A two phase Stefan problem with temperature boundary conditions

JOHN R. CANNON [1] and MARIO PRIMICERIO [2] (*) (**)

Riassunto. - *Si studia un problema di Stefan a due fasi in uno strato piano indefinito, quando sia assegnata la temperatura sui piani che delimitano lo strato stesso.*

Viene dimostrata l'esistenza (in grande) e l'unicità della soluzione sotto ipotesi assai generali sui dati iniziali ed al contorno. Si prova la dipendenza continua e monotona della soluzione dai dati iniziali ed al contorno.

Abstract. - *We studied a two phase Stefan problem in a infinite plane slab, when the temperatures are prescribed on the two limiting planes.*

We proved global existence and uniqueness of the solution under minimal smoothness assumptions upon the initial and boundary data. Furthermore, we demonstrated the continuous and monotone dependence of the solution on the initial and boundary data.

1. - Introduction.

An example of the two phase STEFAN problem considered in this paper is a slab of unit thickness consisting partly of water and partly of ice with the temperature specified on the bounding planes. We consider the simplest case of only one water-ice interface $x = s(t)$. Mathematically, the problem consists of determining two functions, $u(x, t)$ and $v(x, t)$, and a function $x = s(t)$ such that (u, v, s) satisfy

$$(1.1)\qquad \begin{aligned} &L_1(u) \equiv \varkappa_1 u_{xx} - u_t = 0, \qquad 0 < x < s(t), && 0 < t \leq T,\\ &u(0, t) = f(t), \qquad u(s(t), t) = 0, && 0 < t \leq T,\\ &u(x, 0) = \varphi(x), \qquad 0 \leq x \leq b, \qquad s(0) = b, && 0 < b < 1. \end{aligned}$$

$$(1.2)\qquad \begin{aligned} &L_2(v) \equiv \varkappa_2 v_{xx} - v_t = 0, \qquad s(t) < x < 1, && 0 < t \leq T,\\ &v(1, t) = g(t), \qquad v(s(t), t) = 0, && 0 < t \leq T,\\ &v(x, 0) = \psi(x), \qquad b \leq x \leq 1, && \end{aligned}$$

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(1) Mathematics Department, University of Texas at Austin, Texas, 78712, U.S.A.

(2) Università di Firenze, Istituto Matematico «U. Dini», Viale Morgagni 67/A, 50134 Firenze, Italy.

and

$$\dot{s}(t) = - K_1 u_x(s(t), t) + K_2 v_x(s(t), t), \qquad 0 < t \leq T, \tag{1.3}$$

where $\varkappa_i = k_i \rho_i^{-1} c_i^{-1}$, $i = 1, 2$ represent the diffusivities, k_i, $i = 1, 2$, the conductivities, ρ_i, $i = 1, 2$, the densities, c_i, $i = 1, 2$, the heat capacities; $K_i = k_i \rho_2^{-1} L$, $i = 1, 2$, and L is the latent heat; all of the preceding constants are positive. T is an arbitrary but fixed positive number, and the functions $f > 0$, $g < 0$, $\varphi \geq 0$, $\psi \leq 0$, and the value b, $0 < b < 1$, are the boundary and initial data for (1.1), (1.2) and (1.3).

Problems like (1.1), (1.2), and (1.3) have been considered by several authors (see [1, 5, 8, 9, 10, 12, 13, 14, 15, 16]). In each of the references cited the results require considerable smoothness assumptions on the data. In this paper we prove the global existence and uniqueness of the solution and demonstrate the continuous and monotone dependence of the free bonndary $x = s(t)$ on the data. The results of this paper are based upon the maximum principle and the technique of the retarded argument and hence require minimal smoothness requirements upon the data. However, as in [5], it is necessary to require that the data functions are sufficiently small in order that an a priori bound for $\dot{s}$ can be obtained. From our method of argument it is evident that the results generalize to more general parabolic operators for which the maximum principle holds.

2. - Definitions, hypotheses, and reformulation of the free boundary condition.

We begin with a list of the assumptions needed for the existence theorem.

(A) Let $f = f(t)$ and $g = g(t)$ be bounded piecewise continuous functions such that there exist four positive constants α_i, β_i $i = 1, 2$ such that

$$0 < \alpha_1 < f(t) < \beta_1 \quad \text{and} \quad -\beta_2 < g(t) < -\alpha_2 < 0. \tag{2.1}$$

(B) Let $\varphi = \varphi(x)$ and $\psi = \psi(x)$ be piecweise continuous functions such that there exist four positive constants a_i, η_i, $i = 1, 2$ such that

$$0 \leq \varphi(x) \leq a_1(1 - \exp\{\varkappa_1^{-1}\eta_1(x - b)\}) \tag{2.2}$$

and

$$0 \geq \psi(x) \geq - a_2(1 - \exp\{-\varkappa_2^{-1}\eta_2(x - b)\}). \tag{2.3}$$

