5.91) and Lemma 5.21-5.25. If these lemmas are applied to (5.152) the constant N will be seen to depend on bounds for φ_R and $D_j\varphi_R$. Since these are uniformly bounded in R , N can be taken to be independent of R . This proves Lemma 5.35.

PROOF OF THEOREM 5.33 (*completed*). - To complete the proof it must be shown that $D_j f_R$ is bounded in H_0 , uniformly in R , where f_R is defined by (5.150). If $u \in C_0^1(R^n; C^m)$ this follows from Lemma 5.35. For the general case, let $u \in D(\Lambda'') \cap C^1(R^n; C^m)$ and let $\{u_m\} \subset C_0^1(R^n; C^m)$ converge to u in H_0 . Define $\psi = J^*u$, $\psi_m = J^*u_m$ and $f_R^m = \tilde{P}_0 \varphi_R \psi_m - \varphi_R \tilde{P}_0 \psi_m$. Then $f_R^m \rightarrow f_R$ when $m \rightarrow \infty$ because $\tilde{P}_0$ is a bounded operator on H_0 . Moreover, Lemma 5.35 implies that

$$(5.156) \quad \|D_j f_R^m\| \leq N \|\psi_m\|$$

when N is independent of R . $\{\psi_m\}$ is a CAUCHY sequence in H_0 and therefore (5.156) implies that $\{D_j f_R^m\}$ is a CAUCHY sequence in H_0 . Hence $D_j f_R^m \rightarrow w$ in H_0 where w is a weak derivative of f_R in H_0 . But (5.149) implies that $D_j f_R \in H_0$. Hence $w = D_j f_R$ by the uniqueness of weak derivatives. Thus passage to the limit $m \rightarrow \infty$ in (5.156) gives

$$(5.157) \quad \|D_j f_R\| \leq N \|\psi\| = N \|J^*u\| \leq N' \|u\|$$

which completes the proof of Theorem 5.33.

Theorem 5.29 asserts that the coerciveness inequality holds for all $u \in D(\Lambda'') \cap C^1$. The next and final step is to extend it to all $u \in D(\Lambda'')$. This is the content of

THEOREM 5.36. - Under the hypotheses of Theorem 5.8, Λ'' is Λ -coercive, and

$$(5.158) \quad \|u\|_1^2 \leq \mu_1 (\|\Lambda'' u\|^2 + \|u\|^2)$$

for all $u \in D(\Lambda'') = L_{2,m}^1$ where μ_1 is the constant of Theorem 5.14.

The proof is given below after some preliminary lemmas. It will be convenient to define the operator

$$(5.159) \quad \Gamma = \Lambda E^{-1}$$

with domain $D(\Gamma) = \{u : E^{-1}u \in D(\Lambda)\}$. It follows from Theorem 2.6 and (2.41) that $u \in D(\Gamma)$ and $\Gamma u = v$ if and only if

$$(5.160) \quad (E^{-1}u, \Lambda^* \varphi) = (v, \varphi)$$

for all $\varphi \in C_0^1(R^n; C^m)$. Note that $\Lambda^* = A E^{-1}$, by (2.40) and (5.26).

Moreover E^{-1} is selfadjoint on $L_{2,m}$. Thus (5.160) implies

$$\begin{aligned} (v, \varphi) &= (E^{-1}u, AE^{-1}\varphi) = (u, E^{-1}AE^{-1}\varphi) \\ (5.161) \quad &= (u, \Lambda E^{-1}\varphi) = (u, \Gamma\varphi). \end{aligned}$$

Moreover, it is easy to verify that the formal adjoint of Γ on $L_{2,m}$ is $\Gamma^+ = \Gamma$. Thus $u \in D(\Gamma)$ and $\Gamma u = v$ if and only if

$$(5.162) \quad (u, \Gamma^+\varphi) = (v, \varphi)$$

for all $\varphi \in C_0^1(R^n; C^m)$. Hence, $\Gamma = \Gamma_s = \Gamma_w$, by Theorem 2.1.

Let J_k denote FRIEDRICHS' mollifier, defined by (2.19). Then the following lemma is an immediate consequence of Corollary 2.2 and $\Gamma = \Gamma_w = \Gamma_s$.

