Mapping properties of matrix summability methods

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Summary. - *Conditions on a complex matrix A are given which are necessary and sufficient for A to have the property that if x belongs to the complex linear space generated by the set of all convergent real sequences having nonnegative k th differences, then Ax belongs to the complex linear space generated by the set of all convergent real sequences having nonnegative j th differences.*

If q is a positive integer, let BV_q^* denote the complex linear space generated by the set of all convergent real sequences having nonnegative q th differences. The purpose of this paper is to prove the following result.

THEOREM. - A complex matrix $A = (a_{pq})$ has the property that $Ax \in BV_j^*$ whenever $x \in BV_k^*$ if and only if the following conditions hold:

- (1) A has convergent columns,
- (2) $\left\{ \sum_{q=1}^{\infty} a_{pq} \right\}_{p=1}^{\infty} \in BV_j^*$,
- (3) there exists a number L such that

$$\left| \sum_{s_{k-1}=1}^{s_k=n} \sum_{s_{k-2}=1}^{s_{k-1}} \dots \sum_{s_1=1}^{s_2} \sum_{q=1}^{s_1} a_{pq} \right| < \binom{n+k-2}{k-1} L,$$

$$n, p = 1, 2, 3, \dots,$$

- (4) there exists a number M such that

$$\sum_{p=1}^{\infty} \binom{p+j-2}{j-1} \left| \sum_{s_{k-1}=1}^{s_k=n} \sum_{s_{k-2}=1}^{s_{k-1}} \dots \sum_{s_1=1}^{s_2} \sum_{q=1}^{s_1} \Delta_1^j a_{pq} \right| < \binom{n+k-2}{k-1} M,$$

$$n = 1, 2, 3, \dots,$$

where $\Delta_1 a_{pq} = a_{pq} - a_{p+1,q}$, $\Delta_1^2 a_{pq} = \Delta_1 a_{pq} - \Delta_1 a_{p+1,q}$, etc.

We note that BV_1^* is the class of absolutely convergent complex sequen-

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ces (or sequences of bounded variation), and BV_2^* is generated by the set of all convergent real convex sequences.

In [3] it was shown that BV_q^* consists of all convergent complex sequences $z = \{z_p\}$ such that

$$\sum_{p=1}^{\infty} \binom{p+q-2}{q-1} |\Delta^q z_p| < \infty,$$

where $\Delta z_p = z_p - z_{p+1}$, $\Delta^2 z_p = \Delta z_p - \Delta z_{p+1}$, etc.

The following theorem was proved in [1].

THEOREM D. - A complex matrix $A = (a_{pq})$ sums every sequence in BV_k^* if and only if the following conditions hold:

- (i) A has convergent columns,
- (ii) $\{\sum_{q=1}^{\infty} a_{pq}\}_{p=1}^{\infty}$ converges,
- (iii) there exists a number L such that

$$\left| \sum_{s_{k-1}=1}^{s_k=n} \sum_{s_{k-2}=1}^{s_{k-1}} \dots \sum_{s_1=1}^{s_2} \sum_{q=1}^{s_1} a_{pq} \right| < \binom{n+k-2}{k-1} L,$$

$n, p = 1, 2, 3, \dots$

We will need the following lemma from [3].

LEMMA. - If $x \in BV_{q+1}^*$, then

- 1) $\binom{p+r-1}{r} \Delta^r x_p \rightarrow 0$ as $p \rightarrow \infty$, $r = 1, 2, \dots, q$,
- 2) $\sum_{p=1}^{\infty} \binom{p+r-2}{r-1} \Delta^r x_p = x_1 - \lim_n x_n$, $r = 1, 2, \dots, q+1$.

For simplicity of notation we will let

$$v_{pn} = \sum_{s_{k-1}=1}^{s_k=n} \sum_{s_{k-2}=1}^{s_{k-1}} \dots \sum_{s_1=1}^{s_2} \sum_{q=1}^{s_1} a_{pq}, \quad p, n = 1, 2, 3, \dots,$$

and note that

$$\Delta_1^j v_{pn} = \sum_{s_{k-1}=1}^{s_k=n} \sum_{s_{k-2}=1}^{s_{k-1}} \dots \sum_{s_1=1}^{s_2} \sum_{q=1}^{s_1} \Delta_1^j a_{pq}, \quad p, n = 1, 2, 3, \dots.$$

We will now prove the theorem and will show the sufficiency first. Suppose $x \in BV_k^*$. We see from (2) that if x is a constant term sequence, then