(C) Finally, we assume that

$$\begin{gathered} \Gamma = \max(2K_1\varepsilon_1\varkappa_1^{-1}, 2K_2\varepsilon_2\varkappa_2^{-1}) < 1, \text{ where} \\ \varepsilon_1 = \max(\beta_1, a_1) \quad \text{and} \quad \varepsilon_2 = \max(\beta_2, a_2). \end{gathered} \tag{2.4}$$

We note that (2.2) and (2.3) are essentially assumptions of local LIPSCHITZ continuity of the initial data at $x = b$ while (2.4) is the restriction on the data ([3]).

Sometimes it will be necessary to designate (1.1) and (1.2) as an *auxiliary problem* for a given LIPSCHITZ continuous functions $s(t)$. By a *solution of the auxiliary problem*, we mean a pair of functions $u = u(x, t)$ and $v = v(x, t)$ such that

1°. the derivatives appearing in the equations exist and are continuous in their respective domain of definition,

2°. u and v are continuous in the closure of such domains except at points of discontinuity of the data,

3°. for such points of discontinuity

$$0 \leq \underline{\lim}\, u \leq \overline{\lim}\, u < \infty \quad \text{and} \quad -\infty < \underline{\lim}\, v \leq \overline{\lim}\, v \leq 0$$

as each such point is approched from the region in question,

4°. u and v satisfy (1.1) and (1.2) respectively.

Classical analysis of parabolic equations [11] asserts that the solution of the auxiliary problem exists and is unique under the assumptions listed above. Moreover, it follows [4, 11] that $u_x(s(t), t)$ and $v_x(s(t), t)$ exist and are continuous for $0 < t \leq T$. If φ and ψ are continuously differentiable in a neighborhood of b, then $u_x(s(t), t)$ and $v_x(s(t), t)$ exist and are continuous for $0 \leq t \leq T$.

By a solution (u, v, s) of the STEFAN problem (1.1), (1.2) and (1.3), we mean that

1°. $s = s(t)$ is a continuously differentiable function for $0 < t \leq T$ ($0 \leq t \leq T$ if φ and ψ are continuously differentiable in a neighborhood of $x = b$) and continuous for $0 \leq t \leq T$, $s(0) = b$, and $0 < s(t) < 1$,

2°. u and v is the solution to the auxiliary problem for this $s = s(t)$ in the sense specified above, and

3°. u, v and s satisfy (1.3).

We consider now the reformulation of the boundary conditions (1.3). By integrating the operator $L_1(u) = 0$ over the region $0 < \gamma_1 \leq x \leq s(\tau)$, $0 < t_0 \leq \tau \leq t$ and the operator $L_2(v) = 0$ over the region $s(\tau) \leq x \leq \gamma_2 < 1$, $0 < t_0 \leq \tau \leq t$ and multiplying the resulting equations by $K_i \varkappa_i^{-1}$ respectively,

([3]) It is worth to note that, in the case of a water-ice system, (2.4) requires that the initial and boundary temperature lie in the range ($-40°$C, $+40°$C) approximately: a range covering, under normal conditions, the entire range of validity of the description of the fusion process by means of (1.1)-(1.3).

it follows from adding the equations that

$$\int_{t_0}^{t} [K_2 v_x(s(\tau), \tau) - K_1 u_x(s(\tau), \tau)] d\tau = \tag{2.5}$$

$$\int_{t_0}^{t} [K_2 v_x(\gamma_2, \tau) - K_1 u_x(\gamma_1, \tau)] d\tau -$$

$$- K_1 \varkappa_1^{-1} \int_{\gamma_1}^{s(t)} u(x, t) dx - K_2 \varkappa_2^{-1} \int_{s(t)}^{\gamma_2} v(x, t) dx +$$

$$+ K_1 \varkappa_1^{-1} \int_{\gamma_1}^{s(t_0)} u(x, t_0) dx + K_2 \varkappa_2^{-1} \int_{s(t_0)}^{\gamma_2} v(x, t_0) dx.$$