LEMMA 5.37. - If $u \in D(\Gamma)$ then $J_k u \in D(\Gamma)$ and $\Gamma J_k u \rightarrow \Gamma u$ in H where $k \rightarrow \infty$.

COROLLARY 5.38. - If $u \in D(\Lambda)$ then $E^{-1}J_k E u \in D(\Lambda)$ and $\Lambda E^{-1}J_k E u = \Gamma J_k E u$.

PROOF. - It is clear from the definition of $D(\Gamma)$ that $u \in D(\Lambda)$ if and only if $E u \in D(\Gamma)$. Thus $u \in D(\Lambda)$ implies $J_k E u \in D(\Gamma)$, by Lemma 5.37, and this implies that $E^{-1}J_k E u \in D(\Lambda)$ and $\Lambda E^{-1}J_k E u = \Gamma J_k E u$.

LEMMA 5.39. - If $u \in D(\Lambda')$ then $E^{-1}J_k E u \in D(\Lambda')$ and $\lim_{k \rightarrow \infty} \Lambda' E^{-1}J_k E u = \Lambda' u$ in H .

PROOF. - By definition 5.5, $u \in D(\Lambda')$ if and only if $P_0 J^* u \in L_{2,m}^1$. Now $E_0^{-1}J_k = J_k E_0^{-1}$, and it is easily verified that $P_0 J_k = J_k P_0$, because J_k is a scalar operator. Thus

$$\begin{aligned} P_0 J^*(E^{-1}J_k E u) &= P_0 E_0^{-1} E (E^{-1}J_k E u) \\ (5.163) \quad &= P_0 E_0^{-1} J_k E u = P_0 J_k E_0^{-1} E u = P_0 J_k J^* u = J_k (P_0 J^* u) \end{aligned}$$

Moreover, $v \in L_{2,m}^1$ implies $J_k v \in L_{2,m}^1$ by Corollary 2.2. Thus (5.163) implies that if $u \in D(\Lambda')$ then $P_0 J^*(E^{-1}J_k E u) \in L_{2,m}^1$; i.e., $E^{-1}J_k E u \in D(\Lambda')$.

Next, note that

$$\begin{aligned} (5.164) \quad &\|\Lambda'(E^{-1}J_k E u) - \Lambda' u\|^2 = \|\Lambda'(E^{-1}J_k E u - u)\|^2 \\ &= \sum_{j=1}^n \|D_j P_0 J^*(E^{-1}J_k E u - u)\|^2 = \sum_{j=1}^n \|D_j P_0 E_0^{-1} E (E^{-1}J_k E u - u)\|^2 \\ &= \sum_{j=1}^n \|D_j P_0 J_k J^* u - D_j P_0 J^* u\|^2 = \sum_{j=1}^n \|J_k (D_j P_0 J^* u) - D_j P_0 J^* u\|^2. \end{aligned}$$

The last term tends to zero when $k \rightarrow \infty$ because $D_j P_0 J^* u \in H_0$, which proves the convergence statement.

LEMMA 5.40. - If $u \in D(\Lambda'')$ then there exists a sequence $\{u_k\} \subset D(\Lambda'') \cap C^1(R^n; C^m)$ such that $u_k \rightarrow u$ in H and $\Lambda'' u_k \rightarrow \Lambda'' u$ in $L_{2, 2m}$.

PROOF. - Define $u_k = E^{-1} J_k E u$. Then $u_k \rightarrow u$ in H , since E^{-1} is a bounded operator. Moreover, $u_k \in C^1(R^n; C^m)$ because all functions in the range of J_k are in $C^\infty(R^n; C^m)$ and $E^{-1} \in C^1$. To complete the proof it suffices to prove that $u_k \in D(\Lambda') \cap D(\Lambda)$, $\Lambda' u_k \rightarrow \Lambda' u$ in H and $\Lambda u_k \rightarrow \Lambda u$ in H . But $u_k \in D(\Lambda')$ and $\Lambda' u_k \rightarrow \Lambda' u$ in H by Lemma 5.39. Moreover, $u_k \in D(\Lambda)$ and $\Lambda u_k = \Gamma J_k E u$ by Corollary 5.38. Hence, by Lemma 5.37, $\Lambda u_k = \Gamma J_k E u \rightarrow \Gamma E u = \Lambda u$ in H . This completes the proof.