$Ax \in BV_j^*$. Thus since A is a linear transformation we need only consider the case that x is a null sequence. Since (2) of the theorem implies (ii) of Theorem D, we see that the conditions of Theorem D are satisfied, and so Ax converges. Thus in order to show that $Ax \in BV_j^*$, we must show that

$$\sum_{p=1}^{\infty} \binom{p+j-2}{j-1} \left| \sum_{q=1}^{\infty} \Delta_1^j a_{pq} x_q \right| < \infty.$$

We note from (ii) of Theorem D that there exists L_p such that

$\left| \sum_{q=1}^n a_{pq} \right| < L_p$, $n, p = 1, 2, 3, \dots$. Using summation-by-parts, we have

$$\sum_{q=1}^n \Delta_1^j a_{pq} x_q = \sum_{s_1=1}^{n-1} \left(\sum_{q=1}^{s_1} \Delta_1^j a_{pq} \right) \Delta x_{s_1} + \left(\sum_{q=1}^n \Delta_1^j a_{pq} \right) x_n.$$

Consequently

$$\sum_{q=1}^{\infty} \Delta_1^j a_{pq} x_q = \sum_{s_1=1}^{\infty} \left(\sum_{q=1}^{s_1} \Delta_1^j a_{pq} \right) \Delta x_{s_1}, \quad p = 1, 2, 3, \dots,$$

since

$$\left| \sum_{q=1}^n \Delta_1^j a_{pq} x_n \right| < 2^j \sum_{p=1}^{j+1} L_p |x_n| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Again using summation-by-parts, we have

$$\sum_{s_1=1}^n \sum_{q=1}^{s_1} \Delta_1^j a_{pq} x_q = \sum_{s_2=1}^{n-1} \left(\sum_{s_1=1}^{s_2} \sum_{q=1}^{s_1} \Delta_1^j a_{pq} \right) \Delta^2 x_{s_2} + \left(\sum_{s_1=1}^n \sum_{q=1}^{s_1} \Delta_1^j a_{pq} \right) \Delta x_n,$$

and so

$$\sum_{s_1=1}^{\infty} \sum_{q=1}^{s_1} \Delta_1^j a_{pq} x_q = \sum_{s_2=1}^{\infty} \sum_{s_1=1}^{s_2} \sum_{q=1}^{s_1} \Delta_1^j a_{pq} \Delta^2 x_{s_2}, \quad p = 1, 2, 3, \dots,$$

since

$$\left| \sum_{s_1=1}^n \sum_{q=1}^{s_1} \Delta_1^j a_{pq} \Delta x_n \right| < 2^j \sum_{p=1}^{j+1} L_p n |\Delta x_n| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

by 1) of the lemma.

Continuing to use summation-by-parts and 1) of the lemma, we can show that

$$(*) \quad \sum_{q=1}^{\infty} \Delta_1^j a_{pq} x_q = \sum_{n=1}^{\infty} \Delta_1^j v_{pn} \Delta^k x_n, \quad p = 1, 2, 3, \dots$$

We note that neither (3) nor (4) was used to prove (*). Using (*), (4), and interchange of order of summation, we obtain

$$\begin{aligned}
\sum_{p=1}^{\infty} \binom{p+j-2}{j-1} \left| \sum_{q=1}^{\infty} \Delta_1^j a_{pq} x_q \right| &\leq \sum_{p=1}^{\infty} \binom{p+j-2}{j-1} \sum_{n=1}^{\infty} |\Delta_1^j v_{pn}| |\Delta^k x_n| \\
&= \sum_{n=1}^{\infty} |\Delta^k x_n| \sum_{p=1}^{\infty} \binom{p+j-2}{j-1} |\Delta_1^j v_{pn}| \\
&< M \sum_{n=1}^{\infty} \binom{n+k-2}{k-1} |\Delta^k x_n| \\
&< \infty.
\end{aligned}$$

Hence $Ax \in BV_j^*$. This completes the proof of the sufficiency.

Next we will assume that A has the property that $Ax \in BV_j^*$ whenever $x \in BV_k^*$ and prove (1)-(4). From Theorem *D* we see that since A sums every sequence in BV_k^* , then (1) and (3) hold. Clearly (2) holds since $Ax \in BV_j^*$ whenever $x \in BV_k^*$ and $\{1, 1, 1, \dots\} \in BV_k^*$. Assume that (4) does not hold. Suppose there exists a positive integer n such that

$$\sum_{p=1}^{\infty} \binom{p+j-2}{j-1} |\Delta_1^j v_{pn}| = \infty.$$

It is easy to show that the sequence t defined by

$$t_p = \begin{cases} \binom{n+k-p-1}{k-1}, & p = 1, 2, \dots, n, \\ 0, & p > n, \end{cases}$$