Hence, if (u, v, s) is a solution of (1.1), (1.2), and (1.3), then it follows from (1.3) that

$$s(t) = s(t_0) + \int_{t_0}^{t} [K_2 v_x(\gamma_2, \tau) - K_1 u_x(\gamma_1, \tau)] d\tau - \tag{2.6}$$

$$- K_1 \varkappa_1^{-1} \int_{\gamma_1}^{s(t)} u(x, t) dx - K_2 \varkappa_2^{-1} \int_{s(t)}^{\gamma_2} v(x, t) dx +$$

$$+ K_1 \varkappa_1^{-1} \int_{\gamma_1}^{s(t_0)} u(x, t_0) dx + K_2 \varkappa_2^{-1} \int_{s(t_0)}^{\gamma_2} v(x, t_0) dx.$$

holds for any γ_1, γ_2, t_0 satisfying $0 < \gamma_1 \leq s(\tau) \leq \gamma_2 < 1$, $0 < t_0 \leq \tau \leq t$. Conversely, if (u, v, s) satisfies (1.1), (1.2) and (2.6), where s is assumed to be LIPSCHITZ continuous, then since u and v satisfy (1.1) and (1.2) for this s, it follows that u, v and s satisfy (2.5). Hence, (2.6) and (2.5) imply that

$$s(t) - s(t_0) = \int_{t_0}^{t} [K_2 v_x(s(\tau), \tau) - K_1 u_x(s(\tau), \tau)] d\tau \tag{2.7}$$

whence (1.3) follows. In what follows it will be convenient to interchange (1.3) and (2.6) from time to time.

3. - **Existence.**

This section is devoted to the proof of the following result.

THEOREM 1. - Under the hypotheses (A), (B), and (C) of section 2, there exists a solution (u, v, s) of the STEFAN problem (1.1), (1.2), and (1.3) in the sense of section 2 which is defined for all $t > 0$.

PROOF. - We begin with the construction of a family of approximations to a solution of (1.1), (1.2) and (1.3). Let χ^θ be such that for each θ sufficiently small

$$\chi^\theta = \begin{cases} 1, & 0 \leq x \leq b - \theta, \\ 0, & b - \theta < x < b + \theta, \\ 1, & b + \theta \leq x \leq 1. \end{cases} \tag{3.3}$$

Extending φ and ψ to be zero outside of their respective domain of definition, set

$$\varphi_\theta = \chi^\theta \varphi \quad \text{and} \quad \psi_\theta = \chi^\theta \psi. \tag{3.4}$$

Consider the time interval $0 \leq t \leq \theta$ and let (u^θ, v^θ) be the solution of the auxiliary problem with φ and ψ replaced by φ_θ and ψ_θ respectively and with boundary $s^\theta = s^\theta(t) \equiv b$. Next, we define s^θ in the interval $\theta \leq t \leq 2\theta$ by retarding the argument in the boundary condition (1.3) and obtaining

$$\begin{aligned} s^\theta(t) = b + \int_\theta^t [&K_2 v_x^\theta(s^\theta(\tau - \theta), \tau - \theta) - \\ &- K_1 u_x^\theta(s^\theta(\tau - \theta), \tau - \theta)] d\tau, \qquad \theta \leq t \leq 2\theta. \end{aligned} \tag{3.5}$$

Next, we obtain u^θ and v^θ for $\theta \leq t \leq 2\theta$ by solving the auxiliary problem (1.1) and (1.2) for $\theta \leq t \leq 2\theta$ with the «initial» conditions $u^\theta(x, \theta)$ and $v^\theta(x, \theta)$ at $t = \theta$, boundary s given by s^θ for $\theta \leq t \leq 2\theta$, and boundary data f and g for $\theta \leq t \leq 2\theta$. Since $u_x^\theta(s^\theta(t), t)$ and $v_x^\theta(s^\theta(t), t)$ exist [4, 11] and are continuous for $\theta \leq t \leq 2\theta$, it follows that (3.5) can be used to define s^θ for $2\theta \leq t \leq 3\theta$. By induction, s^θ can be defined for $0 \leq t \leq T_\theta$, u^θ can be defined via the auxiliary problem in $0 \leq x \leq s^\theta(t)$ and $0 \leq t \leq T_\theta$, and v^θ can be defined via the auxiliary problem in $s^\theta(t) \leq x \leq 1$ and $0 \leq t \leq T_\theta$, where T_θ is the first positive time that $s^\theta(T_\theta) = \delta$ or $s^\theta(T_\theta) = 1 - \delta$, $0 < \delta < b < 1 - \delta < 1$, and clearly $T_\theta = T_\theta(\delta)$. In the following we demonstrate the

$$T_0 = \inf_{0 \leq \theta < \theta^*} T_\theta > 0$$

for θ^* sufficiently small so that χ^θ is defined. This will follow from the derivation of an apriori bound on s^θ which is independent of θ. Thus in the interval $0 \leq t \leq T_0$ the s^θ will form a bounded equicontinuous family. The ASCOLI-ARZELA theorem will give the existence of the free-boundary s at least for $0 \leq t \leq T_0$. It will then be shown that any solution of (1.1), (1.2) and (1.3) can be continued from t to $t+\sigma$ for any $t \geq T_0$ where σ is independent of t. From this will follow the existence of the solution of the STEFAN problem (1.1), (1.2) and (1.3) for all time.