PROOF OF THEOREM 5.36. - Let $u \in D(\Lambda'')$ and let $\{u_k\}$ be the sequence defined in Lemma 5.40. Then Theorem 5.29 is applicable to $u_k - u_l$ and gives

$$(5.165) \quad \|u_k - u_l\|_1^2 \leq \mu_1^2 (\|\Lambda'' u_k - \Lambda'' u_l\|^2 + \|u_k - u_l\|^2).$$

Thus the convergence $u_k \rightarrow u$ in H and $\Lambda'' u_k \rightarrow \Lambda'' u$ in $L_{2, 2m}^1$, guaranteed by Lemma 5.40, implies that each $D_j u_k$ is a CAUCHY sequence in $L_{2, m}$, and hence $D_j u_k \rightarrow v_j \in L_{2, m}$ for $j = 1, 2, \dots, m$. Clearly, $v_j = D_j u$ in the sense of distribution theory. Thus $u \in L_{2, m}^1$ and passage to the limit $k \rightarrow \infty$ in the inequality

$$(5.166) \quad \|u_k\|_1^2 \leq \mu_1^2 (\|\Lambda'' u_k\|^2 + \|u_k\|^2)$$

gives (5.158). This proves Theorem 5.36.

The goal of this section of the paper was to prove Theorem 5.8, the coerciveness of Λ on $N(\Lambda)^\perp$. This is a corollary of Theorem 5.36.

PROOF OF THEOREM 5.8. - Theorem 5.36 implies that if $u \in D(\Lambda'') = D(\Lambda) \cap D(\Lambda') = L_{2, m}^1$ then

$$(5.167) \quad \|u\|_1^2 \leq \mu_1^2 (\|\Lambda u\|^2 + \|\Lambda' u\|^2 + \|u\|^2).$$

Moreover, Lemma 5.9 implies that $N(\Lambda)^\perp \subset N(\Lambda')$. Thus if $u \in D(\Lambda) \cap N(\Lambda)^\perp$ then $u \in D(\Lambda) \cap N(\Lambda') \subset D(\Lambda) \cap D(\Lambda') = D(\Lambda'')$ and $\Lambda' u = 0$. Hence (5.167) implies

$$(5.168) \quad \|u\|_1^2 \leq \mu_1^2 (\|\Lambda u\|^2 + \|u\|^2)$$

for all $u \in D(\Lambda) \cap N(\Lambda)^\perp$.

§ 6. - Coerciveness Inequalities for Higher Order Derivatives.

The purpose of this section is to generalize the coerciveness theorem for Λ (Theorem 5.8) to a coerciveness theorem for powers of Λ . It is shown below, under suitable assumptions on $E(x)$, that if $u \in D(\Lambda^k) \cap N(\Lambda)^\perp$ where k is a positive integer then all the partial derivatives of u of order k are in H . This is done by generalizing the coerciveness theorem for Λ'' (Theorem 5.36) to one for powers of Λ and Λ' .

Let a general partial derivative operator be denoted by

$$(6.1) \quad D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n},$$

(cf. (4.6)). The coerciveness theorems for higher order operators can be formulated in terms of the SOBOLEV spaces

$$(6.2) \quad L_{2,m}^k = \{u : D^\alpha u \in L_{2,m} \text{ for } |\alpha| \leq k\}.$$

$L_{2,m}^k$ is a HILBERT space with inner product

$$(6.3) \quad (u, v)_k = \int_{R^n} \sum_{|\alpha| \leq k} (D^\alpha u)^* D^\alpha v dx$$

and norm $\|u\|_k = (u, u)_k^{1/2}$. Clearly

$$(6.4) \quad \|u\|_k^2 = \sum_{|\alpha| \leq k} \|D^\alpha u\|^2$$

where $\|\cdot\|$ denotes the norm in $L_{2,m}$.

The principle result of this section is the following generalization of Theorem 5.36.