is an element of BV_k^* , and that $\Delta^k t_q = 1$ if $p = n$, $\Delta^k t_p = 0$ if $p \neq n$. Thus $At \in BV_j^*$. But by (*) we have

$$\begin{aligned}
\sum_{p=1}^{\infty} \binom{p+j-2}{j-1} \left| \sum_{q=1}^{\infty} \Delta_1^j a_{pq} t_q \right| &= \sum_{p=1}^{\infty} \binom{p+j-2}{j-1} \left| \sum_{r=1}^{\infty} \Delta_1^j v_{pr} \Delta^k t_r \right| \\
&= \sum_{p=1}^{\infty} \binom{p+j-2}{j-1} \Delta^k t_n |\Delta_1^j v_{pn}| \\
&= \infty,
\end{aligned}$$

which is a contradiction. Hence

$$\sum_{p=1}^{\infty} \binom{p+j-2}{j-1} |\Delta_1^j v_{pn}| < \infty, \quad n = 1, 2, 3, \dots$$

Since by (2), $\{\sum_{\infty}^{q=1} a_{pq}\}_{\infty}^{p=1} \in BV_j^*$, there exists a number R such that

$$\sum_{p=1}^{\infty} \binom{p+j-2}{j-1} |\Delta_1^j \sum_{q=1}^{\infty} a_{pq}| < R/2.$$

Since we are assuming that (4) does not hold, there exist positive integers q_1 and m_1 such that

$$\sum_{p=1}^{m_1} \binom{p+j-2}{j-1} |\Delta_1^j v_{pq_1}| > \binom{q_1+k-2}{k-1} 2^k (R+2) (k!),$$

and

$$\sum_{p=m_1+1}^{\infty} \binom{p+j-2}{j-1} |\Delta_1^j v_{pq_1}| < \binom{q_1+k-2}{k-1}.$$

There exists $N_1 > q_1$ such that if $n > N_1$, then

$$\sum_{p=1}^{m_1} \binom{p+j-2}{j-1} |\Delta_1^j v_{pn}| < R \binom{n+k-2}{k-1},$$

since $\binom{r+k-2}{k-1}^{-1} v_{pr} \rightarrow \sum_{q=1}^{\infty} a_{pq}$ as $r \rightarrow \infty$, $p = 1, 2, 3, \dots$. Let $q_2 > N_1$ and $m_2 > m_1$ be integers such that $q_2 - q_1 > q_1 - q_0$ ($q_0 = 0$), $q_2^{k-2} > 2q_1^{k-2}$ if $k > 2$,

$$\sum_{p=m_1+1}^{m_2} \binom{p+j-2}{j-1} |\Delta_1^j v_{pq_2}| > \binom{q_2+k-2}{k-1} 2^{k+1} (R+2) (k!),$$

and

$$\sum_{p=m_2+1}^{\infty} \binom{p+j-2}{j-1} |\Delta_1^j v_{pq_2}| < \binom{q_2+k-2}{k-1}.$$

There exists $N_2 > q_2$ such that if $n > N_2$, then

$$\sum_{p=m_1+1}^{m_2} \binom{p+j-2}{j-1} |\Delta_1^j v_{pn}| < R \binom{n+k-2}{k-1}.$$

Let $q_3 > N_2$ and $m_3 > m_2$ be integers such that $q_3 - q_2 > q_2 - q_1$, $q_3^{k-2} > 2q_2^{k-2}$ if $k > 2$,

$$\sum_{p=m_2+1}^{m_3} \binom{p+j-2}{j-1} |\Delta_1^j v_{pq_3}| > \binom{q_3+k-2}{k-1} 2^{k+2} (R+2) (k!),$$

and

$$\sum_{p=m_3+1}^{\infty} \binom{p+j-2}{j-1} |\Delta_1^j v_{pq_3}| < \binom{q_3+k-2}{k-1}.$$

Continue the process to obtain sequences $\{m_p\}_{p=1}^{\infty}$, $\{q_p\}_{p=0}^{\infty}$, and $\{N_p\}_{p=1}^{\infty}$.