Since the function $\dot{s}^\theta$ is continuous for $0 \leq t \leq T_\theta$, it follows that

$$\| \dot{s}^\theta \|_\theta = \sup_{0 \leq t \leq T_\theta} | \dot{s}^\theta(t) | < \infty. \tag{3.6}$$

We demonstrate the following lemma.

LEMMA 1. - For $0 \leq t \leq T_\theta$,

$$| u_x^\theta(s^\theta(t), t) | \leq \varepsilon_1 \left(1 - \exp\left\{ \frac{(\| \dot{s}^\theta \|_\theta + \eta_1)\delta}{\varkappa_1} \right\} \right)^{-1} \cdot \varkappa_1^{-1} (\| \dot{s}^\theta \|_\theta + \eta_1), \tag{3.7}$$

and

$$| v_x^\theta(s^\theta(t), t) | \leq \varepsilon_2 (1 - \exp\{ -(\| \dot{s}^\theta \|_\theta + \eta_2)\varkappa_2^{-1}\delta \})^{-1} \cdot \varkappa_2^{-1}(\| \dot{s}^\theta \|_\theta + \eta_2), \tag{3.8}$$

where $\varepsilon_1 = \max(\beta_1, a_1)$ and $\varepsilon_2 = \max(\beta_2, a_2)$.

PROOF. - Set

$$W_1^\theta(x, t) = \varepsilon_1(1 - \exp\{ -(\| \dot{s}^\theta \|_\theta + \eta_1)\varkappa_1^{-1}\delta \})^{-1} \cdot \left\{ 1 - \exp\left\{ \frac{\| \dot{s}^\theta \|_\theta + \eta_1}{\varkappa_1} (x - s^\theta(t)) \right\} \right\} - u^\theta(x, t) \tag{3.9}$$

for $0 \leq x \leq s(t)$ and $0 \leq t \leq T_\theta$. Direct computation shows that

$$L_1(W_1^\theta) = -\varepsilon_1(1 - \exp\{ -(\| \dot{s}^\theta \|_\theta + \eta_1)\varkappa_1^{-1}\delta \})^{-1} \cdot \exp\left\{ \frac{\| \dot{s}^\theta \|_\theta + \eta^1}{\varkappa_1} (x - s(t)) \right\} \left\{ \frac{(\| \dot{s}^\theta \|_\theta + \eta_1)^2}{\varkappa_1} + \frac{(\| \dot{s}^\theta \|_\theta + \eta_1)\dot{s}^\theta}{\varkappa_1} \right\} < 0 \tag{3.10}$$

since

$$\| \dot{s}^\theta \|_\theta^2 + 2 \| \dot{s}^\theta \|_\theta \eta_1 + \eta_1^2 + \| \dot{s}^\theta \|_\theta \dot{s}^\theta + \eta_1 \dot{s}^\theta > \| \dot{s}^\theta \|_\theta \eta_1 + \eta_1^2 > 0. \tag{3.11}$$

Moreover, we have

$$W_1^\theta(0,\ t) = \varepsilon_1(1 - \exp\{-(\|\dot{s}^\theta\|_\theta + \eta_1)\varkappa_1^{-1}\delta\})^{-1}\cdot \cdot\{1 - \exp\{-\varkappa_1^{-1}(\|\dot{s}^\theta\|_\theta + \eta_1)s^\theta(t)\}\} - f(t) \geq \geq \varepsilon_1 - f(t) \geq \beta_1 - f(t) \geq 0, \tag{3.12}$$

since

$$s^\theta(t) \geq \delta. \tag{3.13}$$

Also

$$W_1^\theta(s^\theta(t),\ t) = 0 \tag{3.14}$$

and

$$W_1^\theta(x,\ 0) \geq \varepsilon_1(1 - \exp\{\varkappa_1^{-1}\eta_1(x-b)\}) - \varphi(x) \geq \geq a_1(1 - \exp\{\varkappa_1^{-1}\eta_1(x-b)\}) - \varphi(x) \geq \geq 0 \tag{3.15}$$

since φ satisfies (2.2). Hence, by the maximum principle $W_1^\theta \geq 0$ in $0 \leq x \leq \leq s^\theta(t)$ and $0 \leq t \leq T_\theta$. Therefore,

$$W_{1x}^\theta(s^\theta(t),\ t) \leq u_x^\theta(s^\theta(t),\ t) \leq 0. \tag{3.16}$$

Since

$$W_{1x}^\theta(s^\theta(t),\ t) = -\varepsilon_1(1 - \exp\{-(\|\dot{s}^\theta\|_\theta + \eta_1)\varkappa_1^{-1}\delta\})^{-1}\cdot\left(\frac{\|\dot{s}^\theta\|_\theta + \eta_1}{\varkappa_1}\right), \tag{3.17}$$

(3.7) follows immediately. Also, the result (3.8) follows from a similar application of the maximum principle.