THEOREM 6.1. - Let k be a positive integer. If the hypotheses of Theorem 5.8 hold and, in addition, the derivatives $D^\alpha E(x)$ are continuous and bounded on R^n for $|\alpha| \leq k$ then

$$(6.5) \quad D(\Lambda^k) \cap D(\Lambda'^k) = L_{2,m}^k$$

and there exists a constant $\mu_k > 0$ such that

$$(6.6) \quad \|u\|_k^2 \leq \mu_k^2 (\|\Lambda^k u\|^2 + \|\Lambda'^k u\|^2 + \|u\|^2)$$

for all $u \in D(\Lambda^k) \cap D(\Lambda'^k)$.

Note that $N(\Lambda)^\perp \subset N(\Lambda') \subset D(\Lambda'^k)$ for $k > 1$. Hence, the following generalization of Theorem 5.8 is an immediate corollary of Theorem 6.1.

THEOREM 6.2. - Under the hypotheses of Theorem 6.1, $D(\Lambda^k) \cap N(\Lambda)^\perp \subset L_{2,m}^k$ and

$$(6.7) \quad \|u\|_k^2 \leq \mu_k^2 (\|\Lambda^k u\|^2 + \|u\|^2)$$

for all $u \in D(\Lambda^k) \cap N(\Lambda)^\perp$.

Theorem 6.1 will be proved by induction on k . For $k = 1$ the assertion of the theorem is true by Theorem 5.36. To complete the induction it will be shown that if the assertion of the theorem is true for an integer $k \geq 1$ then it is also true for $k + 1$. Thus it will be shown that if the $D^\alpha E(x)$ are continuous and bounded for $|\alpha| \leq k + 1$ and

$$(6.8) \quad u \in D(\Lambda^{k+1}) \cap D(\Lambda'^{k+1})$$

then $u \in L_{2,m}^{k+1}$ and there exists a constant μ_{k+1} such that

$$(6.9) \quad \|u\|_{k+1}^2 \leq \mu_{k+1}^2 (\|\Lambda^{k+1} u\|^2 + \|\Lambda'^{k+1} u\|^2 + \|u\|^2).$$

In proving this, the inductive hypothesis is that $u \in L_{2,m}^k$ and (6.7) holds. The proof is given in a series of lemmas.

LEMMA 6.3. - The hypothesis (6.8) implies that $\Lambda^k u$ and $\Lambda'^k u$ are in $L_{2,m}^1$.

PROOF. - Let $v = \Lambda^k u$. Then $v \in D(\Lambda)$ by (6.8). Moreover, $\Lambda'v = \Lambda' \Lambda^k u = (J(-\Delta^{1/2} P_0 J^*)(J^{*-1} \Lambda_0 J) \Lambda^{k-1} u = 0$ because $P_0 \Lambda_0 = 0$. Thus $v \in N(\Lambda') \subset D(\Lambda')$. Hence $v \in D(\Lambda) \cap D(\Lambda') = D(\Lambda'')$ and therefore $v \in L_{2,m}^1$ by Theorem 5.36. Similarly, if $w = \Lambda'^k u$ then $w \in D(\Lambda')$ by (6.8). Moreover, $\Lambda w = \Lambda \Lambda'^k u = E^{-1} J A J^{-1} (J(-\Delta)^{1/2} P_0 J^*) \Lambda'^{k-1} u = E^{-1} J (-\Delta)^{1/2} E_0 \Lambda_0 P_0 J^* \Lambda'^{k-1} u = 0$ because $\Lambda_0 P_0 = 0$. Thus $w \in N(\Lambda) \subset D(\Lambda)$. Hence $w \in D(\Lambda) \cap D(\Lambda') = D(\Lambda'')$ and therefore $w \in L_{2,m}^1$ by Theorem 5.36.

The following notation is used to discuss difference quotients, $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ denotes the vector in R^n with 1 as its j^{th} component and 0 as the remaining components. The translation and difference operators for the j^{th} coordinate are defined by

$$(6.10) \quad \tau_{j,h} u(x) = u(x + h e_j)$$

and

$$(6.11) \quad \delta_{j,h} u(x) = h^{-1} [\tau_{j,h} u(x) - u(x)]$$

where $h \neq 0$ is a real number.