If n is a positive integer and $k \geq 2$, let $\mathcal{L}_n$ be the line which contains the points

$$(q_{n-1} + 1, 2^{1-n} q_n^{2-k}) \text{ and } (q_n + 1, 2^{-n} q_{n+1}^{2-k}).$$

We define a sequence $\{x_p\}$ as follows. If $q_{n-1} + 1 \leq t < q_n + 1$, then x_t is the ordinate of the point on $\mathcal{L}_n$ having abscissa t . If $k=1$, we define a sequence $\{z_p\}$ as follows. If $q_{n-1} + 1 \leq t < q_n + 1$, then $z_t = 2^{-n}$. Clearly $z \in BV_1^*$. In [1], for the case that $k > 2$, a null sequence $x^{(k-2)} \in BV_k^*$ was constructed from x in the following manner. Let

$$x_p^{(r)} = \sum_{q=p}^{\infty} x_q^{(r-1)}, \quad r = 1, 2, \dots, k-2; \quad p = 1, 2, 3, \dots,$$

where $x_q^{(0)} = x_q$, $q = 1, 2, 3, \dots$. It was shown in [1] that

$$\Delta^k x_{q_m}^{(k-2)} > 2^{-m-1} q_m^{1-k}, \quad m = 1, 2, 3, \dots,$$

$\Delta^k x_p^{(k-2)} = 0$ if there is no m such that $p = q_m$, and that

$$x_1^{(k-2)} < 2^{k-2}.$$

In [2], for the case that $k=2$, it was shown that

$$\Delta^2 x_{q_m} > 2^{-m-1} / (q_m - q_{m-1}), \quad m = 1, 2, 3, \dots,$$

$\Delta^2 x_p = 0$ if there is no m such that $p = q_m$, and that

$$\sum_{p=1}^{\infty} p \Delta^2 x_p = 1.$$

Suppose $k \geq 2$. Then

$$\begin{aligned} & \sum_{p=m_r+1}^{m_{r+1}} \binom{p+j-2}{j-1} |\Delta_1^j \sum_{q=1}^{\infty} a_{pq} x_q^{(k-2)}| \\ &= \sum_{p=m_r+1}^{m_{r+1}} \binom{p+j-2}{j-1} |\sum_{n=1}^{\infty} \Delta_1^j v_{pn} \Delta^k x_n^{(k-2)}| \end{aligned}$$

$$\begin{aligned}
&\geq \sum_{p=m_r+1}^{m_r+1} \binom{p+j-2}{j-1} |\Delta_1^j v_{pq_{r+1}} \Delta^k x_{q_{r+1}}^{(k-2)}| \\
&- \sum_{p=m_r+1}^{m_r+1} \binom{p+j-2}{j-1} \left| \sum_{n=1}^r \Delta_1^j v_{pq_n} \Delta^k x_{q_n}^{(k-2)} \right| \\
&- \sum_{p=m_r+1}^{m_r+1} \binom{p+j-2}{j-1} \left| \sum_{n=r+2}^{\infty} \Delta_1^j v_{pq_n} \Delta^k x_{q_n}^{(k-2)} \right| \\
&> \binom{q_{r+1}+k-2}{k-1} 2^{k+r} (R+2) (k!) 2^{-r-2} q_{r+1}^{1-k} \\
&- \sum_{n=1}^r \Delta^k x_{q_n}^{(k-2)} \sum_{p=m_r+1}^{m_r+1} \binom{p+j-2}{j-1} |\Delta_1^j v_{pq_n}| \\
&- \sum_{n=r+2}^{\infty} \Delta^k x_{q_n}^{(k-2)} \sum_{p=m_r+1}^{m_r+1} \binom{p+j-2}{j-1} |\Delta_1^j v_{pq_n}| \\
&> 2^{k-2} (R+2) - \sum_{n=1}^r \binom{q_n+k-2}{k-1} \Delta^k x_{q_n}^{(k-2)} \\
&- R \sum_{n=r+2}^{\infty} \binom{q_n+k-2}{k-1} \Delta^k x_{q_n}^{(k-2)} \\
&\geq 2^{k-2}.
\end{aligned}$$

Similarly, for $k=1$, it is easy to show that

$$\sum_{p=m_r+1}^{m_r+1} \binom{p+j-2}{j-1} |\Delta_1^j \sum_{q=1}^{\infty} a_{pq} z_q| = \sum_{p=m_r+1}^{m_r+1} \binom{p+j-2}{j-1} \left| \sum_{n=1}^{\infty} \sum_{q=1}^n \Delta_1^j a_{pq} \Delta z_n \right| > 2^{k-2}.$$

Hence if $y_p = x_p^{(k-2)}$ for $k \geq 2$, $y_p = z_p$ for $k=1$, then

$$\sum_{p=1}^{\infty} \binom{p+j-2}{j-1} |\Delta_1^j \sum_{q=1}^{\infty} a_{pq} y_q| = \infty,$$

and we have a contradiction, since $y \in BV_k^*$. Thus (4) holds, and the proof is complete.

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