Differentiating (3.5) and applying (3.7) and (3.8) we see that

$$\|\dot{s}^\theta\|_\theta \leq \sum_{i=1}^{2} K_i\varepsilon_i\varkappa_i^{-1}(1 - \exp\{-(\|\dot{s}^\theta\|_\theta + \eta_i)\varkappa_i^{-1}\delta\})^{-1}\cdot(\|\dot{s}^\theta\|_\theta + \eta_i). \tag{3.18}$$

Recalling

$$\Gamma = \max(2K_1\varepsilon_1\varkappa_1^{-1},\ 2K_2\varepsilon_2\varkappa_2^{-2}) < 1. \tag{3.18}$$

and setting

$$\eta = \max(\eta_1,\ \eta_2) \tag{3 20}$$

and

$$\xi = \delta \min(\varkappa_1^{-1}, \varkappa_2^{-1}), \tag{3.21}$$

we obtain

$$\| \dot{s}^\theta \|_\theta \leq \Gamma (1 - \exp\{ - \xi \| \dot{s}^\theta \|_\theta \})^{-1} (\| \dot{s}^\theta \|_\theta + \eta). \tag{3.22}$$

From (3.22) we can demonstrate the following lemma.

LEMMA 2. - For s^θ on $0 \leq t \leq T_\theta$,

$$\| \dot{s}^\theta \|_\theta \leq \max\left(-\frac{1}{\xi} \log\left[\frac{1-\Gamma}{2}\right]; \; \frac{2\Gamma\eta}{1-\Gamma}\right). \tag{3.23}$$

PROOF. - First, suppose that

$$-\frac{1}{\xi} \log\left[\frac{1-\Gamma}{2}\right] < \frac{2\Gamma\eta}{1-\Gamma}, \tag{3.24}$$

Then, either

$$\| \dot{s}^\theta \|_\theta \leq -\frac{1}{\xi} \log\left[\frac{1-\Gamma}{2}\right]. \tag{3.25}$$

or

$$\| \dot{s}^\theta \|_\theta > -\frac{1}{\xi} \log\left[\frac{1-\Gamma}{2}\right]. \tag{3.26}$$

Consider (3.26). From (3.22) we see that

$$\begin{aligned} \| \dot{s}^\theta \|_\theta &\leq \Gamma\left(1 - \exp\left\{-\xi\left(-\frac{1}{\xi}\log\left(\frac{1-\Gamma}{2}\right)\right)\right\}\right)^{-1} \cdot \\ &\quad \cdot (\| \dot{s}^\theta \|_\theta + \eta) = \\ &= \Gamma\left(1 - \frac{1-\Gamma}{2}\right)^{-1} (\| \dot{s}^\theta \|_\theta + \eta) = \\ &= \frac{2\Gamma}{1+\Gamma} (\| \dot{s}^\theta \|_\theta + \eta). \end{aligned} \tag{3.27}$$

Hence,

$$\| \dot{s}^\theta \|_\theta \leq \frac{2\Gamma\eta}{1-\Gamma} \tag{3.28}$$

which demonstrates (3.23) for the case (3.24). For the case

$$\frac{2\Gamma\eta}{1-\Gamma} \leq -\frac{1}{\xi} \log \left| \frac{1-\Gamma}{2} \right| \tag{3.29}$$

the supposition (3.26) leads to the contradiction (3.28). Hence for (3.29), (3.25) holds which demonstrates (3.23).

Select now a δ_0 such that $0 < \delta_0 < b < 1 - \delta_0 < 1$. It is clear that

$$T_0 = \inf_\theta T_\theta(\delta_0) > 0.$$

Moreover, s^θ form a bounded equi-continuous family on $0 \leq t \leq T_0$. By the ASCOLI-ARZELA theorem there exist a sequence s^{θ_j} which converges uniformly to a function $s = s(t)$ on $0 \leq t \leq T_0$ as $\theta_j \to 0$. The function s is LIPSCHITZ continuous, with LIPSCHITZ constant given by the right hand side of (3.23) with δ set equal to δ_0. Let u and v denote the solution of the auxiliary problem for this s. Clearly u^{θ_j} tends to u uniformly as $\theta_j \to 0$ and v^{θ_j} tend to v uniformly as $\theta_j \to 0$. Since