LEMMA 6.4. - If $u \in L_{2,m}^k$ then $\delta_{j,h} u \in D(\Lambda^k) \cap D(\Lambda'^k)$.

This is evident since $\delta_{j, h} u \in L_{2, m}^k = D(\Lambda^k) \cap D(\Lambda'^k)$.

LEMMA 6.5. - Let k be a positive integer and define $\Lambda^0 = I$, $\Lambda'^0 = I$. Then

$$(6.12) \quad \delta_{j, h} \Lambda'^k u = \Lambda'^k \delta_{j, h} u + \sum_{l=1}^k \Lambda'^{k-l} J(-\Delta)^{1/2} P_0(\delta_{j, h} J^*) \tau_{j, h} \Lambda'^{l-1} u$$

and

$$(6.13) \quad \delta_{j, h} \Lambda^k u = \Lambda^k \delta_{j, h} u + \sum_{l=1}^k \Lambda^{k-l} (\delta_{j, h} J^{*-1}) \Lambda_0 \tau_{j, h} \Lambda^{l-1} u$$

for all $u \in L_{2, m}^k$.

PROOF. - The proofs are by induction on k . Note that $\tau_{j, h}(-\Delta)^{1/2} = (-\Delta)^{1/2} \tau_{j, h}$ and $P_0 \tau_{j, h} = \tau_{j, h} P_0$.

PROOF OF (6.12). - To verify (6.12) for $k = 1$ note that

$$(6.14) \quad \delta_{j, h} \Lambda' u = \delta_{j, h} J(-\Delta)^{1/2} P_0 J^* u = J(-\Delta)^{1/2} P_0 \delta_{j, h} (J^* u)$$

and

$$(6.15) \quad \delta_{j, h} (J^* u) = J^* \delta_{j, h} u + (\delta_{j, h} J^*) \tau_{j, h} u.$$

Combining these gives

$$(6.16) \quad \delta_{j, h} \Lambda' u = \Lambda' \delta_{j, h} u + J(-\Delta)^{1/2} P_0 (\delta_{j, h} J^*) \tau_{j, h} u$$

which is (6.12) for $k = 1$. Now if (6.12) holds for k then

$$(6.17) \quad \begin{aligned} \delta_{j, h} \Lambda'^{k+1} u &= \delta_{j, h} \Lambda' (\Lambda'^k u) = \Lambda' \delta_{j, h} (\Lambda'^k u) \\ &+ J(-\Delta)^{1/2} P_0 (\delta_{j, h} J^*) \tau_{j, h} \Lambda'^k u \end{aligned}$$

by the case $k = 1$. Thus, by the inductive hypothesis,

$$(6.18) \quad \begin{aligned} \delta_{j, h} \Lambda'^{k+1} u &= \Lambda'^{k+1} \delta_{j, h} u + \sum_{l=1}^k \Lambda'^{k+1-l} J(-\Delta)^{1/2} P_0 (\delta_{j, h} J^*) \tau_{j, h} \Lambda'^{l-1} u \\ &+ J(-\Delta)^{1/2} P_0 (\delta_{j, h} J^*) \tau_{j, h} \Lambda'^k u \\ &= \Lambda'^{k+1} \delta_{j, h} u + \sum_{l=1}^{k+1} \Lambda'^{k+1-l} J(-\Delta)^{1/2} P_0 (\delta_{j, h} J^*) \tau_{j, h} \Lambda'^{l-1} u \end{aligned}$$

which completes the induction.

PROOF OF (6.13). - Note that

$$\begin{aligned}
 \delta_{j, h} \Lambda u(x) &= \delta_{j, h} J^{*-1} \Lambda_0 u(x) \\
 &= h^{-1} [J^{*-1}(x + h e_j) \Lambda_0 u(x + h e_j) + J^{*-1}(x) \Lambda_0 u(x)] \\
 (6.19) \quad &= J^{*-1}(x) \Lambda_0 h^{-1} [u(x + h e_j) - u(x)] \\
 &\quad + h^{-1} [J^{*-1}(x + h e_j) - J^{*-1}(x)] \Lambda_0 u(x + h e_j) \\
 &= \Lambda \delta_{j, h} u(x) + (\delta_{j, h} J^{*-1}) \Lambda_0 \tau_{j, h} u(x)
 \end{aligned}$$