$$\begin{aligned} s^{\theta_j}(t+\theta_j) - s^{\theta_j}(t_0+\theta_j) = \int_{t_0}^{t} \{ K_2 v_x^{\theta_j}(\gamma_2, \tau) - K_1 u_x^{\theta_j}(\gamma_1, \tau) \} d\tau - \\ - K_1 \varkappa_1^{-1} \int_{\gamma_1}^{s^{\theta_j}(t)} u^{\theta_j}(x, t) dx - K_2 \varkappa_2^{-1} \int_{s^{\theta_j}(t)}^{\gamma_2} v^{\theta_j}(x, t) dx + \\ + K_1 \varkappa_1^{-1} \int_{\gamma_1}^{s^{\theta_j}(t_0)} u^{\theta_j}(x, t_0) dx + K_2 \varkappa_2^{-1} \int_{s^{\theta_j}(t_0)}^{\gamma_2} v^{\theta_j}(x, t_0) dx, \end{aligned} \tag{3.30}$$

it follows that u, v and s satisfy the heat balance (2.6). Hence, (u, v, s) is a solution to the STEFAN problem (1.1), (1.2) and (1.3) for $0 \leq t \leq T_0$.

In order to continue a solution to (1.1), (1.2) and (1.3) to times larger than T_0, it is necessary to obtain some a priori information on solutions of the STEFAN problem (1.1), (1.2) and (1.3). For the moment, it is convenient to assume that there exists a $\mu_0 > 0$, $0 < \mu_0 < b < 1 - \mu_0 < 1$, such that

$$\alpha_1 \mu_0^{-1}(\mu_0 - x) \leq \varphi(x), \qquad 0 \leq x \leq \mu_0, \tag{3.31}$$

and

$$\alpha_2 \mu_0^{-1}((1-\mu_0) - x) \geq \psi(x), \qquad (1-\mu_0) \leq x \leq 1. \tag{3.32}$$

We shall demonstrate later that (3.31) and (3.32) can easily be removed. Note that for consistency here it it must be assumed that a_1 is related to α_1 via

$$a_1(1 - \exp\{-\varkappa_1^{-1}\eta_1 b\}) \geq \alpha_1,$$

and a_2 to α_2 via

$$a_2(1 - \exp\{-\varkappa_2^{-1}\eta_2(1-b)\}) \geq \alpha_2.$$

LEMMA 3. - If s is the free boundary of a solution (u, v, s) of (1.1), (1.2) and (1.3), then there exists a μ_1, $0 < \mu_1 \leq \mu_0$, such that $\mu_1 < s(t) < 1 - \mu_1$ for all t for which the solution is defined.

PROOF. - Since $\mu_0 < b < 1 - \mu_0$, it follows that if s takes on the value μ then there is a first time t^* at which say $s(t^*) = \mu$. Clearly, $\dot{s}(t^*) \leq 0$. But by condition (3.31) it follows from an elementary application of the maximum principle that

$$-u_x(s(t^*), t^*) \geq \alpha_1 \mu^{-1}$$

while

$$v_x(s(t^*), t^*) \geq -A,$$

where A is a constant which depends only on η_2, β_2, and a_2. Hence, from (1.3) we see that

$$\dot{s}(t^*) \geq \alpha_1 \mu^{-1} - A > 0 \tag{3.33}$$

for μ sufficiently small.

By similar reasoning to that of lemmas 1 and 2, the following lemma is valid.

LEMMA 4. - If (u, v, s) is a solution of the STEFAN problem (1.1), (1.2) and (1.3) for $0 \leq t \leq T^*$, then for $0 < t \leq T^*$

$$|u_x(s(t), t)| \leq \varepsilon_1(1 - \exp\{-\varkappa_1^{-1}\eta_1\mu_1\})^{-1} \cdot \varkappa_1^{-1}(\|\dot{s}\| + \eta_1), \tag{3.34}$$

$$|v_x(s(t), t)| \leq \varepsilon_2(1 - \exp\{-\varkappa_2^{-1}\eta_2\mu_1\})^{-1} \cdot \varkappa_2^{-1}(\|\dot{s}\| + \eta_2). \tag{3.35}$$

and

$$\|\dot{s}\| \leq \max\left(-\frac{1}{\xi^*}\log\left[\frac{1-\Gamma}{2}\right], \frac{2\Gamma\eta}{1-\Gamma}\right), \tag{3.36}$$

where $\|\dot{s}\| = \sup_{0<t\leq T^*} |\dot{s}(t)|$, η is defined by (3.20), Γ is defined by (3.19) and

$$\xi^* = \mu_1 \min(\varkappa_1^{-1}, \varkappa_2^{-1}). \tag{3.37}$$

Consider now the u component of a solution of the STEFAN problem (1.1), (1.2) and (1.3). For $t \geq T_0$, it follows from (3.36), (3.34), (2.1) and (2.2) that there exists a positive constant η_3 which is independent of t such that

$$|u_x(x, t)| \leq \eta_3 \tag{3.38}$$

for $\mu_1 \leq x \leq s(t)$. Since $0 \leq u(x, t) \leq \varepsilon_1$ and $u(s(t), t) = 0$, we have

$$0 \leq u(x, t) \leq \begin{cases} \varepsilon_1, & 0 \leq x \leq x^*, \\ \eta_3(s(t) - x), & x^* \leq x \leq s(t), \end{cases} \tag{3.39}$$

where $x^* = s(t) - \eta_3^{-1}\varepsilon_1$ and η_3 is increased if necessary so that $\eta_3^{-1}\varepsilon_1 < 2^{-1}\mu_1$ which implies that x^* is positive. Given $\varepsilon > 0$, consider the function

$$\bar{\varphi}(x) = (\varepsilon_1 + \varepsilon)(1 - \exp\{\eta_4\{x - s(t)\}\}). \tag{3.40}$$