which is (6.13) for $k = 1$. Now if (6.13) holds for k then

$$\begin{aligned}
 \delta_{j, h} \Lambda^{k+1} u &= \delta_{j, h} (\Lambda^k u) \\
 (6.20) \quad &= \Lambda \delta_{j, h} (\Lambda^k u) + (\delta_{j, h} J^{*-1}) \Lambda_0 \tau_{j, h} \Lambda^k u
 \end{aligned}$$

by the case $k = 1$. Thus, by the inductive hypothesis,

$$\begin{aligned}
 \delta_{j, h} \Lambda^{k+1} u &= \Lambda^{k+1} \delta_{j, h} u + \sum_{l=1}^k \Lambda^{k+1-l} (\delta_{j, h} J^{*-1}) \Lambda_0 \tau_{j, h} \Lambda^{l-1} u \\
 &\quad + (\delta_{j, h} J^{*-1}) \Lambda_0 \tau_{j, h} \Lambda^k u \\
 (6.21) \quad &= \Lambda^{k+1} \delta_{j, h} u + \sum_{l=1}^{k+1} \Lambda^{k+1-l} (\delta_{j, h} J^{*-1}) \Lambda_0 \tau_{j, h} \Lambda^{l-1} u
 \end{aligned}$$

which completes the induction.

LEMMA 6.6. - Let $0 < h \leq h_0$. Then there exist constants c'_k and c_k , which depend only on k, h_0, E_0, A_j and the constants $\|D^\alpha E\|_\infty$ for $|\alpha| \leq k+1$, such that

$$(6.22) \quad \sum_{l=1}^k \|\Lambda^{k-l} J(-\Delta)^{1/2} P_0 (\delta_{j, h} J^*) \tau_{j, h} \Lambda^{l-1} u\|^2 \leq c'_k \|u\|_k^2$$

and

$$(6.23) \quad \sum_{l=1}^k \|\Lambda^{k-l} (\delta_{j, h} J^{*-1}) \Lambda_0 \tau_{j, h} \Lambda^{l-1} u\|^2 \leq c_k \|u\|_k^2$$

for all $u \in L_{2, m}^k$.

PROOF. - It is clear that the left-hand side of (6.22), for example, involves the norm of a linear combination of derivatives of u of orders k or less, with coefficients which involve derivatives $D^\alpha \delta_{j, h} J^* = E_0^{-1} D^\alpha \delta_{j, h} E$ of order

k or less. The latter can be majorized by terms of the form $\|D^\alpha E\|$ with $|\alpha| \leq k+1$. The proofs of (6.22) and (6.23) follows by a straightforward induction on k which will not be given here.

The following standard lemma may be found in [2, p. 54].

LEMMA 6.7. - Let $u \in L_{2,m}^k$ and suppose that there exists positive constants C and h_0 such that

$$(6.24) \quad \|\delta_{j,h} u\|_k \leq C$$

for $0 < h \leq h_0$ and $j = 1, 2, \dots, n$. Then $u \in L_{2,m}^{k+1}$ and

$$(6.25) \quad \|D_j u\|_k \leq C$$

for $j = 1, 2, \dots, n$.

LEMMA 6.8. - Let $\|D^\alpha E\|_\infty < \infty$ for $|\alpha| \leq k+1$ and assume that (6.8) holds. Then $u \in L_{2,m}^{k+1}$ and there exists a constant μ'_k , depending only on E_0 , A_j and the $\|D^\alpha E\|_\infty$ for $|\alpha| \leq k+1$, such that

$$(6.26) \quad \|u\|_{k+1}^2 \leq \mu_k'^2 (\|\Lambda^k u\|_1^2 + \|\Lambda'^k u\|_1^2 + \|u\|^2)$$

for all u satisfying (6.8).

PROOF. - By the inductive hypothesis and Lemma 6.4, $\delta_{j,h} u \in D(\Lambda^k) \cap D(\Lambda'^k)$. Hence, the inductive hypothesis implies that (6.6) holds for $\delta_{j,h} u$, which implies

$$(6.27) \quad \|\delta_{j,h} u\|_k^2 \leq \mu_k'^2 (\|\Lambda^k \delta_{j,h} u\|^2 + \|\Lambda'^k \delta_{j,h} u\|^2 + \|\delta_{j,h} u\|^2).$$

Combining this with the results of Lemma 6.5 and 6.6 gives

$$(6.28) \quad \begin{aligned} \|\delta_{j,h} u\|_k^2 &\leq 2\mu_k'^2 (\|\delta_{j,h} \Lambda^k u\|^2 + \|\delta_{j,h} \Lambda'^k u\|^2 + (c_k + c'_k) \|u\|_k^2 + \|\delta_{j,h} u\|^2). \end{aligned}$$

Now Lemma 6.3 implies that the derivatives $D_j \Lambda^k u$ and $D_j \Lambda'^k u$ exist in H , and it follows that $\delta_{j,h} \Lambda^k u \rightarrow D_j \Lambda^k u$ and $\delta_{j,h} \Lambda'^k u \rightarrow D_j \Lambda'^k u$ in H when $h \rightarrow 0$. Thus, the right-hand side of (6.28) is bounded for $0 < h \leq h_0$. It follows by Lemma 6.7 $u \in L_{2,m}^{k+1}$. Moreover, passage to the limit $h \rightarrow 0$ in (6.28) gives

$$(6.29) \quad \begin{aligned} \|D_j u\|_k^2 &\leq 2\mu_k'^2 (\|D_j \Lambda^k u\|^2 + \|D_j \Lambda'^k u\|^2 + (c_k + c'_k) \|u\|^2 + \|D_j u\|^2). \end{aligned}$$

Summing over $j = 1, 2, \dots, n$, adding $\|u\|^2$ to both sides, and noting that $\|\Lambda^k u\|_1^2 \leq \|\Lambda^k u\|_1^2$ and $\|\Lambda'^k u\|_1^2 \leq \|\Lambda'^k u\|_1^2$ gives

$$\begin{aligned} & \|u\|_{k+1}^2 \\ (6.30) \quad & \leq 2\mu_k^2(\|\Lambda^k u\|_1^2 + \|\Lambda'^k u\|_1^2 + n(c_k + c'_k)\|u\|^2 + \|u\|_1^2). \end{aligned}$$

Inequality (6.26) follows immediately from (6.30) and the following lemma.

LEMMA 6.9 - Let $k \geq 1$ be an integer and let j be an integer such that $0 \leq j \leq k$. Then given $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that for all $u \in L_{2,m}^{k+1}$

$$(6.31) \quad \|u\|_j^2 \leq \varepsilon \|u\|_{k+1}^2 + C\|u\|^2.$$

PROOF. - It can be shown by an elementary argument that for each $\varepsilon > 0$ there exists a $C = C(\varepsilon)$ such that

$$(6.32) \quad \sum_{|\alpha| \leq j} p^{2\alpha} \leq \varepsilon \sum_{|\alpha| \leq k+1} p^{2\alpha} + C$$

for all $p \in R^n$. (6.31) follows immediately from (6.32) by the PLANCHEREL theory.

LEMMA 6.10. - Let $\varepsilon > 0$. Then (6.8) implies that there exists a constant C_1 , depending only on E_0 , A_j and $\|D^\alpha E\|_\infty$ for $|\alpha| \leq k-1$, and a constant $C_1(\varepsilon)$ such that

$$\begin{aligned} & \|\Lambda^k u\|_1^2 \\ (6.33) \quad & \leq \mu_1^2[\|\Lambda^{k+1} u\|^2 + \varepsilon C_1 \|u\|_{k+1}^2 + C_2(\varepsilon)\|u\|^2]. \end{aligned}$$

PROOF. - Lemma 6.3 implies that $\Lambda^k u \in L_{2,m}^1$. Hence by Theorem 5.36 applied to $\Lambda^k u$,

$$\begin{aligned} & \|\Lambda^k u\|_1^2 \leq \mu_1^2[\|\Lambda^{k+1} u\|^2 + \|\Lambda' \Lambda^k u\|^2 + \|\Lambda^k u\|^2] \\ (6.34) \quad & = \mu_1^2[\|\Lambda^{k+1} u\|^2 + \|\Lambda^k u\|^2] \end{aligned}$$

since $\Lambda' \Lambda^k = 0$. Now $\|\Lambda^k u\|^2 \leq C_1 \|u\|_k^2$ where C_1 depends on E_0 , A_j and $\|D^\alpha E\|_\infty$ for $|\alpha| \leq k-1$ only. Combining this with (6.31) and (6.34) gives (6.33).