From (3.39) it is clear that there exists an $\eta_4 = \eta_4(\varepsilon, \eta_3^{-1}\varepsilon_1)$ sufficiently large such that

$$0 \leq u(x, t) \leq \bar{\varphi}(x). \tag{3.41}$$

Similarly, there exists an η_5 such that

$$0 \geq v(x, t) \geq \bar{\psi}(x). \tag{3.42}$$

where

$$\bar{\psi}(x) = -(\varepsilon_2 + \varepsilon)(1 - \exp\{-\eta_5(x - s(t))\}). \tag{3.43}$$

Suppose now that a solution (u, v, s) of the STEFAN problem (1.1), (1.2) and (1.3) exists for $0 \leq t \leq T^*$, where $T^* \geq T_0$. Then, for ε sufficiently small and fixed, (3.41) and (3.42) imply that the discussion of Lemma 1, Lemma 2, and the paragraph following Lemma 2 can be duplicated with ε_1 replaced by $\varepsilon_1 + \varepsilon$, ε_2 replaced with $\varepsilon_2 + \varepsilon$, η_1 replaced with η_4, η_2 replaced with η_5, and δ_0 replaced with $2^{-1}\mu_1$. Hence, there exists a positive σ which does not depend upon t such that the solution (u, v, s) of the STEFAN problem (1.1), (1.2) and (1.3) can be continued to the interval $T^* \leq t \leq T^* + \sigma$. By induction, it follows that the solution (u, v, s) exists for $0 \leq t \leq T$ for any arbitrary T.

The removal of (3.31) and (3.32) is quite easy. Recall that the argument for the existence of (u, v, s) for $0 \leq t \leq T_0$ did not require the assumptions (3.31) and (3.32). Hence, for data satisfying only conditions (A), (B) and (C) of section 2, a solution exists for $0 \leq t \leq T_0$. At $t = T_0$, $u > 0$ for $0 \leq x < s(T_0)$ and $v < 0$ for $s(T_0) < x \leq 1$. Consequently the existence of a $\mu_0 > 0$ and the conditions (3.31) and (3.32) at $t = T_0$ follow from the continuity of u and v and the conditions (2.1). Reconsideration of the problem for $t \geq T_0$ and the previous arguments yields the result stated in Theorem 1.

4. - Stability.

Let (u_i, v_i, s_i), $i = 1, 2$, denote solutions of the STEFAN problem (1.1), (1.2) and (1.3) for the respective data f_i, g_i, φ_i, ψ_i, and b_i, $i = 1, 2$, which satisfy assumptions (A), (B), and (C) of section 2. Assume that

(D) φ_i and ψ_i, $i = 1, 2$, are continuously differentiable in $0 \leq x \leq \gamma$ and $1 - \gamma \leq x \leq 1$, $\gamma > 0$, respectively.

We state the following result.

THEOREM 2. - There exists a constant C which depends upon T, $\varkappa_i$, K_i, a_i, β_i, α_i, η_i, $i = 1, 2$, γ and $\min(b_1, b_2, (1 - b_1), (1 - b_2))$ such that for $0 \leq t \leq T$

$$
\begin{aligned}
(4.1)\qquad | s_1(t) - s_2(t) | \leq C \{ & \sup_{0 \leq \tau \leq t} | f_1(\tau) - f_2(\tau) | + \\
& + \sup_{0 \leq \tau \leq \gamma} | g_1(\tau) - g_2(\tau) | + \sup_{0 \leq x \leq 1} | \Phi_1(x) - \Phi_2(x) | + \\
& + \sup_{0 \leq x \leq \gamma} | \varphi_1'(x) - \varphi_2'(x) | + \sup_{1-\gamma \leq x \leq 1} | \psi_1'(x) - \psi_2'(x) | + \\
& + | b_1 - b_2 | \},
\end{aligned}
$$

where

$$
\Phi_i(x) = \begin{cases} K_1 \varkappa_1^{-1} \varphi_i(x), & 0 \leq x \leq b_i, \\ K_2 \varkappa_2^{-1} \psi_i(x), & b_i \leq x \leq 1. \end{cases}
$$

PROOF. - A straight forward application of the technique used in [3] yields

$$
\begin{aligned}
(4.2)\qquad | s_1(t) - s_2(t) | \leq C_1 \Big\{ & | b_1 - b_2 | + \int_0^1 | \Phi_1(x) - \Phi_2(x) | \, dx + \\
& + \int_0^t K_1 \left| \frac{\partial u_1}{\partial x}(\gamma_1, \tau) - \frac{\partial u_2}{\partial x}(\gamma_1, \tau) \right| d\tau +
\end{aligned}
$$

$$+\int_0^t K_2 \left| \frac{\partial v_1}{\partial x}(\gamma_2, \tau) - \frac{\partial v_2}{\partial x}(\gamma_2, \tau) \right| d\tau \Big\}$$

for $0 \leq t \leq T$, $0 < \gamma_1 < \gamma$ and $1 - \gamma < \gamma_2 < 1$. Next, $\frac{\partial u_1}{\partial x}(\gamma_1, \tau) - \frac{\partial u_2}{\partial x}(\gamma_2, \tau)$ and $\frac{\partial v_1}{\partial x}(\gamma_2, \tau) - \frac{\partial v_2}{\partial x}(\gamma_2, \tau)$ can easily be estimated in terms of the data and $\sup_{0 \leq \eta \leq \tau} | s_1(\eta) - s_2(\eta) |$. Consequently, (5.1) follows from an application of GRONWALL's lemma via [2, lemma 2 page 380].

5. - Uniqueness.

As a corollary to the stability result we have the following uniqueness theorem.

THEOREM 3. - Under the hypotheses (A), (B), (C), and (D), the solution (u, v, s) of the STEFAN problem (1.1), (1.2) and (1.3) is unique.

6. - Monotone dependence.

The following result is a consequence of the maximum principle and the stability theorem.

THEOREM 4. - Under the assumptions (A), (B), (C), and (D) if (u_i, v_i, s_i); $i = 1, 2$, are solutions of the STEFAN problem (1.1), (1.2), and (1.3) corresponding to the data f_i, g_i, φ_i, ψ_i, and b_i and if $f_1 \leq f_2$, $g_1 \leq g_2$, $\varphi_1 \leq \varphi_2$, $\psi_1 \leq \psi_2$, and $b_1 \leq b_2$, then $s_1(t) \leq s_2(t)$ for all $t > 0$.

PROOF. - Suppose first that $b_1 < b_2$. Then, $s_1(t) < s_2(t)$. If not, then there exists a first time t^* such that $s_1(t^*) = s_2(t^*)$ and $\dot{s}_1(t^*) \geq \dot{s}_2(t^*)$. By the maximum principle, $u_2 - u_1 > 0$ in $0 < x < s_1(t)$, $0 < t \leq t^*$. At $(s_1(t^*), t^*)$, $u_2 - u_1 = 0$. Hence, by the parabolic version of HOPF's lemma,

$$\frac{\partial u_2}{\partial x}(s_1(t^*), t^*) < \frac{\partial u_1}{\partial x}(s_1(t^*), t^*).$$

Similarly,

$$\frac{\partial v_2}{\partial x}(s_1(t^*), t^*) > \frac{\partial v_1}{\partial x}(s_1(t^*), t^*).$$

By (1.3), it follows that $\dot{s}_2(t^*) > \dot{s}_2(t)$. Hence, $s_1(t) < s_2(t)$. The remainder of the theorem follows from the stability by considering the solutions of (1.1), (1.2), and (1.3) for $b_2 + \varepsilon$, f_2, g_2, φ_2, and ψ_2 and by letting ε tend to zero.

7. - Asymptotic behavior.

Referring to the asympotic behavior result in [8; p. 71] if f and g are such that

$$\int_1^\infty \{ (f(\tau) - f_\infty)^2 + (g(\tau) - g_\infty)^2 \} \, d\tau < \infty, \tag{7.1}$$

where f_∞ is a positive constant, $\alpha_1 < f_\infty < \beta_1$ and g_∞ is a negative constant, $-\beta_2 < g_\infty < -\alpha_2$, then the solution (u, v, s) of problem (1.1), (1.2), and (1.3) tends asymptotically to the steady state solution

$$\begin{aligned} u_\infty &= b_\infty^{-1} f_\infty (b_\infty - x), \qquad 0 \le x \le b_\infty, \\ v_\infty &= (1 - b_\infty)^{-1} (-g_\infty)(b_\infty - x), \qquad b_\infty \le x \le 1, \end{aligned} \tag{7.2}$$

where

$$b_\infty = K_1 f_\infty (K_1 f_\infty + K_2 (-g_\infty))^{-1}. \tag{7.3}$$

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