LEMMA 6.11. - Let $\varepsilon > 0$. Then (6.8) implies that there exists a constant C_1 , depending only on E_0 , A_j and $\|D^\alpha E\|_\infty$ for $|\alpha| \leq k$, and a constant $C_4(\varepsilon)$ such that

$$(6.35) \quad \|\Lambda'^k u\|_1^2 \leq \mu_1^2[\|\Lambda^{k+1} u\|^2 + \varepsilon C_3 \|u\|_{k+1}^2 + C_4(\varepsilon)\|u\|^2].$$

PROOF. - Lemma 6.3 implies that $\Lambda'^k u \in L_{2,m}^1$. Hence by Theorem 5.36 applied to $\Lambda'^k u$,

$$\begin{aligned}\|\Lambda'^k u\|_1^2 &\leq \mu_1^2 [\|\Lambda \Lambda'^k u\|^2 + \|\Lambda'^{k+1} u\|^2 + \|\Lambda'^k u\|^2] \\ &= \mu_1^2 [\|\Lambda'^{k+1} u\|^2 + \|\Lambda'^k u\|^2]\end{aligned}$$

since $\Lambda \Lambda'^k = 0$. Now, it is easy to verify by induction that $\|\Lambda'^k u\|^2 \leq C_3 \|u\|_k^2$ for all $u \in L_{2,m}^k$ where C_3 depends on E_0 , A_j and $\|D^\alpha E\|_\infty$ for $|\alpha| \leq k$ only. Combining with (6.31), (6.36) gives (6.35).

PROOF OF THEOREM 6.1. - (6.8) and the inductive hypothesis imply that Lemmas 6.8, 6.10 and 6.11 hold. Combining these results gives

$$\begin{aligned}\|u\|_{k+1} &\leq \mu_k' \{ \mu_1^2 [\|\Lambda^{k+1} u\|^2 + \|\Lambda'^{k+1} u\|^2 + \varepsilon(C_1 + C_3)\|u\|_{k+1}^2 \\ (6.38) \quad &+ (C_2(\varepsilon) + C_4(\varepsilon))\|u\|^2] + \|u\|^2 \}.\end{aligned}$$

Hence

$$\begin{aligned}(1 - \mu_k'^2 \mu_1^2 (C_1 + C_3)\varepsilon) \|u\|_{k+1}^2 \\ (6.39) \quad &\leq \mu_k'^2 \mu_1^2 [\|\Lambda^{k+1} u\|^2 + \|\Lambda'^{k+1} u\|^2] \\ &+ (\mu_k'^2 + \mu_k'^2 (C_2(\varepsilon) + C_4(\varepsilon))) \|u\|^2.\end{aligned}$$

Choosing ε so that $1 - \mu_k'^2 \mu_1^2 (C_1 + C_3)\varepsilon = \frac{1}{2}$ implies that (6.9) holds with a suitable constant μ_{k+1} . This completes the induction and the proof of Theorem 6.1.

§ 7. - Concluding Remarks.

The coerciveness theorems proved in this paper suggest a number of new problems in the theory of partial differential equations. One obvious problem is to generalize the coerciveness theorem to general first-order operators L which are nonelliptic and have constant deficit. It is also of interest to extend the results to operators of higher order. In this connection it would be desirable to use the DOUGLIS-NIRENBERG notion of ellipticity [9] so that the equations of the system can have different orders. Another problem which is important for applications to problems in the theory of wave propagation and scattering is to extend the coerciveness theorems to boundary value problems for nonelliptic operators in bounded and unbounded domains. In this case, if the method of augmented operators is used it will be necessary to find suitable boundary conditions for the augmented operators.

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